THE MINIMAL MODELS OF $G(2,4)$, $G(2,5)$, $G(2,6)$, $G(3,6)$

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Abstract. We calculate the minimal models of the complex Grassmannians $G(2,4)$, $G(2,5)$, $G(2,6)$, and $G(3,6)$

1. The complex Grassmannian $G(2,4)$

Consider the Grassmannian of complex k-planes in $\mathbb{C}^{k+n}$, denoted by $G(k,k+n)$. Observe that the group $U(k+n)$ acts transitively on $G(k,k+n)$, and the stabilizer of a fixed k-plane $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ is $U(k) \times U(n)$. So, the Grassmannian is a homogeneous space,

$$G(k,k+n) = U(k+n)/(U(k) \times U(n)).$$

Now, the inclusion $U(k) \times U(n) \hookrightarrow U(k+n)$ induces a map

$$BU(k) \times BU(n) \to BU(k+n)$$

on classifying spaces. The homotopy fiber of this map is $U(k+n)/(U(k) \times U(n))$. (The homotopy fiber of any inclusion of Lie groups is of this form.)

So, we have the fibration

$$U(k+n)/(U(k) \times U(n)) \to BU(k) \times BU(n) \to BU(k+n).$$

Considering minimal models, we obtain an algebraic fibration. Let us focus on the case of $k = n = 2$ (so we are modelling $G(2,4)$). The minimal model of $BU(2)$ is given by $\Lambda(a_1,a_2)$, and so the model of $BU(2) \times BU(2)$ is given by $\Lambda(a_1,a_2) \otimes \Lambda(b_1,b_2) = \Lambda(a_1,a_2,b_1,b_2)$. Let us denote the model of $BU(4)$ by $\Lambda(c_1,c_2,c_3,c_4)$. The differential is zero in all of these algebras, and the variables a_i, b_i, c_i are of degree equal to twice their index (they are universal Chern classes). The notation Λ is used for the free exterior (graded-commutative) algebra on a given set of generators in prescribed degrees.

On the level of cohomology (or equivalently, minimal models, since everything is formal), the map $BU(2) \times BU(2) \to BU(4)$ is given by $H^*(BU(4)) \to H^*(BU(2)) \otimes H^*(BU(2))$, that is,

$$\Lambda(c_1,c_2,c_3,c_4) \to \Lambda(a_1,a_2,b_1,b_2).$$

Let us figure out what the variables c_i are mapped to. If a complex rank 4 bundle E splits into the sum of two complex rank 2 bundles, $C = A \oplus B$, then on Chern classes we have

$$1 + c_1(C) + c_2(C) + c_3(C) + c_4(C) = (1 + c_1(A) + c_2(A))(1 + c_1(B) + c_2(B)).$$

Denoting by a_i, b_i, c_i the ith Chern class of A,B,or C respectively, we obtain the equations

$$\begin{aligned} c_1 &= a_1 + b_1, \\ c_2 &= a_2 + b_2 + a_1 b_1, \\ c_3 &= a_1 b_2 + a_2 b_1, \\ c_4 &= a_2 b_2. \end{aligned}$$

These relations tell us how the map of minimal models $\Lambda(c_1, c_2, c_3, c_4) \to \Lambda(a_1, a_2, b_1, b_2)$ is prescribed.

Now we turn this map $\Lambda(c_1, c_2, c_3, c_4) \to \Lambda(a_1, a_2, b_1, b_2)$ into a fibration. That is, we will find a differential graded algebra E containing $\Lambda(c_1, c_2, c_3, c_4)$ with a quasi-isomorphism to $\Lambda(a_1, a_2, b_1, b_2)$ such that this diagram commutes,

$$\begin{array}{ccc} & E & \\ \nearrow & & \searrow \\ \Lambda(c_1, c_2, c_3, c_4) & \longrightarrow & \Lambda(a_1, a_2, b_1, b_2) \end{array}$$

This should be thought of as dual to the mapping path space construction, turning any map $X \to Y$ into a fibration. Just as the homotopy fiber is the fiber of the map from the mapping path space to Y, here the algebraic model of the homotopy fiber is the fiber of the inclusion $\Lambda(c_1, c_2, c_3, c_4) \hookrightarrow E$, which is defined to be $E/\text{ideal}(c_1, c_2, c_3, c_4)$.

