Instantons and four manifolds

Preparation for minor topic in orals exam

Introduction/Glossary.

In the category of topological manifolds, we can consider the question of obtaining piecewise-linear (PL) structures, and perhaps even smooth structures on a given manifold.

Denote the group of homeomorphisms, PL isomorphisms, and diffeomorphisms of $\mathbb{R}^n$ by Top_n, PL_n, and Diff_n respectively. Taking their limits we obtain the groups we call Top, PL, Diff, which have classifying spaces. For a topological manifold X we obtain a map $X \to \mathrm{BTop}$. A smooth structure implies PL implies topological structure, and for dimension $n \geq 5$, lifts (up to homotopy) of this classifying map to BPL and BDiff correspond to isotopy classes of PL and smooth structures, respectively. The homotopy fibers of the maps $\mathrm{BO} \sim \mathrm{BDiff} \to \mathrm{BPL} \to \mathrm{BTop}$ are

- PL/O, whose homotopy groups are the groups Γ_k of diffeomorphism classes of S^k. These groups are 0 for $k \leq 6$.
- Top/PL, which is a $K(\mathbb{Z}_2, 3)$. The obstruction to lifting obtained here is the Kirby-Siebenmann invariant in $H^4(-, \mathbb{Z}_2)$.

For $n = 3$ the categories are equivalent: every topological manifold has a PL structure, and every PL manifold has a smooth structure. For $n = 4$, PL coincides with Diff. The obstructions to obtaining a smooth structure are, among other things, the Kirby-Siebenmann invariant, the signature divided by 8 (mod 16) if the manifold is spin, the form of the intersection form (by Donaldson's theorem), etc. For $n = 5$, the three categories are distinct. Lifting from Top to PL has a single obstruction, which is the Kirby-Siebenmann invariant. Lifting from PL to Top has obstructions with coefficients in the groups Γ_k.

It should be emphasized that BO is homotopy equivalent to BDiff. If we are interested in almost complex structures, we consider the map $\mathrm{BU} \to \mathrm{BO}$, which has homotopy fiber O/U, which turns out be homotopy equivalent to ΩO (whose homotopy groups we know by Bott periodicity).

Chapter 1. Fake $\mathbb{R}^4$.

As we will see, using Donaldson's theorem we can conclude that there is no smooth four-manifold with intersection form $E_8 \oplus E_8$. However, there is a smooth four-manifold with intersection form $E_8 \oplus E_8 \oplus 3H$, namely the $K3$ complex surface (unique up to diffeomorphism). By doing surgery on the $K3$ to remove the homology corresponding to $3H$, we will obtain an exotic $\mathbb{R}^4$. The procedure is as follows:

Denote the homology generated by the spheres in the $K3$ by $a_1, a_2, a_3, b_1, b_2, b_3$. We use the following result.

Result 1. The homology generated by these spheres is represented by a collared topological embedding of $X = 3(S^2 \times S^2) - D^4$ in $K3$, which is smoothly equivalent to an embedding of X in $3(S^2 \times S^2)$. Denote the embedded X along in $3(S^2 \times S^2)$ by $j(X)$. We see that $3(S^2 \times S^2) - j(X)$ has a single end, due to the collar around the boundary of $j(X)$, homeomorphic to $S^3\times[0,\infty)$ and $H_2(3(S^2\times S^2)-j(X),\mathbb{Z})=0$ since all the H_2 was in the collar. Also, $3(S^2\times S^2)-j(X)$ is simply connected. Now we apply the following result to conclude that $3(S^2\times S^2)-j(X)$ is homeomorphic to $\mathbb{R}^4$

Result 2. A non-compact four-manifold V without boundary, which is simply connected and has zero second homology, and has a single end homeomorphic to $S^3\times[0,\infty)$, is homeomorphic to $\mathbb{R}^4$.

Now consider the embedding of X in $K3$. Denote that by $i(X)$. Suppose there was a smoothly embedded S^3 between the borders of the collar around $i(X)$, surrounding X. Then we could do surgery to $K3$ to remove this second homology generated by X, and we would obtain a smooth manifold with intersection form $E_8\oplus E_8$. That cannot be. On the $3(S^2\times S^2)$ side, since the embeddings of X are smoothly equivalent, we conclude that there is no S^3 smoothly embedded so as to lie within the borders of the collar $j(X)$. But then we conclude that the compact set $3(S^2\times S^2)-$ (collar around $j(X)$), living inside the homeomorphic-to-$\mathbb{R}^4$ space $3(S^2\times S^2)-j(X)$, is not surrounded by any smoothly embedded three-sphere (since otherwise we would obtain the contradictory three-sphere on the $K3$ side). Standard $\mathbb{R}^4$ has the property that any compact subset is surrounded by a smoothly embedded three-sphere, so we conclude that what we have here is a fake (i.e. "exotic") $\mathbb{R}^4$.

Chapter 2. The Yang-Mills equation.

Connections, curvature, and gauge transformations

A principal G-bundle P over a manifold X is assigned with a covering $\mathcal{O}_\alpha$ of X and transition maps $s_{\beta\alpha}:\mathcal{O}_\alpha\cap\mathcal{O}_\beta\to G$ such that the cocycle condition is satisfied. We can associate to P a vector bundle $\eta=P\times_G V$ by choosing a vector space V on which G acts (preferrably freely), and forming the disjoint union $\mathcal{O}_\alpha\times V/\sim$, where $(x,v)\sim(y,w)$ if $x=y$ and $w=s_{\beta\alpha}v$. Denoting the representation by $G\overset{\rho}{\to}Aut(V)$, this vector bundle is obtained by pulling back the universal bundle via $X\to BG\overset{B\rho}{\to}BAut(V)$.

Now we define an object that allows us to identify the fibers across a principal or associated bundle intrinsically. We will work with associated vector bundles most of the time, and so only define the following notions in their case. A *covariant derivative* (or *connection*) is an operator

$$D:\Gamma(\eta)\to\Gamma(\eta\otimes T^*(X))=\Omega^1(\eta).$$

Our bundles will have metrics on them, and we require that D satisfies the Leibniz rule.

Locally, on $\mathcal{O}_\alpha$, covariant derivatives look like $d+A_\alpha$, where

$$A_\alpha:\mathcal{O}_\alpha\to T^*(\mathcal{O}_\alpha)\otimes\mathfrak{g}.$$

Here we can start thinking of our associated vector bundle right away as $P\times_G\mathfrak{g}$, where G acts on $\mathfrak{g}$ via the adjoint action. That is, $(p,V)\sim(p,L_{g^{-1}*}R_{g*}V)$. On overlaps we have the transformation law

$$A_\alpha=s_{\beta\alpha}^{-1}ds_{\beta\alpha}+s_{\beta\alpha}^{-1}A_\beta s_{\beta\alpha}.$$

This tells us that the difference of two connections transforms via the adjoint action of G, i.e. it is a section of $\Omega^1(\mathrm{ad}\eta)$. So, the space of all connections, denote it $\mathcal{A}$, is an affine space. Note it is contractible, in particular.

A connection can also be thought of as a $\mathfrak{g}$-valued 1-form on the tangent bundle of the total space P (as in *Spin Geometry*). This connection 1-form defines a projection onto the tangent space $\mathfrak{g}$ of the fibers G, and so its kernel gives a horizontal distribution of tangent subspaces, projecting isomorphically down to the tangent space of the base manifold X. Such a connection 1-form on a principal $SO(n)$ bundle gives a covariant derivative D on X as defined above via the following result.

Proposition 4.4 in Spin Geometry. Let ω be a connection 1-form on the principal $SO(n)$ bundle of frames $P_{SO}(E)$, where E is a vector bundle over X. Then ω determines a unique covariant derivative D on E via

$$De_i = \sum_j \mathcal{E}^* \omega_{ij} \otimes e_j,$$

where $\mathcal{E} = (e_1, \ldots, e_n) : X \to P_{SO}(E)$ is a local frame.

We can consider the complex

$$\Omega^0(\eta) \xrightarrow{D} \Omega^1(\eta) \xrightarrow{D} \Omega^2(\eta) \xrightarrow{D} \cdots$$

obtained by tensoring the de Rham complex $\Omega^i \xrightarrow{d} \Omega^{i+1}$ with the bundle η. To make sense of D for $i \geq 1$, define $D(\sigma \otimes \theta) = D\sigma \wedge \theta + \sigma \otimes d\theta$ for a section σ of η and $\theta \in \Omega^i$. Now we can consider the composition $D^2 : \Omega^0(\eta) \to \Omega^2(\eta)$. Locally, this operator looks like $dA + A \wedge A$, and, denoting it F_α, transforms as $F_\alpha = s_{\beta\alpha}^{-1} F_\beta s_{\beta\alpha}$. So we can consider it a section of $\Lambda^2 T^* X \otimes \eta$ transforming under the adjoint action, i.e. we obtain an element $F \in \Omega^2(\mathrm{ad}\eta)$ which we call the textitcurvature of D.

Now we consider the Hodge star operator $* : \Omega^2 \to \Omega^2$ on a four manifold M with given metric, defined by $\int_M \alpha \wedge *\beta = \int_M g(\alpha, \beta)$. Since $*^2 = 1$, it is in particular a normal operator and Ω^2 decomposes into a sum of its eigenspaces $+1$ and -1, denoted Ω^2_+ and Ω^2_-. This decomposition extends to $\Omega^2(\xi)$ for any vector bundle ξ. In particular, the curvature F of a given connection D decomposes into its self-dual and anti-self-dual pieces, $F = F_+ + F_-$.

We define the *gauge transformations* of a principal G bundle P to be bundle maps s that are G-equivariant, i.e. $s(p.g) = s(p).g$ (the action is from the right).

Lemma 4.1.2 in Morgan's Gauge Theory and the Topology of Four-Manifolds Lectures.
The group of gauge transformations is naturally isomorphic to the group of sections of the bundle $\mathrm{Ad}(P) = P \times_G G$.

Proof. Given a gauge transformation $P \xrightarrow{s} P$, observe that for every p there is an element, call it $\psi(p) \in G$, such that $s(p) = p.\psi(p)$. Define a section $P \to P \times_G G$ by $x \mapsto [(p, \psi(p)]$for any p in the fiber above x. Observe that

$$pg.\psi(pg) = s(pg) = s(p).g = p.\psi(p)g,$$

and so by freeness of the action we conclude $\psi(pg) = g^{-1}\psi(p)g$, and so the section is indeed well defined. □

So, the space of gauge transformations, denote it $\mathcal{G}$, has a natural group structure, when thinking of it as the sections of $\mathrm{Ad}\eta$. This is an infinite-dimensional Lie group with Lie algebra $Lie(\mathcal{G}) = \Gamma(\mathrm{ad}\eta)$.

Gauge transformations act on covariant derivatives and curvature. Indeed, for $s \in \mathcal{G}$, define $s.D = s^*(D) = s^{-1} \circ D \circ s$, which on sections $\sigma \in \Gamma(\eta)$ is $(s.D)\sigma = s^{-1}(D(s\sigma))$. We have

$$s^*(A_\alpha) = s^{-1}ds + s^{-1}A_\alpha s.$$

On curvature, $s^*F = s^{-1}Fs$, where s only moves around the section part and acts trivially on the form part.

