CLOSED LIE GROUPS ARE RATIONALLY PRODUCTS OF ODD SPHERES

ABSTRACT. Following Félix-Halperin-Thomas section 12(a) Example 3, we show that a closed Lie group (meaning, a Lie group which is closed as a manifold) has minimal model given by the exterior algebra in generators in odd degrees, with trivial differential. In fact, the proof that follows works for any closed manifold with a continuous multiplication $m : X \times X \to X$.

Let X be a closed Lie group with multiplication $m : X \times X \to X$. The multiplication induces a map on rational cohomology, which by Künneth we can write as

$$m^* : H^* \to H^* \otimes H^*,$$

where $H^* = H^*(X)$. Let H^+ denote the positive degree elements in $H^*(X)$. Denote by V the vector space of indecomposable elements in H^+, i.e. those that cannot be written as the sum of products of positive degree elements. We have the decomposition $H^+ = V \oplus (H^+ \cdot H^+)$ (meaning, every element is a sum of indecomposable and decomposables). The inclusion of vector spaces $V \hookrightarrow H^+$ induces a map of exterior algebras

$$\varphi : \Lambda V \to H^*.$$

Observe that this map is surjective, since indecomposables are certainly hit, and decomposables can be decomposed into indecomposables. Now let us show that is also injective. We do this inductively over the filtration

$$\Lambda V^{\leq 0} \subset \Lambda V^{\leq 1} \subset \Lambda V^{\leq 2} \subset \cdots,$$

where $\Lambda V^{\leq n}$ denotes the exterior algebra on those elements of V which have degree at most n. Certainly φ is injective on $\Lambda V^{\leq 0}$, since there we have only the constants. Suppose φ is injective on $\Lambda V^{\leq n-1}$ and take a $w \in \Lambda V^{\leq n}$ such that $\varphi(w) = 0$. Since H^* is finitely generated, we can write w as a finite sum,

$$w = \sum_I v_1^{k_1} \cdots v_r^{k_r} a_I,$$

where the v_i are a basis for the degree n elements in V, a is in $\Lambda V^{\leq n-1}$, and I an index set with elements k_j.

We want to show that $w = 0$. We consider $m^*(\varphi(w))$, which is the sum of all $a \otimes b$ such that $m(a, b) = \varphi(w)$. There are many elements that multiply to w, but let us consider only those such that the first factor, a, is an element of degree n in V. So, we are considering the projection of $m^*(\varphi(w)) \in H^* \otimes H^*$ in $V^n \otimes H^*$. This projected element looks like

$$\sum_{i=1}^{r} \big(\pm v_i \otimes \varphi(\sum_I k_i v_1^{k_1} \cdots v_i^{k_i - 1} \cdots v_r^{k_r} a_I)\big).$$

Here we chipped off one v_i from the right side of the tensor and put it to the left. We can do this once for every instance of v_i on the right, hence the factor of k_i in the smaller sum. The factor of $\pm$ on the left shows up due to graded commutativity in moving v_i out to the left. Now, since $\varphi(w)$ was assumed 0, this sum is 0. The v_i are linearly independent, so we have, for every i, $\varphi(\sum_I k_i v_1^{k_1} \cdots v_i^{k_i-1} \cdots v_r^{k_r} a_I) = 0$. Within this inductive step, we can do another induction on the degree of w (meaning, the highest degree pure element that shows up in w written as a sum of elements) for the desired conclusion that $\varphi(w) = 0$ implies $w = 0$. The base case is trivial since then we are only dealing with constants. Now by this second layer of induction, from $\varphi(\sum_I k_i v_1^{k_1} \cdots v_i^{k_i-1} \cdots v_r^{k_r} a_I) = 0$ we conclude $\sum_I k_i v_1^{k_1} \cdots v_i^{k_i-1} \cdots v_r^{k_r} a_I = 0$ since we have decreased the degree of w by n. Now, assuming none of the products of powers of the v_i's are zero (due to, say, graded commutativity), these products of powers are linearly independent in their common grading. Therefore, all the a_I are 0, and hence $w = 0$. That, or all the exponents attached to the v_i are zero, in which case w is a sum of a_I's, which would mean $w \in \Lambda V^{\leq n-1}$, and so we could apply our first layer of induction to conclude $w = 0$.

Therefore, φ defines an isomorphism of dga's from ΛV to H^*. Note that since H^* is zero for sufficiently high gradings (since X is a manifold), this isomorphism with the exterior algebra ΛV tells us that there can be no elements of even degree in V (since otherwise we would have elements of arbitrarily high grading in ΛV and thus in H^*).

