GEOMETRIC FORMALITY IS NOT A RATIONAL HOMOTOPY INVARIANT

ABSTRACT. We show by example that geometric formality is not a rational homotopy invariant.

We will use [Kotschick, Corollary 11]:

Theorem 0.1. *A closed oriented four–manifold with $b_1 = 1$ and $b_2 = 0$ is geometrically formal if and only if it fibers over a circle.*

We note that $S^1 \times S^3$ is geometrically formal. Now take any rational homology four–sphere X that is not homotopy equivalent to S^4 itself. We note that such a rational homology four–sphere is necessarily not simply-connected. Now consider $S^1 \times S^3 \# X$. A rational homology sphere is orientable, and so $(S^1 \times S^3)\#X$ is an orientable four–manifold with $b_1 = 1$ and $b_2 = 0$. If it were geometrically formal, then there would be a three–manifold M such that there is a fiber bundle $M \to (S^1 \times S^3)\#X \to S^1$. The long exact sequence in homotopy gives us a short exact sequence of groups $1 \to \pi_1 M \to \pi_1(S^1 \times S^3)\#X \to \mathbb{Z} \to 0$. Note that $\pi_1(S^1 \times S^3)\#X \cong \mathbb{Z} * \pi_1 X$; the map $\mathbb{Z} * \pi_1 X \to \mathbb{Z}$ factors through the abelianization of $\mathbb{Z} * \pi_1 X$, which is $\mathbb{Z} \oplus H_1(X;\mathbb{Z})$. Since X is a rational homology sphere, $H_1(X;\mathbb{Z})$ is a torsion group, and so we conclude that the map $\mathbb{Z} * \pi_1 X \to \mathbb{Z}$ is the projection which maps any generator of $\pi_1 X$ to 0 and $\pm 1 \in \mathbb{Z}$ to $\pm 1 \in \mathbb{Z}$ (since the map is surjective).

Lemma 0.2. *(Corey Bregman) The kernel of this projection map $\mathbb{Z} * \pi_1 X \to \mathbb{Z}$ is not finitely generated.*

Proof. If $\mathbb{Z} = \langle t \rangle$, then the kernel is generated by expressions of the form $t^n h t^{-n}$, where $h \in H$ and $n \in \mathbb{Z}$. The normal form for elements of a free product means that distinct copies $t^n H t^{-n}$ intersect trivially. One can then define a surjection from the kernel to $\bigoplus_{n \in \mathbb{Z}} H$ via the map sending $t^n h t^{-n}$ to h in the nth copy of H. The latter is not finitely generated, and so the kernel is not finitely generated. □

It follows that $\pi_1 M$ is not finitely generated; a contradiction since M is a closed manifold.

References

[1] Kotschick, D., 2001. On products of harmonic forms. Duke Mathematical Journal, 107(3), pp.521-531.