Chern classes of $SU(2)$ bundles

We are actually only interested in principal $U(1)$ and $SU(2)$ bundles. The corresponding Lie algebras are $i\mathbb{R}$ (abelian) and $su(2)$ = traceless skew-hermitian 2x2 matrices. The first Chern class $c_1(\eta)$

classifies the $U(1)$ bundle η, and, over four-manifolds, the second Chern class $c_2(\eta)$ classifies the $SU(2)$ bundle η. These classifications follow since $U(1)$ bundles are classified by maps to $\mathbb{CP}^\infty$ which classifies $H^2(X,\mathbb{Z})$, and $SU(2)$ bundles are classified by maps to $BSU(2)$ which is a $K(\mathbb{Z},4)$ from the perspective of four-manifolds, and so such maps correspond to $H^4(X,\mathbb{Z})$.

Chern-Weil theory tells us $c(\eta) = [\det(1+\frac{i}{2\pi}F)] \in H^*(X,\mathbb{Z})$, where F is the curvature of *any* (!) connection D on η. Consider the case of η a $U(1)$ bundle. The curvature, recall, is a section of $\Lambda^2 T^*X \otimes i\mathbb{R} = \Lambda^2 T^*X$, so it is just an ordinary global 2-form α. Then the above Chern-Weil formula gives us $c(\eta) = 1 + \frac{i}{2\pi}\alpha$, so $c_1(\eta) = \frac{i}{2\pi}[\alpha]$, and this classifies η topologically.

If η is an $SU(2)$ bundle, then its curvature F is an $su(2)$-valued 2-form, i.e. it has the form

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are ordinary 2-forms. Since this matrix is traceless, being in $su(2)$, we have $a + d = 0$. Chern-Weil tells us

$$\begin{aligned} c(\eta) &= \det(1 + \frac{1}{2\pi}F) \\ &= 1 - (\text{trace}(F))\frac{i}{2\pi} + (\det(F))(\frac{i}{2\pi})^2, \end{aligned}$$

and so $c_1(\eta) = 0$ and $c_2(\eta) = -\frac{1}{4\pi^2}(ad - bc) = \frac{1}{4\pi^2}(a^2 + bc)$. Note that all forms are 2-forms here, so they commute. Observe that

$$\begin{aligned} \text{trace}(F \wedge F) &= \text{trace}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \wedge \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \\ &= \text{trace}\left(\begin{smallmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{smallmatrix}\right) \\ &= a^2 + d^2 + 2bc = 2(a^2 + bc), \end{aligned}$$

and so

$$c_2(\eta) = \frac{1}{8\pi^2}\text{trace}(F \wedge F).$$

The number $k = -\frac{1}{8\pi^2}\int_M \text{trace}(F \wedge F)$ is called the *topological charge* of the bundle. In the proof of Donaldson's theorem, we will be considering $SU(2)$ bundles with topological charge $+1$.

Let us consider now consider how an $SU(2)$ bundle η might split into complex line bundles. These splittings will correspond to singularities in the moduli space of anti-self-dual connections we will construct over a given simply connected smooth four-manifold M. Suppose $\eta = L_1 \oplus L_2$. Then on Chern classes we have $c_1(\eta) = 0 = c_1(L_1) + c_1(L_2)$, and so $c_1(L_1) = -c_1(L_2)$, and hence by the classification of line bundles via first Chern class, $L_2 = L_1^{-1}$. Let us write from now on L instead of L_1. On second Chern classes we have $c_2(\eta) = c_2(L) + c_2(L^{-1}) + c_1(L)c_1(L^{-1}) = 0 + 0 - c_1(L)^2$. So, the topological charge k is equal to $c_1(L)^2$, i.e. it is equal to $\omega(\alpha,\alpha)$, where ω is the intersection form of the manifold, and $\alpha = \pm c_1(L)$. From here we obtain the following result.

Proposition 2.11 Freed-Uhlenbeck. The $SU(2)$ bundle η with topological charge k splits iff $\omega(\alpha,\alpha) = k$ holds for some α. The number of such splittings $\eta = L \oplus L^{-1}$, corresponding to line bundles L with $c_1(L) = \alpha$, is equal to half the number of solutions α (since α and $-\alpha$ give the same splitting).

The Yang-Mills functional

Given an $SU(2)$ bundle η over a four manifold M with metric, we can define an inner product on all $\Omega^i(\eta)$ by combining the one on the manifold with the ad-invariant inner product $(A, B) =$

trace$(AB^*) = -$trace(AB) on $su(2)$. Define the *Yang-Mills* functional evaluated on a connection D on η to be

$$\text{YM}(D) = \int_M (F_D, F_D).$$

Let us consider extremal points of the functional. Note

$$\begin{aligned} F_{D+tA}\sigma &= (D+tA)(D+tA)\sigma = (DD)\sigma + tA(D\sigma) + A(D\sigma) + t^2(A\wedge A)\sigma \\ &= F_D\sigma + t(DA)\sigma + t^2(A\wedge A)\sigma \end{aligned}$$

and so

$$\begin{aligned} \text{YM}(D+tA)'(0) &= (\int_M (F_{D+tA}, F_{D+tA}))'(0) \\ &= (\int_M (F + t(DA) + t^2(A\wedge A), F_D + t(DA) + t^2(A\wedge A))'(0) \\ &= (\int_M (F,F) + 2t(DA,F) + (\text{higher order terms in t}))'(0) \\ &= 2\int_M (DA, F_D) \\ &= 2\int_M (A, D^*F_D). \end{aligned}$$

Here we define $D^* : \Omega^2(\eta) \to \Omega^1(\eta)$ as the formal adjoint to D. It turns out $D^* = -*D*$. So, a critical point D of the Yang-Mills functional satisfies $D^*F_D = 0 = D*F$. By the Bianchi identity, it is always true that $DF_D = 0$. Call these critical points D *Yang-Mills connections* and their curvatures F_D *Yang-Mills fields.* Note the analogy with Hodge theory and the definition of harmonic forms ($dh = d^*h = 0$).

Let us find the absolute minimum and maximum values that YM could possibly achieve. Take a connection D and decompose its curvature $F = F_+ + F_-$. The spaces Ω^2_+ and Ω^2_- are orthogonal, so we have $\text{YM}(D) = \int_M (F,F) = \int_M (F_+,F_+) + \int_M (F_-,F_-)$. Similarly we have trace$(F\wedge F) =$ trace$(F_+\wedge F_+) +$ trace$(F_-\wedge F_-)$. For any F we have $|F|^2 = -\int_M \text{trace}(F\wedge *F)$ by combination of the inner products. Now trace$(F_+\wedge F_+) =$ trace$(F_+\wedge *F_+)$ and trace$(F_-\wedge F_-) = -$trace$(F_-\wedge *F_-)$, and so by integrating we obtain

$$-8\pi^2 k = \int \text{trace}(F,F) = -|F_+|^2 + |F_-|^2,$$

that is,

$$8\pi^2 k = \int_M |F_+|^2 - |F_-|^2.$$

In our case, $k = 1$. So, $\text{YM}(D) = \int |F_+|^2 + |F_-|^2$ and $\int |F_+|^2 - |F_-|^2 = 8\pi^2$. So, $YM(D) \geq 8\pi^2$ with equality when $F_- = 0$, i.e. at self-dual connections.

So, if D is an absolute minimum point of YM, D is self-dual. Minima of the Yang-Mills functional do not have to be absolute minima, but when they are, the solutions of the second order equation $DF_D = D*F_D = 0$ are solutions of the first order *self-dual Yang-Mills equations* $F = *F$.

We are interested in self-dual Yang-Mills fields, or *instantons.* The space of instantons is gauge-invariant, and we consider the moduli space $\mathcal{M} = \{D \in \mathcal{A}, F_D = *F_D\}/\mathcal{G}$.

Line bundles

We consider the case of $U(1)$ bundles λ, which we should understand since splittings of $SU(2)$ bundles will be important. Note that the Lie algebra of $U(1)$ is abelian, and so the adjoint bundle ad$\lambda =$

$L \times_{U(1)} i\mathbb{R}$ is just $M \times i\mathbb{R}$, and so the curvature f of any connection d is an ordinary 2-form. The Yang-Mills equations become $df = d^* f = 0$, which are the equations for harmonic 2-forms. Since $c_1(\lambda) = \frac{i}{2\pi} f$ by Chern-Weil, we get the following result.

Theorem 2.20 Freed-Uhlenbeck. If λ is a line bundle, then the curvature of any Yang-Mills connection d is the unique harmonic 2-form f such that $[f] = -2\pi i c_1(\lambda)$.

Example. Consider a line bundle L over $S^2 \times S^2$. By the previous theorem, the curvature of any Yang-Mills connection on L should be a harmonic 2-form representing some multiple of c_1. But now suppose we want instantons, i.e. self-dual Yang-Mills connections. The curvature of such a connection would be a self-dual harmonic two-form. The space of harmonic forms $\mathcal{H} = H^2(S^2 \times S^2, \mathbb{C})$ splits into $\mathcal{H}^+ \oplus \mathcal{H}^-$, self-dual and anti-self-dual, where the dimensions of these spaces are b^+ and b^-, the numbers of positive/negative eigenvalues of the intersection forms. Note that for $S^2 \times S^2$ we have that $\mathcal{H}^+$ is a line in a two dimensional space. As we vary the metric on the base, this lines moves around the two dimensional ambient space. So, for a generic metric, this line will miss the integral cohomology lattice, and so no self-dual harmonic form can represent the first Chern class of the line bundle. Therefore, for a generic metric on $S^2 \times S^2$, there are no instantons (or anti-instantons, since $b^+ > 0$).

However, there are always Yang-Mills connections (i.e. solutions to the Yang-Mills equations) on a line bundle. First observe that for any connection d_A on a line bundle, locally $d_A = d + a$ for some $a \in \Omega^1(\mathrm{ad}L)$. We think of d_A as an operator $\Omega^1(\mathrm{ad}L) \to \Omega^2(\mathrm{ad}L)$ and show that it has the same effect as d. Indeed, for $\sigma \otimes V \in \Omega^1(\mathrm{ad}L)$, we have $(d + a).\sigma \otimes V = d\sigma \otimes V \pm sigma \otimes a.V$. Now, a acts on V via the derivative of the adjoint action of G on $\mathfrak{g}$ (for general group G and Lie algebra $\mathfrak{g}$ now). The derivative of the adjoint action $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is $\mathrm{ad} : \mathfrak{g} \to gl(\mathfrak{g})$ given by $\mathrm{ad}(X).V = [X, V]$. Bracketing in $i\mathbb{R}$ is trivial (in our case $\mathfrak{g} = i\mathbb{R}$), and so this action is trivial, so we have $\sigma \otimes a.V = 0$ and we obtain that $d_A = d + a$ acts the same as d. Consequently, d_A^* acts the same as d^*. So solving the Yang-Mills equation means finding a connection satisfying $d^* F_A = 0$, since that is the same as $d_A^* F_A = 0$. Now choose a base connection A_0. Any other connection on the line bundle is of form $A = A_0 + a$ for some $a \in \Omega^1(\mathrm{ad}L)$. We want to solve $d^* F_A = 0$, that is, $d^* F_{A_0+a} = 0$. Observe

$$\begin{aligned} F_{A_0+a} &= d_{A_0+a}(A_0 + a) = d(A_0 + a) \\ &= dA_0 + da = d_{A_0} A_0 + da \\ &= F_{A_0} + da. \end{aligned}$$

Now, $d^* F_{A_0+a} = d^*(F_{A_0} + da) = d^* F_{A_0} + d^* da$. By the Hodge decomposition, we have $F_{A_0} = d\eta + d^*\omega + h$, where h is harmonic. It follows that $d^* F_{A_0} = d^* d\eta$, and so $a = -\eta$ solves the desired equations, giving a Yang-Mills connection. So, any line bundle on $S^2 \times S^2$ with any metric on the base has solutions to the Yang-Mills equations (minima of the Yang-Mills functional), but by perturbing the metric we can guarantee that there are no instantons.