In general, if we have that the cohomology algebra $H^*(X)$ of a space X is isomorphic to some exterior algebra ΛV via a map $\varphi : \Lambda V \to H^*$, we can define a new map to the rational polynomial forms on X by mapping an element w of ΛV to a form whose class in cohomology is $\varphi(w)$. Since φ is an isomorphism, and the cohomology algebra of ΛV is itself, this map is a quasi-isomorphism. Therefore ΛV is a minimal model for the space X.

So, we have shown that the minimal model of a Lie group is an exterior algebra on odd generators with trivial differential.

As an immediate consequence of the above, we obtain the following:

Corollary 0.1. *All of the even homotopy groups of a closed Lie group G are purely torsion.*

Proof. The only non-zero rational homotopy group of an odd sphere S^n is $\pi_n(S^n) \otimes \mathbb{Q} = \mathbb{Q}$, since the minimal model of such a sphere is given by $\Lambda(x_n)$, the exterior algebra in one generator in degree n. From the multiplicativity of the π_k functors we conclude that $\pi_k(G) \otimes \mathbb{Q}$ can be non-zero only for odd k. □

Corollary 0.2. *A closed simply connected Lie group G does not admit a symplectic structure.*

Proof. Again from the minimal model we conclude that $H^2(G, \mathbb{Q}) = 0$. (Observe that for this conclusion it suffices to assume just that there is at most one factor of S^1 in the decomposition of G as a product of odd spheres.) Therefore, any element $\omega \in H^2(G, \mathbb{Z})$ is torsion. A symplectic form would have to have some positive power be a non-trivial element in top cohomology, and therefore could not be torsion. Therefore G cannot be symplectic. □

Now let us work out an example to illustrate how to see exactly which odd spheres show up in the decompositions of some commonly used Lie groups. This method will use knowledge of the rational cohomology ring of the classifying space of the group.

Example 0.3. Let us figure out the rational homotopy type of $SO(6)$. In order to do so, we use that $SO(6)$ is the homotopy fiber of the map $\{\cdot\} \to BSO(6)$ (where $\{\cdot\}$ is the set with one element, thought of as $ESO(6)$). The cohomology algebra of $BSO(6)$ is the exterior algebra $\Lambda(p_1, p_2, e_6)$ with trivial differential, where p_1 and p_2 are Pontryagin classes (in degrees 4 and 8), and e_6 is the Euler class (in degree 6). (We omit p_3 from the algebra since $e_6^2 = p_3$.) In this situation, where the cohomology algebra is an exterior algebra with trivial differential, the minimal model and cohomology algebra coincide (in particular, the space considered is formal).

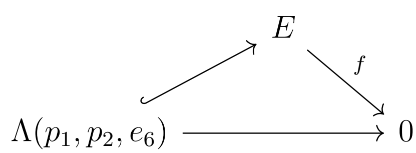
Consider now the induced map on cohomology algebras (or, minimal models)

$$H^*(BSO(6), \mathbb{Q}) \to H^*(\{\cdot\}, \mathbb{Q}),$$

that is,

$$\Lambda(p_1, p_2, e_6) \longrightarrow \{0\}.$$

To find the homotopy fiber of a map between topological spaces, we convert that map into a fibration and take its fiber. In this dual, algebraic situation, converting this map into a fibration means finding a differential graded algebra E containing $\Lambda(p_1, p_2, e_6)$ with a quasi-isomorphism f to the zero algebra, such that this diagram commutes:



We want E to be quasi-isomorphic to the zero algebra, so we add variables to $\Lambda(p_1, p_2, e_6)$ to kill everything in cohomology. Namely, introduce η_3, η_7, and η_5 such that $d\eta_3 = p_1$, $d\eta_7 = p_2$, and $d\eta_5 = e_6$. So, we will take E to be the exterior algebra $\Lambda(p_1, p_2, e_6, \eta_3, \eta_7, \eta_5)$ with differential given by $d\eta_3 = p_1$, $d\eta_7 = p_2$, $d\eta_5 = e_6$. The map $f : E \to 0$ is necessarily the zero map, and it is a quasi-isomorphism by construction of E. (Observe that E has trivial cohomology. All the closed forms were made exact by introducing these new variables, and these new variables are not closed.)

The homotopy fiber of the map $\{\cdot\} \to BSO(6)$ has model equal to the "fiber" of the inclusion $\Lambda(p_1, p_2, e_6) \hookrightarrow E$. This "fiber" is by definition the quotient of E by the ideal generated by the generators in the "base", i.e. the ideal generated by p_1, p_2, e_6. We obtain $\Lambda(\eta_3, \eta_7, \eta_5)$ with trivial differential as our fiber. This algebra is also the minimal model of $S^3 \times S^5 \times S^7$, so rationally

$$SO(6) \underset{\mathbb{Q}}{\cong} S^3 \times S^5 \times S^7.$$