Now we have to construct this E to contain $\Lambda(c_1, c_2, c_3, c_4)$ and to be quasi-isomorphic via some map f to $\Lambda(a_1, a_2, b_1, b_2)$. In order to do this, prescribe $f(c_i)$ to be the corresponding polynomial in a_i, b_i given above. Introduce a variable $\overline{a_i}$ in degree 2 so that $f(\overline{a_1}) = a_1$. Note that now we have a_1 and b_1 in the image of f, since we have $a_1 + b_1$ thanks to c_1 and a_1 is the image of $\overline{a_1}$. Next, introduce a variable $\overline{a_2}$ such that $f(\overline{a_2}) = a_2$. This gets us a_1, b_1, a_2, b_2 in the image of f, due to c_1,c_2,$\overline{a_1}$, $\overline{a_2}$. So, we have made f surjective. Now we look at its kernel, and introduce variables to kill these elements in cohomology. Since c_3 has to be mapped to $a_1 b_2 + a_2 b_1$, we see that we necessarily have

$$f(c_3 - \overline{a_1} c_2 + 2\overline{a_1 a_2} + \overline{a_1}^2 c_1 - \overline{a_1}^3 + \overline{a_2} c_1) = 0.$$

Similarly, since $f(c_4) = a_2 b_2$, we get that

$$f(c_4 - \overline{a_2} c_2 + \overline{a_2}^2 + \overline{a_1 a_2} c_1 - \overline{a_1}^2 \overline{a_2}) = 0.$$

To get rid of this non-injectivity on cohomology, we introduce variables η_5 and η_7 in E (in degrees 5 and 7, respectively) with the prescription that

$$\begin{aligned} d\eta_5 &= c_3 - \overline{a_1} c_2 + 2\overline{a_1 a_2} + \overline{a_1}^2 c_1 - \overline{a_1}^3 + \overline{a_2} c_1, \\ d\eta_7 &= c_4 - \overline{a_2} c_2 + \overline{a_2}^2 + \overline{a_1 a_2} c_1 - \overline{a_1}^2 \overline{a_2}. \end{aligned}$$

Setting $f(\eta_5) = f(\eta_7) = 0$, we have made f injective as well on cohomology, and thus a quasi-isomorphism.

So, our E is given by $\Lambda(c_1, c_2, c_3, c_4, \overline{a_1}, \overline{a_2}, \eta_5, \eta_7)$ with the first six variables closed, and the differential on η_5 and η_7 given as above. Now we take the "fiber" of this map, that is, we take $\Lambda(c_1, c_2, c_3, c_4, \overline{a_1}, \overline{a_2}, \eta_5, \eta_7)$ mod the elements in the ideal of the c_i, with corresponding modification of the differential. Our fiber is given by $\Lambda(\overline{a_1}, \overline{a_2}, \eta_5, \eta_7)$, where $d\overline{a_1} = d\overline{a_2} = 0$ and

$$d\eta_5 = 2\overline{a_1 a_2} - \overline{a_1}^3,$$
$$d\eta_7 = \overline{a_2}^2 - \overline{a_1}^2\overline{a_2}.$$

Renaming the variables to $c_1 = \overline{a_1}, c_2 = \overline{a_2}, u = \eta_5, v = \eta_7$, we have concluded that the homotopy fiber of the map $BU(2) \times BU(2) \to BU(4)$, that is $G(2,4)$, has minimal model

$$\Lambda(c_1, c_2, u, v), \text{ where } dc_1 = dc_2 = 0, du = -c_1(c_1^2 - 2c_2), dv = c_2^2 - c_1^2 c_2.$$

The cohomology ring structure is read off from this algebra, and so are, for example, the rational homotopy groups (whose ranks correspond to the number of generators of the algebra in a given degree). We have that $\pi_2(G(2,4)) \otimes \mathbb{Q} = \pi_4(G(2,4)) \otimes \mathbb{Q} = \pi_5(G(2,4)) \otimes \mathbb{Q} = \pi_7(G(2,4)) \otimes \mathbb{Q} = \mathbb{Q}$, and all the other homotopy groups are purely torsion.

2. Other complex Grassmannians

Similar computations to the one carried out above give us the following minimal models for some Grassmannians:

- Model$(G(2,5)) = \Lambda(c_1, c_2, \eta_7, \eta_9)$, with $dc_1 = dc_2 = 0$, $d\eta_7 = c_1^2 c_2 + c_2^2 - c_1^4$, $d\eta_9 = c_1^5 + 2c_1 c_2^2 - c_1^3 c_2$.
- Model$(G(2,6)) = \Lambda(c_1, c_2, \eta_9, \eta_{11})$, with $dc_1 = dc_2 = 0$, $d\eta_9 = 3c_1 c_2^2 - 2c_1^3 c_2$, $d\eta_{11} = c_2^3 - 2c_1^4 c_2 + c_1^6$.
- Model$(G(3,6)) = \Lambda(c_1, c_2, c_3, \eta_7, \eta_9, \eta_{11})$, with $dc_1 = dc_2 = dc_3 = 0$, $d\eta_7 = c_1^4 - 3c_1^2 c_2 + 2c_1 c_3 + c_2^2$, $d\eta_9 = 2c_2 c_3 - 2c_1 c_2^2 + c_1^3 c_2 - c_1^2 c_3$, $d\eta_{11} = c_3^2 + c_1^3 c_3 - 2c_1 c_2 c_3$.