Hypersurfaces in $\mathbb{C}P^5$ are non-zero in the oriented cobordism ring

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We show that no smooth hypersurface in $\mathbb{C}P^5$ bounds an oriented 9-manifold in the sense of oriented cobordism.

Consider a degree d hypersurface $X \in \mathbb{C}P^5$. From the short exact sequence of bundles

$$0 \to TX \to T\mathbb{C}P^5|_X \to \mathcal{O}(d)|_X \to 0$$

(where $\mathcal{O}(d)$ is the line bundle corresponding to the normal direction in a tubular neighborhood of X in $\mathbb{C}P^5$), we obtain the equation

$$1 + 6a + 15a^2 + 20a^3 + 15a^4 = (1 + c_1 + c_2 + c_3 + c_4)(1 + da)$$

on Chern classes, where the left side is the truncated expansion of $c(\mathbb{C}P^5) = (1+a)^6$ and we are writing a instead of i^*a in the equation (where i is the inclusion of X in $\mathbb{C}P^5$). Solving for $c_i = c_i(X)$ we obtain

$$\begin{aligned}
c_1 &= (6-d)a \\
c_2 &= (d^2 - 6d + 15)a^2 \\
c_3 &= (-d^3 + 6d^2 - 15d + 20)a^3 \\
c_4 &= (d^4 - 6d^3 + 15d^2 - 20d + 15)a^4.
\end{aligned}$$

The Hirzebruch signature formula tells us that the signature of an eight manifold can be computed via the Pontryagin classes,

$$\tau(X) = \langle \frac{1}{45}(7p_2 - p_1^2), [X] \rangle.$$

Using the relation

$$1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots)$$

we obtain

$$\begin{aligned}
p_1 &= c_1^2 - 2c_2, \\
p_2 &= 2c_4 + c_2^2 - 2c_1c_3.
\end{aligned}$$

Now a tedious but direct calculation gives us

$$\frac{1}{45}\langle (7p_2 - p_1^2), [X] \rangle = \frac{3}{45}\langle (2d^4 - 10d^2 + 23)a^4, [X] \rangle.$$

Recall that we have been writing a instead of i^*a. Observe that $\langle i^*a^4, [X] \rangle = \langle a^4, i_*[X] \rangle = d$ by Bézout (a line in $\mathbb{C}P^5$ intersects a degree d hypersurface in d points), so we obtain

$$\tau(X) = \frac{3}{45}(2d^5 - 10d^3 + 23d).$$

Observe that $2d^5 - 10d^3 + 23d$ has no positive integer roots. Therefore no hypersurface in $\mathbb{C}P^5$ has signature 0. In particular, no hypersurface in $\mathbb{C}P^5$ is zero in the oriented cobordism ring Ω^{SO}.

Note, however, that for even d we have that X is 0 in the *unoriented* cobordism ring. Indeed, the relevant Stiefel-Whitney numbers are $w_2^4, w_4^2, w_2w_6, w_2^2w_4, w_8$ (the odd Stiefel-Whitney classes are zero since X is complex), and we can evaluate these on the fundamental class by applying $c_i = w_i \pmod 2$, evaluating the corresponding product of Chern classes on the fundamental class, and modding out by 2. For example,

$$\begin{aligned}
\langle w_2w_6, [X]\rangle &= \langle (6-d)(-d^3+6d^2-15d+20)a^4, [X]\rangle \\
&= d(6-d)(-d^3+6d^2-15d+20) \\
&= 0 \pmod 2
\end{aligned}$$

if d is even (and a factor of d shows up in all the Stiefel-Whitney numbers).