Example. Now we consider the moduli space of Yang-Mills connections on a line bundle L over a simply-connected four manifold M. Fix a Yang-Mills connection d_0 which exists by the above example. Any other connection (and so any other Yang-Mills connection) will be of the form $d_0 + a$ for some $a \in \Omega^1(\mathrm{ad}L)$. Since $U(1)$ is abelian, we have $\mathrm{ad}L = M \times i\mathbb{R}$, and so any Yang-Mills connection will have the form $d_0 + ia$ for some ordinary 1-form $a \in \Omega^1$. Computing the curvature and using uniqueness from Theorem 2.20 gives us that a is a closed form. Namely, $(d + ia)(d + ia) = dd + 2ida - a \wedge a$, and since a is an ordinary 1-form, $a \wedge a = 0$, and since $dd = (d + ia)(d + ia)$ (by uniqueness of curvature), we conclude $da = 0$. Since M is simply connected, $H^1(M, \mathbb{Z}) = 0$, and so $a = dv$ for some function v. Now let us consider the action of gauge transformations on our base connection d_0. Note that gauge transformations are sections of the bundle $\mathrm{Ad}L = L \times_{U(1)} U(1) = M \times U(1)$. So, gauge transformations are functions of M to the circle. Since $[M, S^1] = H^1(M, \mathbb{Z}) = 0$, every such map has a logarithm, and so any gauge transformation s is of the form $s = e^{iu}$ for some real valued function u.

Consider now how s acts on d_0,

$$\begin{aligned} s.d_0 &= s^{-1}ds + s^{-1}d_0 s \\ &= ie^{-iu}ue^{iu} + e^{-iu}d_0e^{iu} \\ &= d_0 + iu. \end{aligned}$$

So, any other Yang-Mills connection is in the gauge orbit of d_0, i.e. the moduli space is a point. Note by the previous example that instantons (self-dual Yang-Mills connections) might not exist at all.

Now let us return to the case of an $SU(2)$ bundle η over the four manifold M. This bundle possibly has multiple splittings of the form $\eta = L \oplus L^{-1}$. We look at connections that respect a given splitting and the corresponding gauge orbits. To do this we need a preliminary lemma.

Lemma 2.21 Freed-Uhlenbeck. The intersection form ω of M is positive definite (write $\omega > 0$) iff there are no asd harmonic 2-forms on M.

Proof. Decompose a given harmonic 2-form f into its self-dual and anti-self-dual parts, $f = f_+ + f_-$. Then we have

$$\begin{aligned} ||f||^2 = (f,f) &= \int f \wedge *f \\ &= \int (f_+ + f_-) \wedge *(f_+ + f_-) = \int f_+ \wedge f_+ - \int f_- \wedge f_- \\ &= \omega(f_+, f_+) - \omega(f_-, f_-). \end{aligned}$$

So, if f were an asd harmonic 2-form while $\omega > 0$, we would have $||f||^2 = -\omega(f_-, f_-) < 0$, and conversely if there are no asd harmonic 2-forms, then $\omega(f,f) = \omega(f_+, f_+) = ||f||^2 > 0$ $\square$

Now suppose we consider a splitting $\eta = L \oplus L^{-1}$. Yang-Mills connections on L and L^{-1} induce a Yang-Mills connection on η, and if we assume $\omega > 0$, we have that these connections are self-dual (since they are represented by harmonic 2-forms). A split self-dual connection on η is self-dual and hence Yang-Mills when restricted to L and L^{-1}. If M is simply connected, then these split connections are all gauge equivalent by the example above. Therefore we have the following result.

Proposition 2.22 Freed-Uhlenbeck. Suppose M is a simply connected manifold with $\omega(M) > 0$. Then for a splitting $\eta = L \oplus L^{-1}$ of an $SU(2)$ bundle, there is a unique (up to gauge) self-dual Yang-Mills field $F = \left(\begin{smallmatrix} f & 0 \\ 0 & -f \end{smallmatrix}\right)$ respecting the splitting.

Donaldson's theorem

Take a smooth, simply connected, closed four manifold M with $\omega(M) > 0$ and consider the $SU(2)$ bundle over it with topological charge $k = 1$. The following properties of the topology of the moduli space $\mathcal{M}$ of self-dual connections modulo gauge equivalence are proved throughout the book:

- Denote by m half the number of solutions to $\omega(\alpha, \alpha) = 1$. Then for a generic metric on M there are m points such that the moduli space with these points removed is a smooth five-manifold. These points correspond to topological splittings $\eta = L \oplus L^{-1}$.
- Small neighborhoods of these m points are homeomorphic (diffeomorphic away from the point itself) to a cone on $\mathbb{CP}^2$.
- The moduli space $\mathcal{M}$ is orientable.
- The moduli space is non-empty after removing the m singular points. In fact, there is a collar $(0, \varepsilon) \times M$ inside $\mathcal{M}$, and $\mathcal{M} \cup M$, away from the singular points, is a smooth manifold with boundary.

- The moduli space with the end of the collar attached, $\bar{\mathcal{M}} = \mathcal{M} \cup M$ is compact.

Cutting off the tips of the cones by the singular points, we obtain an oriented cobordism, via $\bar{\mathcal{M}}$, between M and a disjoint union, that is a connect sum (equal in cobordism), of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}}^2$'s. From the proof it will follow that all the singular points are in the same connected component as the collar, and so they will all share the same orientation as M inherited from the connected component. So we have the cobordism $M \sim m\mathbb{CP}^2$.

Now, consider the intersection form $\omega > 0$ of our simply connected smooth closed four-manifold M. Its signature is equal to its rank (since it is positive definite), and its signature is equal to m, since signature is an oriented cobordism invariant, and $\sigma(m\mathbb{CP}^2) = \sigma((1) \oplus \cdots \oplus (1)) = m$. But note also that the number of $\mathbb{CP}^2$'s is half the number of solutions to $\omega(\alpha, \alpha) = 1$. Such a solution (mod $\pm$) gives a splitting over the integers (!) of ω into $\omega = \mathbb{Z}\langle\alpha\rangle \oplus \alpha^{\perp}$. Splitting off m times (which is the rank), gives us a diagonalization of the matrix to the identity matrix (where the i, i slot is $\omega(\alpha, \alpha) = 1$). This argument (assuming all those properties of the moduli space of instantons mod gauge) gives us the main result.

Donaldson's diagonalization theorem. A smooth closed simply connected four-manifold with positive intersection form is such that its intersection form is diagonalizable *over the integers* to the identity matrix.

The assumption of being simply connected can be relaxed to demanding that there are no nontrivial homomorphisms from the fundamental group to $SU(2)$, since then every flat $SU(2)$ bundle over M is trivial (and this is all that is necessary in the collar/analytic part of the proof). For example, finite simple non-abelian groups are covered by this assumption. Groups like $\mathbb{Z}_2$ are excluded, though. Donaldson's result has been subsequently extended to include arbitrary fundamental groups.

Donaldson's result can be combined with Freedman's theorem and the classification of indefinite forms, along with Rochlin's theorem, to conclude the non-smoothability of many topological manifolds, and to conclude that two given smooth four-manifolds (everything is simply connected again) are homeomorphic. Freedman's theorem states that given any unimodular symmetric bilinear form over the integers, there is a simply connected four-manifold with that intersection form. (The intersection form determines the homotopy type of the four-manifold, by Whitehead. A simply connected four manifold is obtained by gluing on a four-disk to a wedge of two-spheres. The gluing is determined up to homotopy by a map from S^3 to the wedge of S^2's, and this map is the sum of Whitehead products corresponding to the intersection form.) Back to Freedman's theorem, if the intersection form is even, then there is only one homeomorphism class of such manifolds, and some manifolds inside a given class may be smoothable while others may not. If the intersection form is odd, then the manifolds fall into two distinct homeomorphism classes, at most one of which may contain smoothable manifolds. So, for any odd intersection form we obtain a least one homeomorphism class of non-smoothable manifolds. Other results we can use to think about intersection forms are:

- *van der Blij's lemma.* An even intersection form has signature divisible by 8.
- *Rochlin's theorem.* The intersection form of a smooth closed spin four-manifold is divisible by 16. (Being spin implies the intersection form is even. The converse is true if $H_1(M, \mathbb{Z})$ has no 2-torsion. In particular, the converse is true if the manifold is simply connected. An example of a closed smooth four-manifold with even intersection form which is not spin is the Enriques surface obtained by quotienting a smooth $K3$ by a free $\mathbb{Z}_2$ action ($H_1(\text{Enriques}, \mathbb{Z}) = \mathbb{Z}_2$).
- *Serre's classification of unimodular forms.* Indefinite odd forms diagonalize (over $\mathbb{Z}$, as always) to a diagonal matrix with $+1$'s and -1's on the diagonal. Indefinite even forms are equivalent to $mE_8 \oplus nH$, where H is the intersection form of $S^2 \times S^2$ (Rank and signature of E_8 are both

8, while rank of H is 2 and signature of H is 0.) Definite odd forms diagonalize to positive or negative the identity matrix.

Now we can combine these results to obtain some corrolaries:

- The intersection form E_8 does not correspond to a smooth simply connected manifold. This follows from Rochlin's theorem since the signature of such a manifold would be 8, which is not divisible by 16.
- The intersection form $2E_8$ does not correspond to a smooth manifold. This is not excluded by Rochlin but by Donaldson, since it is positive definite and even. Donaldson's theorem implies that if a smooth manifold has positive definite intersection form, it must be odd.
- Combining Freedman and Donaldson we get that every smooth simply connected four-manifold is homeomorphic to $mE_8 \oplus nH$ (with some restrictions on m and n, for ex. $n > 0$ by Donaldson) or $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$, in the case of indefinite forms, or $m\mathbb{CP}^2$ or $m\overline{\mathbb{CP}}^2$, in the case of definite forms.
- Two smooth simply connected four-manifolds are homeomorphic iff their intersection forms have the same rank, parity, and signature.

