Representing cohomology of low-dimensional manifolds by maps to manifolds

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Recall that the integral cohomology of a space X is in natural bijection with homotopy classes of maps to an Eilenberg-Maclane space, i.e.

$$H^k(X, \mathbb{Z}) = [X, K(\mathbb{Z}, k)].$$

If the space X is a manifold of dimension n, then by cellular approximation we have that homotopy classes of maps from X to $K(\mathbb{Z}, k)$ are in bijection with homotopy classes of maps from X to the $n+1$ skeleton of $K(\mathbb{Z}, n)$,

$$[X, K(\mathbb{Z}, k)] = [X, K(\mathbb{Z}, k)^{n+1}].$$

We exploit this fact and construct models of Eilenberg-Maclane spaces that look like manifolds up to some low-dimensional skeleton, in order to represent the cohomology of X by maps to this manifold skeleton.

First of all, note that S^1 is a $K(\mathbb{Z}, 1)$. Therefore $H^1(X, \mathbb{Z}) = [X, S^1]$.

Next, recall that a model for $K(\mathbb{Z}, 2)$ is given by $\mathbb{C}P^\infty$. Since $(\mathbb{C}P^\infty)^{(2n+1)} = \mathbb{C}P^n$, we conclude that $H^2(X, \mathbb{Z}) = [X, \mathbb{C}P^{\lfloor \frac{n+1}{2} \rfloor}]$.

Representing the top cohomology of a manifold is also easy. Form a model for $K(\mathbb{Z}, n)$ by taking an S^n and attaching $n+2$ cells and higher to kill the higher homotopy groups. Note that, as constructed, $K(\mathbb{Z}, n)^{(n+1)} = S^n$. Therefore, for an n-manifold X, we have

$$H^n(X, \mathbb{Z}) = [X, K(\mathbb{Z}, n)] = [X, K(\mathbb{Z}, n)^{(n+1)}] = [X, S^n].$$

Observe that we have described all the cohomology of manifolds up to dimension 4 by maps to other manifolds, save for H^3 of a four manifold. If we could find a manifold M with $\pi_1(M) = 0, \pi_2(M) = 0, \pi_3(M) = \mathbb{Z}, \pi_4(M) = 0$, then we would have $H^3(X, \mathbb{Z}) = [X, M]$. Indeed, we could make a model for $K(\mathbb{Z}, 3)$ by attaching 6-cells and higher to kill off the fifth homotopy group of M and above, and we would have

$$[X, M] = [X, M^{(5)}] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}).$$

Suppose M is a manifold with the above listed homotopy groups. We first show that the dimension of M must be at least eight. Note that M is orientable, since it has trivial fundamental group. Obviously, M cannot be of dimension one (i.e. a circle). Every two-manifold is either a sphere, which has π_2, or has non-trivial π_1. A three-manifold with π_1 and π_2 zero, and π_3 equal to $\mathbb{Z}$ is a homotopy S^3, which has $\pi_4(S^3) = \mathbb{Z}_2$. By the Hurewicz theorem we conclude that $H_1(M) = 0, H_2(M) = 0, H_3(M) = \mathbb{Z}, H_4(M) = 0$, where the conclusion about H_4 follows from the surjectivity of π_4 onto H_4. (This observation for homology would have also eliminated the cases of dimensions one and two.) Since $H_4 = 0$, M cannot be a four manifold. If M were a five manifold, Poincare duality would imply that $H_2 = \mathbb{Z}$. Now, if M were of dimension six, we would have that its middle homology group H_3 is $\mathbb{Z}$. However, the intersection form on an oriented manifold of dimension two mod four is skew-symmetric, and hence the underlying free abelian group must have even rank, a contradiction. Lastly, M cannot be of dimension seven since that would imply $H_4(M) = \mathbb{Z}$ by duality.

Now we exhibit an eight dimensional manifold satisfying the above properties. Consider the eight manifold $SU(3)$. All the special unitary groups are simply connected, and π_2 of a Lie group vanishes, so we have $\pi_1(SU(3)) = \pi_2(SU(3)) = 0$. From the long exact sequence in homotopy for the fibration

$$SU(2) \to SU(3) \to S^5$$

we conclude $\pi_3(SU(3)) = \pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$. Finally, from the fibration

$$SU(3) \to SU(4) \to S^7$$

we have $\pi_4(SU(3)) = \pi_(SU(4))$. The fibration

$$SU(4) \to SU(5) \to S^9$$

tells us $\pi_4(SU(4)) = \pi_4(SU(5))$. And so on, inductively we conclude $\pi_4(SU(3)) = \pi_4(SU(n))$ for all $n \geq 4$. Since $U(n) = SU(n) \times S^1$, we conclude $\pi_4(SU(3)) = \pi_4(SU(n)) = \pi_4(U(n))$ for all $n \geq 4$. In other words,

$$\pi_4(SU(3)) = \operatorname{colim} \pi_4(U(n)) = \pi_4(\operatorname{colim} U(n)) = \pi_4(U),$$

which is 0 by Bott periodicity.

In conclusion, the third integral cohomology of a four manifold is represented by maps to $SU(3)$.

For manifolds of dimension four or less, we thus have the following table.

	dim 1	dim 2	dim 3	dim 4
H^1	S^1	S^1	S^1	S^1
H^2		S^2	$\mathbb{C}P^2$	$\mathbb{C}P^2$
H^3			S^3	$SU(3)$
H^4				S^4