The dimension of the moduli space is computed abstractly by the Atiyah-Singer index theorem. For a principal G bundle η over a closed four-manifold M, we have that the dimension of the moduli space of instantons mod gauge is

$$\dim(\mathcal{M}_P) = p_1((\mathrm{ad}P) \otimes \mathbb{C})[M] - \dim(G)(1 - b_1 + b_2^-).$$

In our case of $G = SU(2)$, we have the following computation for the Pontryagin class involved. Denote by f the map $X \to BSU(2)$ that classifies our bundle η. Then the classifying map for $\mathrm{ad}\eta$ is obtained by composing this with the map on classifying spaces obtained by applying the classifying space functor to the map $SU(2) \overset{\rho}{\to} \mathrm{Aut}(su(2)) = SO(3)$. This map just turns out to be the 2-to-1 cover $SU(2) \to SO(3)$. So on classifying spaces, $BSU(2) \overset{B\rho}{\to} BSO(3)$, the induced map on cohomology is multiplication by 2. Now, $\mathrm{ad}\eta$ is classified by $B\rho \circ f$ from M to $BSO(3)$. We have the following:

$$\begin{aligned}
p_1(\mathrm{ad}\eta \otimes \mathbb{C}) &= -c_2((\mathrm{ad}\eta \otimes \mathbb{C}) \otimes \mathbb{C}) \\
&= -c_2((\mathrm{ad}\eta \otimes \mathbb{C}) \oplus (\overline{\mathrm{ad}\eta \otimes \mathbb{C}})) \\
&= -2c_2(\mathrm{ad}\eta \otimes \mathbb{C}) + c_1^2(\mathrm{ad}\eta \otimes \mathbb{C}) \\
&= 2p_1(\mathrm{ad}\eta).
\end{aligned}$$

The class $c_1^2(\mathrm{ad}\eta \otimes \mathbb{C})$ is zero since it is a torsion class in the top cohomology of M. Now, pulling back to $BSU(2)$, we have:

$$\begin{aligned}
p_1(\mathrm{ad}\eta \otimes \mathbb{C}) &= 2p_1(\mathrm{ad}\eta) \\
&= 4p_1(\eta) = 4(c_1^2 - 2c_2)(\eta) \\
&= -8c_2(\eta) = 8k.
\end{aligned}$$

So, in our case of the $SU(2)$ bundle with topological charge 1, and assuming $\omega(M) > 0$ (so $b_2^- = 0$), and $\pi_1(M) = 0$ (so $b_1 = 0$), we have that the dimension of the moduli space of instantons mod gauge is $8 - 3(1 - 0 + 0) = 5$.

Chapter 3. Manifolds of connections.

Here we show that the moduli space $\mathcal{M}$ is a smooth manifold (of dimension 5) away from the singular points for a generic metric on the base manifold M. The idea of the proof is to construct a slice to the $\mathcal{G}$-action on the space of connections away from the reducible connections. This will show that $\widehat{\mathcal{X}} = \widehat{\mathcal{A}}/\mathcal{G}$ is a manifold. However, it will not be true that $\widehat{\mathcal{M}} \subset \widehat{\mathcal{X}}$ is a manifold for any choice of metric on M. So, to get around this problem, we parametrize the space of metrics, call this parametrized space $\mathcal{C}$, and look at the product $\widehat{\mathcal{A}} \times \mathcal{C}$. Now, consider the pairs $(D, g) \in \widehat{\mathcal{A}} \times \mathcal{C}$ such that $*_g F_D = F_D$, i.e. D is self-dual with respect to the metric g. Denote this space $\widehat{\mathcal{SD}} \subset \widehat{\mathcal{A}} \times \mathcal{C}$. We will exhibit $\widehat{\mathcal{SD}}$ as the preimage of a regular value of some smooth function, so it will be a manifold. The slices constructed on $\widehat{\mathcal{X}}_g$ also exist on all of $\widehat{\mathcal{SD}}$, and thus we obtain a manifold $\widehat{\mathcal{SD}}/\mathcal{G}$. Now, we can look at $\widehat{\mathcal{SD}}/\mathcal{G}$ as a union of its slices $\widehat{\mathcal{SD}}/\mathcal{G} = \bigcup_{g \in \mathcal{C}} \mathcal{M}_g$. Applying the Smale-Sard theorem to the projection onto the metric coordinate, we will conclude that $\mathcal{M}_g$ is a smooth manifold for a generic metric. We will compute the dimensions of these smooth manifolds $\mathcal{M}_g$ to be five.

Sobolev spaces

In demonstrating these results, we will want elliptic operators to be invertible on the spaces involved. Spaces of class C^∞ are not good for this, since they are not Banach and elliptic operators do not invert on them. So we replace C^∞ spaces with C^k or Sobolev spaces, and argue that the results proved there carry over to the smooth case.

For any bundle ξ over our base manifold, we define the Sobolev space $H_l(\xi)$ as the space of sections whose (weak) derivatives of order $\leq l$ are integrable. That is, $H_l(\xi)$ is the Hilbert space completion of $\Gamma(\xi)$, the space of smooth sections, with respect to the inner product given by $(a, b) = \int_M \sum_{i \leq l} (D^i a, D^i b)$. By the Sobolev embedding theorem, $H_l(\xi) \subset C^k(\xi)$ as soon as $l > \frac{\dim M}{2} + k$, so in our case, $H_l(\xi)$ is contained in the continuous sections if $l > 2$. By Rellich, $H_l(\xi)$ is compactly contained in $H_k(\xi)$ for any $l > k$.

We define analogous objects to the ones we've been considering, in this Sobolev context. We set $\mathcal{G}_l = H_l(\mathrm{Ad}\eta), \mathcal{A}_l = D_0 + H_l(\mathrm{ad}\eta \otimes T^*M) = \Omega^1(\mathrm{ad}\eta)_l$ (where D_0 is any base connection in $\mathcal{A}$). Then $\mathcal{G}_l$ acts smoothly on $\mathcal{A}_{l-1}$.

Curvature is here an operator $F : \Omega^1(\mathrm{ad}\eta)_l \to \Omega^2(\mathrm{ad}\eta)_{l-1}$, where $F(D) = F_D$. Let us compute the differential of F at a point D. Take a curve $\gamma(t)$ in $\mathcal{A}$ such that $\gamma(0) = D$, and $\gamma'(0) = A$. For example, we can take $\gamma(t) = D + tA$ (since $\mathcal{A}$ is just a linear space). Then the differential ∂F evaluated at $A \in T_D\mathcal{A} = \mathcal{A}$ is equal to $F(\gamma(t))'(0)$, which satisfies

$$\begin{aligned} F(\gamma(t))'(0)(\sigma) &= F(D + tA)'(0)(\sigma) = (D + tA)(D + tA)'(0)(\sigma) \\ &= (DD + t(A(D-) + D(A-)) + t^2 A \wedge A)'(0)(\sigma) = DA(\sigma). \end{aligned}$$

So, $\partial F(D)(A) = DA$, meaning ∂F at D is the linear map $\Omega^1(\mathrm{ad}\eta)_l \to \Omega^2(\mathrm{ad}\eta)_{l-1}$ defined by $A \mapsto DA$. It can be shown by elliptic regularity arguments that the topology of $\widehat{\mathcal{M}} \subset \widehat{\mathcal{A}}_{l-1}/\mathcal{G}_l$ is independent of l as long as $l > 2$. (This should follow from the fact that is F_D is self-dual, then there an $s \in \mathcal{G}_l$ such that $s^*(D)$ is a smooth connection.)

Reducible connections

We say a connection D is reducible if $D = d_1 \oplus d_2$ corresponding to connections on some splitting $\eta = L \oplus L^{-1}$ of the bundle. The following result lets us understand how to detect reducible connections. (We will drop the Sobolev subscripts everywhere from now on.)

Theorem 3.1 Freed-Uhlenbeck. Assume D is not a flat connection. (This is the case for all connections on a $k = 1$ bundle since $c_2 \neq 0$, and by Chern-Weil a flat bundle would have vanishing Chern class.)

Denote by $\mathcal{G}_D$ the stabilizer of $\mathcal{G}$ in D. Then the following are equivalent:

(a) D is reducible

(b) $\mathcal{G}_D/\mathbb{Z}_2 \neq 1$

(c) $\mathcal{G}_D/\mathbb{Z}_2 = U(1)$

(d) $D : \Omega^0(\text{ad}\eta) \to \Omega^1(\text{ad}\eta)$ has kernel

Proof.

- (a) implies (b). For a reducible connection, the circle $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ stabilizes D. These gauge transformations are just sections (along with their inverse) of the adjoint bundle to a principal circle bundle, which is trivial. This circle contained in the stabilizer of D is the circle of constant sections of this adjoint bundle.

- (b) implies (c). Analagously to the (d) implies (a) case, we can show that any $s \in \mathcal{G}_D$, thought of as a section of $\eta \times_{SU(2)} SU(2)$, has unequal constant eigenvalues and so defines a splitting $\eta = \lambda_1 \oplus \lambda_2$, from where we see that s lies in a circle of $\mathcal{G}_D$. So, $\mathcal{G}_D$ is at least $U(1)$. If it is larger than a circle, then the holonomy of the connection D would be smaller than $U(1)$, since it is contained in $SU(2)$ and centralized by a circle. Then the holonomy would have to be discrete, but any connection with discrete holonomy is flat (look at small loops at any point), contrary to the assumption ($c_2 \neq 0$).

- (c) implies (d). Take a path $\gamma(t)$ in the stabilizer of D such that $\gamma(0) = +1$ and denote $u = \gamma'(0)$. The vector u is in the Lie algebra to $\mathcal{G}$, which is $\Omega^0(\text{ad}\eta)$. Differentiating the equation $\gamma(t).D = D$, that is $\gamma(t)^{-1} \circ D \circ \gamma(t) = D$ leads us to $D\gamma'(0) = 0$, i.e. $Du = 0$.

- (d) implies (a). Take an element $u \in \ker D$. Pointwise, u is an element of the Lie algebra to $SU(2)$, i.e. it is a pointwise skew-hermitian traceless matrix. So, take its purely imaginary eigenvalues (as functions of the point) $\pm i\lambda$, a consider a neighborhood where $\lambda > 0$. Take a smoothly varying unit length eigenvector e for $+i\lambda$ on this neighborhood. So, we have $ue = i\lambda e$ and $(e, e) = 1$. Differentiating $ue = i\lambda e$ we obtain

$$\begin{aligned} D(ue) &= (Du)e + uDe = uDe \\ &= iD(\lambda e) = i(d\lambda e + \lambda De) \\ &= id\lambda e + i\lambda De, \end{aligned}$$

  and taking inner product with e we obtain $(uDe, e) = id\lambda(e, e) + i\lambda(De, e)$. Observe that by differentiating the equation $(e, e) = 1$ we obtain $\text{Re}(De, e) = 0$, and so by taking the imaginary part in the previous equation we conclude $\text{Imaginary}((uDe, e)) = \text{Imaginary}(id\lambda) + \text{Imaginary}(i\lambda(De, e))$, i.e. $d\lambda = \text{Imaginary}((uDe, e)) = -\text{Imaginary}((De, ue)) = \lambda\text{Re}((De, e)) = 0$ since $u^* = -u$. So λ is a constant function, and so e is defined everywhere. For each of $+i\lambda$ and $-i\lambda$ we get a globally defined eigenvector and so obtain a splitting $\eta = \lambda_1 \oplus \lambda_2$. From the previous equations we can also conclude $De = 0$, so D splits as $D = d_1 \oplus d_2$ for some connections d_i on the λ_i. That is, D is a reducible connection. □

Corollary. $\mathcal{G}/\mathbb{Z}_2$ acts freely on the irreducible connections $\mathcal{A}$.

A slice theorem

To approach proving that our moduli space is a manifold, we note some useful sufficient conditions for the quotient of a manifold by a free group action to be a manifold. Say M is some smooth (possibly infinite dimensional) manifold, and G a (possibly infinite dimensional group) acting freely on M. Then, if through each point in M we can find a *slice* of the action, the quotient M/G is a manifold. A slice of an action at a point $x \in M$ is an open submanifold $N \subset M$ "orthogonal" to the action at all its points, that is, for each $y \in N$ we have $T_y M = T_y N \oplus T_y(G.y)$. We also require that G identifies no two points in N (i.e. the projection to M/G is injective).

In our case, let us first take $M = \widehat{\mathcal{A}}$ and $G = \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}} = \mathcal{G}/\mathbb{Z}_2$ is the group of gauge transformations modulo its center (so $\tilde{\mathcal{G}}$ acts freely on $\widehat{\mathcal{A}}$, by the previous corollary). At a point $D \in \mathcal{A}$ we want to see what the slice should look like. Luckily the space $\widehat{\mathcal{A}}$ has an inner product inherited from the base manifold and $su(2)$. We look at the local orbit $\tilde{\mathcal{G}}.D$ and note that the derivative of the action, for a u in $\Omega^0(\mathrm{ad}\eta)$, i.e. the Lie algebra of $\tilde{\mathcal{G}}$, is given by $D \to Du$. Locally any other connection is of the form $D + A$ for $A \in \Omega^1(\mathrm{ad}\eta)$, and we want the slice to consist of A such that $(Du, A) = 0$ for all u, i.e. $(D^*A, u) = 0$ for all $u \in \Omega^0(\mathrm{ad}\eta)$. So, our slice at $D \in \widehat{\mathcal{A}}$, which we will think of as the tangent space at D in the quotient $\widehat{\mathcal{X}}$ (by virtue of the following theorem) look like

$$\mathcal{X}_D = \{A \in \Omega^1(\mathrm{ad}\eta) | D^*A = 0\}.$$

Theorem 3.2. Freed-Uhlenbeck, "slice theorem". At each $D \in \widehat{\mathcal{A}}$, there is a neighborhood diffeomorphic to $\ker D^* \times \tilde{\mathcal{G}}$. That is, there is a neighborhood U of D and a map $s : U \to \tilde{\mathcal{G}}$ such that $s(D')^*D' \in D + \mathcal{X}_D$ for all $D' \in U$ (s pulls D' back to the slice at the origin connection D). Furthermore, the map $\Phi : U \to \mathcal{X}_D \times \tilde{\mathcal{G}}$ maps U diffeomorphically to a neighborhood of $(0, id)$, and it is $\tilde{\mathcal{G}}$-equivariant. The neighborhood U can also be chosen to be $\tilde{\mathcal{G}}$-invariant.

The parametrized moduli space

The moduli space will not always be a manifold – a perturbation of the metric might be necessary. We parametrize the space of metrics in a tractable way by choosing a base metric g on M, and setting $\mathcal{C}$ to be the space of C^k automorphisms (remember the assumption of being in Hilbert spaces) of the tangent bundles, $\mathcal{C} = C^k(GL(TM))$. For $\varphi \in \mathcal{C}$, we obtain the corresponding metric by taking $\varphi^* g$. All metrics can be obtained in this way since the space of metrics is the space of sections of a bundle with fiber $GL(4)/O(4)$. Every metric will in fact be obtained multiple times, but this will not affect the genericity conclusions that follow in this chapter.

Now consider the map P_- that projects a 2-form onto its anti-self-dual part, i.e. $P_-(\theta) = \frac{1}{2}(\theta - *\theta)$. This, of course, depends on the underlying metric g. Projection onto the anti-self-dual with respect to the metric $\varphi^* g$ is given by $\varphi^* P_-(\varphi^{-1})^*$. Consider now this map, which detects whether a given connection is self-dual with respect to a metric, $\mathcal{P} : \widehat{\mathcal{A}} \times \mathcal{C} \to \Omega^2_-(\mathrm{ad}\eta)$, given by $\mathcal{P}(D, \varphi) = P_-((\varphi^{-1})^* F_D)$. Note that, as desired, $\mathcal{P}(D, \varphi) = 0$ iff F_D is self-dual with respect to the metric $\varphi^* g$. We have the following crucial result.

Theorem 3.4 Freed-Uhlenbeck. The map $\mathcal{P}$ is smooth and 0 is a regular value (that is, the differential $\partial\mathcal{P}$ is surjective at points which are mapped to 0). Therefore $\widehat{\mathcal{SD}} = \mathcal{P}^{-1}(0)$ is a smooth manifold.

A word on the proof. Since $\mathcal{P}$ is quadratic in D and φ, it is smooth. Now let us look at what $\partial\mathcal{P}$ looks like at a point (D, φ). Note that the Lie algebra of $\mathcal{C}$, denote it $Lie(\mathcal{C})$, is equal to $C^k(End(TM))$. Now, the differential $\partial\mathcal{P}_{(D,\varphi)} : T_D(\widehat{\mathcal{A}}) \oplus T_\varphi(\mathcal{C}) \to \Omega^2_-(\mathrm{ad}\eta)$ can be written as

$$\partial_1\mathcal{P}_{(D,\varphi)} \oplus \partial_2\mathcal{P}_{(D,\varphi)} : \Omega^1(\mathrm{ad}\eta) \oplus Lie(\mathcal{C}) \to \Omega^2_-(\mathrm{ad}\eta),$$

since the tangent space to a connection is $\Omega^1(\mathrm{ad}\eta)$ and $\Omega^2_-(\mathrm{ad}\eta)$ is a linear space. A quick computation shows that $\partial_1\mathcal{P} : \Omega^1(\mathrm{ad}\eta) \to \Omega^2_-(\mathrm{ad}\eta)$ is given by $\partial_1\mathcal{P}(A) = P_-((\varphi^{-1})^* DA)$, and also $\partial_2\mathcal{P}(r) = P_-((\varphi^{-1})^*(r^* F_D))$ for $r \in Lie(\mathcal{C})$. If (D, φ) is mapped to zero under $\mathcal{P}$, then $\partial_1\mathcal{P}$ fits into the elliptic complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\eta) \xrightarrow{D} \Omega^1(\mathrm{ad}\eta) \xrightarrow{\partial_1\mathcal{P}} \Omega^2_-(\mathrm{ad}\eta) \longrightarrow 0,$$

which will feature heavily in what's to come.

For now, $\widehat{\mathcal{SD}}$ is a manifold. To obtained our desired parametrized moduli space, we have to mod out by gauge.

Theorem 3.16 Freed-Uhlenbeck. The parametrized moduli space $\hat{S}D/\mathcal{G}$ is manifold.

Proof. We consider the behavior of the map $\mathcal{P}$ restricted to a slice at a point $(D,\varphi) \in \mathcal{P}^{-1}(0) = \widehat{\mathcal{SD}}$. This restriction $\overline{\mathcal{P}}$ has domain $\{D + A, D^*A = 0\} \times \mathcal{C}$ (and D^* is the adjoint of D with respect to the metric $\varphi^* g$). Recall the elliptic complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\eta) \xrightarrow{D} \Omega^1(\mathrm{ad}\eta) \xrightarrow{\partial_1 \mathcal{P}} \Omega^2_-(\mathrm{ad}\eta) \longrightarrow 0,$$

and observe that $\ker D^* \perp \mathrm{image}\, D$ (in fact, $\Omega^1(\mathrm{ad}\eta) = \ker D^* \oplus \mathrm{image}\, D$), $\partial_1 \overline{\mathcal{P}}_{(D,\varphi)} = \partial_1 \mathcal{P}_{(D,\varphi)}|_{\ker D^*}$ by domain consideration, $\partial_2 \overline{\mathcal{P}}_{(D,\varphi)} = \partial_2 \mathcal{P}_{(D,\varphi)}$, and $\partial \mathcal{P}_{(D,\varphi)}$ is onto. It follows that $\partial \overline{\mathcal{P}}_{(D,\varphi)}$ is onto as well. Surjectivity is an open condition, so $\partial \overline{\mathcal{P}}_{(-,-)}$ is surjective in a neighborhood U of (D,φ). So, $(\overline{\mathcal{P}})^{-1}(0) \cap U$ is a manifold, and by the slice theorem we can identify this with a neighborhood of the image of (D,φ) in the quotient space $\widehat{\mathcal{X}}$. We conclude that $\widehat{\mathcal{SD}}/\mathcal{G}$ is a manifold (a submanifold of $\widehat{\mathcal{X}} \times \mathcal{C}$ (we could have previously shown that $\widehat{\mathcal{X}}$ is a manifold itself). □

The moduli space

Now we consider our parametrized moduli manifold $\widehat{\mathcal{SD}}/\mathcal{G} = \cup_{\varphi \in \mathcal{C}} \mathcal{M}_{\varphi^* g}$ and the projection π onto the $\mathcal{C}$ coordinate. We need the following generalization of Sard's theorem.

Sard-Smale theorem. Let $E \xrightarrow{\pi} \mathcal{C}$ be a Fredholm map between Banach manifolds. Then the set of regular values of π is a countable intersection of open dense sets.

The preimage of any regular value is a manifold of dimension equal to the index of the map π (i.e. the index of the linear map $\partial\pi$ at any point). We will take any self-dual connection D (existence guaranteed by the existence of a neighborhood around singular points in the case of reducible connections existing, and by Taubes' result otherwise), and declare the base metric to be the metric which D is self-dual with respect to. We will compute the index of π at the orbit of this point (D, id), where id is the base metric, to be 5. The index is constant along paths, so we will conclude the following desired result.

Theorem 3.17 Freed-Uhlenbeck "Transversality theorem". For a generic metric g, the moduli space $\widehat{\mathcal{M}}_g$ is a five-manifold.

Proof. We compute the index of π at $\overline{(D,id)}$ (the gauge orbit of (D,id)). The tangent space of $\widehat{\mathcal{SD}}$ at the point $(D,id) \in \widehat{\mathcal{SD}}$ can be obtained by taking a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ contained in $\widehat{\mathcal{SD}}$, so $\mathcal{P}(\gamma(t)) = 0$, and concluding $T_{(D,id)}\widehat{\mathcal{SD}} = \{(A,r) \in \Omega^1(\mathrm{ad}\eta) \oplus Lie(\mathcal{C}) | \partial_1\mathcal{P}(A) + \partial_2\mathcal{P}(r) = 0\}$. Now, the tangent space at $\overline{(D,id)}$ in the quotient is obtained by looking at the slice of connections annihilated by D^*,

$$T_{\overline{(D,id)}}\widehat{\mathcal{SD}}/\mathcal{G} = \{(A,r) | \partial_1\mathcal{P}(A) + \partial_2\mathcal{P}(r) = 0 \text{ and } D^*A = 0\}.$$

Since $\partial\pi(A,r) = r$, we have $\ker \partial\pi = \{(A,r)\}\partial_1\mathcal{P}(A) = 0, D^*A = 0, r = 0\}$ since the sum of the partial derivatives is 0 and in the kernel there is no movement in the metric direction. We also have $\mathrm{image}\,\partial\pi = (\partial_2\mathcal{P})^{-1}(\mathrm{image}\,\partial_1\mathcal{P}|_{D^*A=0})$. Consider the elliptic complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\eta) \xrightarrow{D} \Omega^1(\mathrm{ad}\eta) \xrightarrow{\partial_1 \mathcal{P}} \Omega^2_-(\mathrm{ad}\eta) \longrightarrow 0,$$

and recall that $\partial_1\mathcal{P} = P_-D$. So, $\partial_1\mathcal{P}|_{\mathrm{image}\, D} = 0$, so the restriction of $\partial_1\mathcal{P}$ to the kernel of D^* is the full first partial derivative. Therefore $\mathrm{image}\,\partial\pi = (\partial_2\mathcal{P})^{-1}(\mathrm{image}\,\partial_1\mathcal{P})$. Denote the dimensions of the homology of the elliptic complex by h^0, h^1, h^2. The cokernel of $\partial_1\mathcal{P}$ has dimension h^2 by definition, and it is straightforward to see that image $\partial\pi$ has the same codimension. Taking h^1 to be the dimension of the harmonic forms in $\Omega^1(\mathrm{ad}\eta)$ (i.e. forms such that $D^*- = D- = 0$), we see that the dimension of the kernel of $\partial\pi$, which is the space of these harmonic forms, is h^1. Since we are calculating at an irreducible connection D, we have from a previous theorem that D has no kernel, and hence $h^0 = 0$. So, the index of $\partial\pi$ is equal to $h^1 - h^2$. By the Atiyah-Singer index theorem, the Euler characteristic of this complex, $h^0 - h^1 + h^2$, is $3(1 - b_1 + b_2^-) - 8k = -5$, and so we conclude that the index of π is 5. □

To do away with all the Sobolev subscripts in our conclusions about the topology of the moduli space, the following result helps.

Proposition 3.20. Freed-Uhlenbeck. The set of C^k metrics for which $\mathcal{M}$ is a manifold is open and dense, and therefore contains smooth and analytic metrics.

Corollary 3.21. Freed-Uhlenbeck. If the intersection form ω is indefinite, then for a generic (i.e. an open dense set of) metrics, there are no line bundle solutions to the self-dual or anti-self-dual equations.

Proof. We saw a proof of this in one of the examples in Chapter 2, using the fact that a codimension 1 subspace of the space of harmonic forms generically misses the integral cohomology lattice where the Chern classes live. Alternatively, we can consider here the (simpler) elliptic complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d_-} \Omega^2 \longrightarrow 0,$$

which is of this form since the bundle $\mathrm{ad}\eta$ is trivial. Now, the formula for the Euler characteristic of this complex by Atiyah-Singer is $1(1 - b_1 + b_2^-) - p_1(\mathrm{ad}\eta \otimes \mathbb{C}) = 1 + b_2^-$, since the adjoint bundle is trivial and the manifold is assumed simply connected. As before, the dimension of the moduli space is the index of the projection map, which turns out to be $h^1 - h^2$. Since h^0 is the dimension of the vector space of constants here, and $h^0 - h^1 + h^2 = 1 + b_2^-$, we obtain that the dimension of the moduli space is $-b_2^-$. So, there are no self-dual connections generically. By reversing the orientation of the manifold and switching the roles of b_2^- and b_2^+ we conclude that there are generically no anti-self-dual connections either.

Chapter 4. Cones on $\mathbb{CP}^2$.

Slices revisited

Now we consider a reducible connection D. The Lie algebra of $\mathcal{G}_D$ is the one-dimensional kernel of the operator $D : \Omega^0(\mathrm{ad}\eta) \to \Omega^1(\mathrm{ad}\eta)$. So, $\mathcal{G}_D = \{e^{i\theta u}\}$, for $u \in \ker D$. Consider again the elliptic complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\eta) \xrightarrow{D} \Omega^1(\mathrm{ad}\eta) \xrightarrow{P_- D} \Omega^2_-(\mathrm{ad}\eta) \longrightarrow 0,$$

and observe that, since the complex is $\mathcal{G}_D$-equivariant, the group $\mathcal{G}_D$ acts on the cohomology groups H^0, H^1, H^2 of this complex. The action on H^0 is trivial since $\mathcal{G}_D$ fixes D, and we will consider the action on H^1 and H^2 later. We equate H^1 and H^2 with the respective spaces of harmonic forms.

The slices $\mathcal{X}_D = \{A \in \Omega^1(\mathrm{ad}\eta), D^*A = 0\}$ give local charts near irreducible connections, while for reducible connections we also have to consider the stabilizing circle $\mathcal{G}_D$. The action of an $s \in \mathcal{G}_D$ on $A \in \mathcal{X}_D$ is given by $g.A = g^{-1}(D + A)g = g^{-1}Ag$ since $Dg = 0$.

Theorem 4.4. Freed-Uhlenbeck. Let D be a reducible connection and $\overline{D}$ its gauge orbit. Then any small neighborhood U of $\overline{D}$ is homeomorphic to $\mathcal{X}_D/\mathcal{G}_D$ and smooth away from $\overline{D}$.

Structure of the singular point

Locally the moduli space is the quotient by gauge orbits of the kernel of the map $\mathcal{P}$ restricted to a metric, call it P_-F, which maps $A \in \mathcal{X}_D$ to $P_-(DA + A \wedge A)$. The same is true around singular points. The linearization (i.e., differential) of P_-F is the map P_-D, which is Fredholm. So, P_-F is a nonlinear Fredholm map. To understand it, we have the following lemma (which is crucial to proving the Smale-Sard theorem).

Lemma 4.7. Freed-Uhlenbeck. Suppose $\Psi : X \to Y$ is a nonlinear Fredholm map between Hilbert spaces mapping 0 to 0. Then Ψ is locally (around the origin) equivalent to the sum of its differential at the origin and a nonlinear map with finite dimensional range. That is, there is a splitting $X = \ker d\Psi_0 \oplus X'$, $Y = \operatorname{image} d\Psi_0 \oplus Y'$ (where $\ker \Psi_0$ and Y' are finite-dimensional since Ψ is Fredholm) and a map $\phi : X \to Y'$ such that there is an origin-preserving diffeomorphism identifying Ψ with $d\Psi_0 + \phi$. Furthermore, if a group G acts orthogonally on X and Y, and Ψ is G-equivariant, then X' and Y' are G-invariant and ϕ is G-equivariant as well.

We apply this to the map $P_-F : \ker D^* \to \Omega^2(\mathrm{ad}\eta)$, where we split the domain and codomain as $P_-F : \ker P_-D|_{\ker D^*} \oplus X' \to \operatorname{image} P_-D \oplus (\operatorname{image} P_-D)^\perp$. Note by elementary linear algebra that $(\operatorname{image} P_-D)^\perp = \ker(P_-D)^*$. The linearization at 0 of P_-F is $P_-D|_{\ker D^*}$, and since $\mathcal{G}_D$ acts by rotation (and thus orthogonally), we have that the conditions of the lemma are satisfied. So, the moduli space locally near a reducible D is of the form $(P_-D + \overline{\phi})^{-1}(0)/\mathcal{G}_D$ for some $\mathcal{G}_D$-equivariant map $\overline{\phi} : \ker D^* \to \ker(P_-D)^*$. Restricting to the kernel of P_-D gives us the following result.

Corollary 4.8. Freed-Uhlenbeck. Near a reducible connection D, the moduli space is homeomorphic to $\phi^{-1}(0)/\mathcal{G}_D$ for some $\mathcal{G}_D$-equivariant local map $\phi : H^1 \to H^2$.

Since from $P_-F = P_-D + \phi$ (up to diffeomorphism) it follows that $d\phi_0 = 0$, and so in order to conclude that the moduli space is a manifold, we need $H^2 = 0$.

Proposition 4.9. Freed-Uhlenbeck. For a reducible connection D there are real isomorphisms $H^1 = \mathbb{C}^q$ and $H^2 = \mathbb{C}^p \oplus P_-H^2_{\text{de Rham}}(M)$, under which $S^1/\mathbb{Z}_2$ acts via the standard action (as S^1) on the complex vector spaces, and trivially on $P_-H^2_{\text{de Rham}}(M)$. If $P_-H^2_{\text{de Rham}}(M) = 0$, then $q = p + 3$.

Proof. First let us determine the fixed points of the $S^1 (= S^1/\mathbb{Z}_2)$ action on H^1. Take $A \in H^1$, and a generator u of the one-dimensional kernel of D. Then taking a path $\gamma(t)$ in the circle $\mathcal{G}_D$, we can set it so that $\gamma'(0) = u$. Differentiating $\gamma(t).A = \gamma(t)^{-1} A \gamma(t)$ gives us $[A, u] = 0$. From here it should follow that $A = \theta \otimes u$ for an ordinary 1-form θ. We should also be able to conclude that $d^*\theta = 0$ and that $d\theta$ is self-dual. Then we would have

$$\int_M |d\theta|^2 = \int_M d\theta \wedge *d\theta = \int_M d\theta \wedge d\theta = \int_M d(\theta \wedge d\theta) = 0.$$

Therefore $d\theta = 0$, and so θ is harmonic, and since $H^1(M, \mathbb{R}) = 0$, we have $\theta = 0$ and so $A = 0$. So, the S^1 action on H^1 of the elliptic complex has no fixed points. Similarly, a fixed point in H^2 has the form $\alpha \otimes u$ for some self-dual ordinary two form α. In our scenario, a lemma from Chapter 2 shows that this space is trivial. In any case, all the fixed points of the S^1 action of H^2 lie in $P_-H^2(M, \mathbb{R})$.

Now, some general representation theory tells us that any representation of S^1 on a vector space V can be decomposed into $V_\mathbb{C} \oplus V_\mathbb{R}$, where the action on the real part is trivial and the action on the complex part is a sum of actions on $\mathbb{C}$. Furthermore, the only irreducibile representation of a circle which is free away from 0 is the standard rotation on $\mathbb{C}$. Therefore, $H^1 = \mathbb{C}^q \oplus \{\text{fixed-points } V_\mathbb{R}\}$ and $H^2 = \mathbb{C}^p \oplus \{\text{fixed-points } V_\mathbb{R}\}$. By the previous discussion on fixed points, we conclude the statement of the theorem. The fact that $q = p + 3$ follows $h^0 = 1$ and the Euler characteristic of the elliptic complex being -5 by Atiyah-Singer. □

Now we can discern what the moduli space looks like at reducible connections where $H^2 = 0$ (and so $H^1 = \mathbb{C}^3$.

Corollary 4.10. Freed-Uhlenbeck. If $H^2_D = 0$, then near the origin $\phi^{-1}(0)/\mathcal{G}_D$ is homeomorphic to a cone on $\mathbb{CP}^2$ (and diffeomorphic off the vertex).

Now, in case $H^2 \neq 0$, we can apply the following result, where we do a slight perturbation of $\mathcal{M}$ to make ϕ surjective.

Theorem 4.11. Freed-Uhlenbeck. There exists a perturbation of $\mathcal{M}$ so that locally around a self-dual connection, $\mathcal{M}$ is homeomorphic (diffeomorphic away from the vertex) to a cone on $\mathbb{CP}^2$.

Chapter 5. Orientability.

Index bundles

Here we prove that the moduli space $\mathcal{M}$ is orientable, meaning, the manifold part $\widehat{\mathcal{M}}$ is orientable. (Here we are assuming we have put a nice metric on the base manifold M already. In this chapter $\widehat{\mathcal{SD}}$ will refer to the self-dual connections with respect to this fixed nice metric.) Recall the total space of irreducible connections modulo gauge equivalence, $\widehat{\mathcal{X}}$. We show that $\widehat{\mathcal{M}}$ is orientable, i.e. $T\widehat{\mathcal{M}}$ is orientable, by producing an orientable bundle ξ over $\widehat{\mathcal{X}} \supset \widehat{\mathcal{M}}$ extending $T\widehat{\mathcal{M}}$.

This bundle ξ will be the equivariant index bundle of the elliptic complex (D^*, P_-F) considered heavily in Chapters 3 and 4. We will prove that it is orientable by showing that its base $\widehat{\mathcal{X}}$ is simply connected. The bundle ξ will only be a virtual bundle, i.e. an element of the real K-theory of compact submanifolds of $\widehat{\mathcal{X}}$. To determine orientability of a bundle it is enough to restrict attention to compact subsets, and by doing so we have the following nice isomorphism for K-theory: Take any infinite dimensional separable Hilbert space H. Consider the map

$$[X, \text{Fredholm}(H)] \xrightarrow{\text{ind}} KO(X)$$

(where X is any compact manifold) defined by $\text{ind}(\psi)_x = \ker\psi_x - \text{coker}\psi_x$, where the right hand side is the fiber of a virtual bundle at a point. These fibers $\ker\psi$ and $\text{coker}\psi$ might not glue together well to form two vector bundles whose difference will be our virtual bundle, but by perturbing ψ slightly (and remaining in the same homotopy class), we indeed do obtain a well-defined virtual bundle $\text{ind}(\psi)$.

For a virtual bundle $E - E'$, define its Stiefel-Whitney classes by $w_i(E) - w_i(E')$. We say the virtual bundle is orientable if w_1 is 0. This is equivalent to requiring that w_1 of the ordinary vector bundle $E \oplus E'^{\perp}$ is zero (the orthogonal bundle exists if we assume the base to be compact).

Now we construct this bundle over $\widehat{\mathcal{X}}$ extending $T\widehat{\mathcal{M}}$. Take a point $\overline{D} \in \widehat{\mathcal{M}}$. The tangent space of $\widehat{\mathcal{M}}$ at this point is

$$E_D = \ker(D^* \oplus P_-D) : \Omega^1(\text{ad}\eta) \to \Omega^0(\text{ad}\eta) \oplus \Omega^2_-(\text{ad}\eta),$$

where $D \in \widehat{\mathcal{SD}}$ is any lift of $\overline{D}$. Thus we can consider the vector bundle $E_{\widehat{\mathcal{SD}}}$ with fibers these kernels. This is a 5-dimensional real bundle over $\widehat{\mathcal{SD}}$, and the gauge group $\mathcal{G}$ acts as transition data, and this action on the total space of the bundle covers the action of $\mathcal{G}$ on $\widehat{\mathcal{SD}}$. So, we have $T\widehat{\mathcal{M}} = E_{\widehat{\mathcal{SD}}}/\mathcal{G}$. Now we extend this construction to all of $\widehat{\mathcal{X}} = \widehat{\mathcal{A}}/\mathcal{G}$. First, on the upstairs ambient space $\widehat{\mathcal{A}}$ containing $\widehat{\mathcal{SD}}$, we define

$$L(D) = D^* \oplus P_-D : \Omega^1(\text{ad}\eta) \to \Omega^0(\text{ad}\eta) \oplus \Omega^2_-(\text{ad}\eta)$$

for any $D \in \widehat{\mathcal{A}}$. This operator $L(D)$ is elliptic and therefore Fredholm. (Note that, if we go over to Sobolev spaces of sections instead of smooth sections, the domain and codomain of $L(D)$ are Hilbert spaces.) We can thus consider the parametrized family of elliptic operators (where the parametrization is over $\widehat{\mathcal{A}}$),

$$L : \widehat{\mathcal{A}} \times \Omega^1(\text{ad}\eta) \to \widehat{\mathcal{A}} \times \Omega^0(\text{ad}\eta) \oplus \Omega^2_-(\text{ad}\eta).$$

The gauge group acts on the Hilbert spaces involved via conjugation, and the operator L is equivariant with respect to this action, so we have a well defined operator on the quotient,

$$\overline{L} : (\hat{A} \times \Omega^1(\mathrm{ad}\eta)/\mathcal{G} \to (\hat{A} \times \Omega^0(\mathrm{ad}\eta) \oplus \Omega^2_-(\mathrm{ad}\eta))/\mathcal{G}.$$

Note that $(\hat{A} \times \Omega^1(\mathrm{ad}\eta)/\mathcal{G}$ fibers over $\widehat{\mathcal{X}} = \widehat{\mathcal{A}}/\mathcal{G}$ with fiber $\Omega^1(\mathrm{ad}\eta)$, and $(\hat{A} \times \Omega^0(\mathrm{ad}\eta) \oplus \Omega^2_-(\mathrm{ad}\eta))/\mathcal{G}$ fibers over $\widehat{\mathcal{X}}$ with fiber $\Omega^0(\mathrm{ad}\eta) \oplus \Omega^2_-(\mathrm{ad}\eta)$. Since the Fredholm operator above each point of $\widehat{\mathcal{A}}$ in L was of index 5, the same is true for $\overline{L}$. We thus define the virtual bundle ξ over $\widehat{\mathcal{X}}$ given by $\xi_x = \ker \overline{L}_x - \mathrm{coker}\overline{L}_x$. At points $x \in \widehat{\mathcal{M}}$, the cokernel is trivial and so this bundle is just the kernel bundle considered above, hence an extension of $T\widehat{\mathcal{M}}$.

Components of $\mathcal{G}$

In order to show that $\widehat{\mathcal{X}}$ is simply connected (which will imply that ξ and thus $T\widehat{\mathcal{M}}$ is orientable, as wanted), we consider the space of gauge transformations. We do this because of the existence of a principal fibration $\mathcal{G}/\mathbb{Z}_2 \to \widehat{\mathcal{A}} \to \widehat{\mathcal{X}}$ giving by modding out the action of $\mathcal{G}/\mathbb{Z}_2$ on $\widehat{\mathcal{A}}$, which will give us a long exact sequence in homotopy relating $\pi_1(\widehat{\mathcal{X}})$ to $\pi_0(\mathcal{G}/\mathbb{Z}_2)$. We consider $\tilde{\mathcal{G}} = \mathcal{G}/\mathbb{Z}_2$ instead of $\mathcal{G}$, since $\mathcal{G}$ has an ineffective $\mathbb{Z}_2$ action on the connections, corresponding to elements $\{+1, -1\}$ which constitute the center of $SU(2)$. These elements, thought of as trivial sections of the bundle $\eta \times_{SU(2)} SU(2)$, act trivially on connections since $s.A = s^{-1}Ds + s^{-1}As = 0 + A$ since $Ds = 0$ (s is a constant section) and the element corresponding to s in $SU(2)$ is in the center, so its adjoint action on $su(2)$ is trivial. The action of fiber on total space should be effective to consider a principal fibration. Compare $\mathbb{Z}$ acting ineffectively on S^n extending the effective $\mathbb{Z}_2$ action.

Let us make the preliminary observation that $\widehat{\mathcal{A}}$ has trivial homotopy groups, since it is obtained from the contractible space $\mathcal{A}$ by removing sets of infinite codimension (compare $S^\infty - \{pt\} = \mathbb{R}^\infty$ both being contractible). So, from the long exact in homotopy for the fibration $\tilde{\mathcal{G}} \to \widehat{\mathcal{A}} \to \widehat{\mathcal{X}}$ we conclude $\pi_1(\widehat{\mathcal{X}}) = \pi_0(\tilde{\mathcal{G}})$.

To compute $\pi_0(\tilde{\mathcal{G}})$, we first consider $\pi_0(\mathcal{G})$. When we start considering gauge transformations in a moment, we will think of them as sections of an $SU(2)$ bundle over the base manifold M. Observe that if we puncture M, what we obtain is homotopy equivalent to a 3-complex, and the classifying map of an $SU(2)$ bundle to $BSU(2)$ is homotopically trivial, since $BSU(2)$ is 3-connected. Therefore the considered $SU(2)$ bundle is obtained by patching together two trivializations: one over a small neighborhood of the point we chose to puncture, and the other away from a smaller neighborhood of this point. Call the neighborhood near the point M^+, and the other M^-. Suppose we chose them so that $M^+ \cap M^- = S^3 \times (0,1)$. So, our principal $SU(2)$ bundle η is obtained as $\eta = M^+ \times SU(2) \sqcup M^- \times SU(2)$ where $(m^+, g) \sim (m^-, g')$ iff $m^+ = m^-$ and $g' = h(m^+)g$. Since $c_2(\eta) = -1$, the map h on each slice of S^3 along $(0,1)$ is a degree -1 map, so we can take it to be $h(x,t) = x^{-1}$.

Suppose further that we chose M^+ and M^- so that $M^+ - M^-$ is a 4-ball. We show that the space of gauge transformations $\mathcal{G}$ has the same number of connected components as the space $\mathcal{G}_0$ of those which are constantly $+1$ on this four-ball B^4. Consider the inclusion $\mathcal{G}_0 \overset{i}{\hookrightarrow} \mathcal{G}$. First we show the inclusion is surjective on connected components. Take a $g \in \mathcal{G}$, and show there is an element of $\mathcal{G}$ that is constantly $+1$ on B^4 homotopic to it. This element is easy to obtain, just move g along the connected $SU(2)$ fibers over to $+1$. Now we show the map is injective. Suppose two elements of the gauge group which are $+1$ on B^4 are homotopic via a path in $\mathcal{G}$. We show that this path can be deformed to one in $\mathcal{G}_0$. Indeed, at each fiber $SU(2)$ above points in B^4, this given path gives us a loop based at $+1$. Since $SU(2)$ is simply connected, we can contract these loops and obtain a path of gauge transformations contained in $\mathcal{G}_0$.

The space $\mathcal{G}_0$ is easier to deal with. Note that any $s \in \mathcal{G}_0$ consists of two maps, s^- and s^+ from M^- and M^+ respectively to $SU(2)$, where on the overlap $s^-(x,t) = h(x,t).s^+(x,t)$. The action here is conjugation (since these are sections of $\mathrm{Ad}\eta$, not η), so we have $s^-(x,t) = x^{-1}s^+(x,t)x$. Here $t = 0$

corresponds to the boundary of B^4, and $t=1$ corresponds to the outer boundary of M^+. For $t=0$, note $s^+(x,0)=+1$ by definition of $\mathcal{G}_0$, and so $s^-(x,0)=+1$. The map s^- contains all the information of s, in fact, since the only part of M outside of the domain of s^- is B^4 over which s is constantly $+1$. Observe that we have $s^- \in [(M^-, \partial M^-), (SU(2), +1)] = [(M, pt), (S^3, pt)] = [M, S^3]$.

So, $\pi_0(\mathcal{G}) = \pi_0(\mathcal{G}_0) = [M, S^3]$. Observe that $[M, S^3]$ is at most $\mathbb{Z}_2$. Indeed, by crushing the complement of a small disk in M (call this map σ), we obtain a factoring through S^4 of any map $M \to S^3$. Since $\pi_4(S^3) = \mathbb{Z}_2$, this tells us that $[M, S^3]$ is surjected onto by $\mathbb{Z}_2$. A result of Steenrod tells us that $[M, S^3] = 0$ if the intersection form $\omega(M)$ is odd, and $\mathbb{Z}_2$ if it is even (with the bijection to $[S^4, S^3]$ provided by σ^*).

The element -1

There is another, much simpler, fibration at play here, $\mathbb{Z}_2 \to \mathcal{G} \to \tilde{\mathcal{G}}$. The long exact sequence in homotopy terminates with $\pi_0(\mathbb{Z}_2) \to \pi_0(\mathcal{G}) \to \pi_0(\tilde{\mathcal{G}}) \to 0$. If the intersection form $\omega(M)$ is odd, then by Steenrod we have $[M, S^3] = 0$, and as we saw, $[M, S^3] = \pi_0(\mathcal{G}_0) = \pi_0(\mathcal{G})$. So, we obtain $\pi_1(\widehat{\mathcal{X}}) = \pi_0(\tilde{\mathcal{G}}) = 0$ in the case of odd intersection form, and we are done with our proof of orientability of $\widehat{\mathcal{M}}$. In the case of even intersection form, we have the exact sequence $\mathbb{Z}_2 \xrightarrow{j_*} \mathbb{Z}_2 \to \pi_0(\tilde{\mathcal{G}}) \to 0$. To show that $\pi_0(\tilde{\mathcal{G}}) = 0$ as desired, we have to show that j_* is surjective. That is, the gauge transformation -1 (when thought of as a section of the bundle Adη) is not homotopic to $+1$ in $\mathcal{G}_0$.

First we push -1 to an element of $\mathcal{G}_0$ in the following way. Choose a vector in $su(2)$, apply the exponential map, and obtain a circle subgroup $S^1 = \{e^{i\Lambda\theta}\}$ in $SU(2)$. Define $\gamma(t)$ to be the path from $+1$ to -1 in this circle. Push -1 to an element $s \in \mathcal{G}_0$ defined by $s^+ = +1$ on B^4, $s^+ = \gamma(t)$ on $S^3 \times (0,1)$, $s^- = x^{-1}\gamma(t)x$ on $S^3 \times (0,1)$, $s^- = -1$ on $M^+ - M^-$. This element s is in the same path component as -1 in $\mathcal{G}$, as wanted. In $\pi_0(\mathcal{G}_0)$, the element s is represented by s^-. We show that this is obtained by pulling back a generator for $[S^4, S^3]$ via a degree one collapsing map $M \to S^4$. Indeed, define σ to be the map that crushes B^4 to the north pole of S^4, and everything outside M^+ to the south pole. Then s^- factors as $s^- = u\sigma$, where $u : S^4 \to S^3$ is defined by $u(x, \varphi) = x^{-1}e^{i\varphi\Lambda}x$. Here we think of S^4 as given by an S^3-coordinate x and a polar angle φ ($\varphi = 0$ is the north pole, $\varphi = \pi$ is the south pole). Then u factors as the inverse of the suspension of the Hopf map $S^3 \to S^2$ followed by a diffeomorphism of S^3, hence it is a generator of $\pi_4(S^3)$. Explicitly, u factors as $(x, \varphi) \mapsto (\mathrm{hopf}(x^{-1}), \varphi) \mapsto x^{-1}e^{i\varphi\Lambda}x$. Therefore $s^- = \sigma^*(u)$ is nontrivial, and so j_* is surjective, so $\pi_0(\tilde{\mathcal{G}}) = \pi_1(\widehat{\mathcal{X}}) = 0$, and thus $\widehat{\mathcal{M}}$ is orientable.

Chapter 10. The technique of Fintushel and Stern.

Suppose we consider $SO(3)$ bundles over a smooth four-manifold M as opposed to $SU(2)$ bundles. Principal $SO(3)$ bundles over four-manifolds are classified by p_1 and w_2. Informally, this is true since two $SO(3)$ bundles E and F with coinciding p_1 and w_2 are such that $w_2(E-F)=0$, and so $E-F$ lifts to an $SU(2)$ bundle, classified by c_2 which is some multiple of $p_1(E-F)=0$, hence trivial. (A formal proof is in Appendix E.)

Let us consider the moduli space of connections on an $SO(3)$ bundle ξ over M. In order to obtain a moduli space that isn't already covered by the $SU(2)$ case, we suppose $w_2(\xi) \neq 0$. Recall the formula for the dimension of the moduli space, given by $p_1(\mathrm{ad}\eta \otimes \mathbb{C}) - \dim(G)(1 - b_1 + b_2^-)$. Observe that for $SO(3)$ bundles, ad$\eta = \eta$, so we have

$$p_1(\mathrm{ad}\eta \otimes \mathbb{C}) = p_1(\eta \otimes \mathbb{C}) = p_1(\eta \oplus \eta) = 2p_1(\eta).$$

Here we used that $\eta \otimes \mathbb{C}$ is real-isomorphic to $\eta \oplus \eta$, and the fact that $2p(E \oplus F) = 2p(E)p(F)$, but since we are in H^4 of a four-manifold there is no torsion, so we have $p(E \oplus E) = 2p(E)$. So, assuming as in Donaldson's theorem that our manifold has $H^1(M, \mathbb{Z}) = 0$ and $\omega(M) > 0$, the formula for the dimension of the moduli space becomes $2p_1(\xi)[M] - 3$. Let us consider $SO(3)$ bundles ξ

with $p_1(\xi)[M] = 2$ then, so we have a particularly simple, one-dimensional moduli space of self-dual connections modulo gauge equivalence.

This moduli space over a closed smooth four-manifold with vanishing first Betti number and positive definite intersection form has the following properties analogous to those of the space considered in Donaldson's theorem. Namely:

Let ξ be an $SO(3)$ bundle over such an M with $p_1(\xi)[M] = 2$. Then we have the following properties of the moduli space of self-dual on ξ modulo gauge:

- Let m be half the number of solutions to $\omega(\alpha, \alpha) = 2$ with $\alpha \bmod 2 = w_2$. Then for a generic metric on M, there exist m "singular" points such that the moduli space with these points removed is a smooth one dimensional manifold.
- For a generic metric on M, there are neighborhoods of the singular points that are homeomorphic (diffeomorphic away from the points themselves) to an open real interval (which is a cone on a point, i.e. $\mathbb{C}/S^1$).
- The moduli space is compact.

Observe that the description of the neighborhoods of singular points guarantees that these points are not isolated.

Now we prove a special case of a result of Fintushel and Stern, addressing non-smoothability.

Fintushel and Stern's theorem on non-smoothability; a special case. Let M be a closed oriented topological four-manifold with $\omega(M) > 0$ and $H_1(M, \mathbb{Z}) = 0$. If there exists an $\alpha \in H^2(M, \mathbb{Z})$ such that $\omega(\alpha, \alpha) = 2$ and $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in H^2(M, \mathbb{Z})$ with $\omega(\beta, \beta) = \omega(\gamma, \gamma) = 1$, then M is not smoothable.

Proof. We assume $H_1(M, \mathbb{Z}) = 0$ since then $H^2(M, \mathbb{Z})$ has no torsion, so statements like $\omega(x, x) = 0$ will let us conclude that x itself is 0 (which we will need). Conclusions like that cannot be made in the presence of torsion, since $\omega(x, x) = 0$ for any torsion class in $H^2(M, \mathbb{Z})$.

Now, take an $SO(2) = U(1)$ bundle over M, call it λ, with $c_1(\lambda) = \alpha$. Define $\xi = \lambda \oplus \varepsilon^1_{\mathbb{R}}$. Note that $p_1(\xi) = p_1(\lambda \oplus \varepsilon^1) = p_1(\lambda) = c_1(\lambda)^2 - 2c_2(\lambda) = c_1(\lambda)^2 = \alpha^2$, so $p_1(\xi)[M] = \omega(\alpha, \alpha) = 2$. (Also observe that $\pm\alpha = w_2 \bmod 2$.) Now we are in the conditions of the bullet points above, if M was smooth. In particular, the number of solutions to $\omega(x, x) = 2$ with $x \bmod 2 = w_2(\xi)$ is divisible by 4, since half of the number of such solutions is equal to the number of boundary points of a smooth one-manifold, which thus must be an even number itself. We will show that the assumed $\pm\alpha$ are in fact the only solutions under the given assumptions, so the number of solutions is not divisible by 4, giving a contradiction with smoothability.

Take any other solution to the pair of equations $\omega(x, x) = 2$ and $x \bmod 2 = w_2(\xi)$. Since there is no torsion, it is of the form $\alpha + 2\beta$ for some β. First of all, note that we have

$$\begin{aligned} 2 = \omega(\alpha + 2\beta, \alpha + 2\beta) &= \omega(\alpha, \alpha) + 4\omega(\alpha, \beta) + 4\omega(\beta, \beta) \\ &= 2 + 4(\omega(\alpha, \beta) + \omega(\beta, \beta)), \end{aligned}$$

from which we conclude that $\omega(\alpha, \beta) = -\omega(\beta, \beta)$. Now, note that

$$\begin{aligned} \omega(\alpha + \beta, \alpha + \beta) &= \omega(\alpha, \alpha) + \omega(\beta, \beta) + 2\omega(\alpha, \beta) \\ &= 2 + \omega(\beta, \beta) - 2\omega(\beta, \beta) \\ &= 2 - \omega(\beta, \beta), \end{aligned}$$

and since $\omega \geq 0$, we have $\omega(\beta, \beta) \leq 2$. Now we consider the three possible cases of $\omega(\beta, \beta)$.

- If $\omega(\beta, \beta) = 0$, then $\beta = 0$ and the putative alternate solution is just α.
- If $\omega(\beta, \beta) = 1$, then consider $\omega(\alpha + \beta, \alpha + \beta) = 2 + \omega(\beta, \beta) + 2\omega(\alpha, \beta) = 2 - \omega(\beta, \beta) = 1$. Now note that $-\beta$ satisfies $\omega(-\beta, -\beta) = 1$, and we have a decomposition $\alpha = -\beta + (\alpha + \beta)$ of the type forbidden by hypothesis. So this case cannot happen.
- If $\omega(\beta, \beta) = 2$, then $\omega(\alpha + \beta, \alpha + \beta) = 2 + 2 - 2 \cdot 2 = 0$, and so $\alpha + \beta = 0$, thus the putative alternate solution $\alpha + 2\beta$ is just $-\alpha$.

□

As an application, the topological manifold corresponding to the intersection form $E_8 \oplus E_8$ is not smoothable.