# A NILMANIFOLD IS EITHER FORMAL OR HAS A NON-TRIVIAL TRIPLE MASSEY PRODUCT

Abstract. We review the standard result [1] that a nilmanifold is formal if and only if it is a torus, and we do an explicit example of computing the minimal model of a nilmanifold. We then observe that the non-formality of a nilmanifold is detected by a triple Massey product.

## 1. Nilmanifolds and formality

Take any simply connected nilpotent Lie group $G$ with a cocompact discrete subgroup $\Gamma$ (cocompact meaning $G/\Gamma$ is compact). We call the closed manifold $M = G/\Gamma$ a *nilmanifold*. A theorem of Malcev guarantees that such a subgroup exists if and only if there is a basis for the Lie algebra of $G$ in which the structure constants are rational numbers.

Now consider $\Omega^G(G)$, the dga (over $\mathbb{R}$) of forms on $G$ that are left-invariant with respect to all elements of $G$. Denote by $\Omega^\Gamma(G)$ the forms on $G$ that are left-invariant just with respect to $G$. Note that $\Omega^\Gamma(G)$ are exactly those forms which descend to forms on the nilmanifold $M$. A theorem of Nomizu tells us that the inclusion $\Omega^\Gamma(G) \hookrightarrow \Omega^G(G)$ induces an isomorphism on cohomology. Therefore the real minimal models of these two dga's are isomorphic.

Let us therefore compute Model($\Omega^G(G)$) (and so we will have the real minimal model Model($M$) of $M$). The model of a Lie group can be obtained from the complex of left-invariant vector fields that compute its Lie algebra cohomology. Namely, take a basis of left-invariant one-forms $\{x_1, x_2, \ldots, x_n\}$ (where $n = \dim(M)$) dual to the left-invariant vector fields, ordered so that $dx_1 = dx_2 = \cdots = dx_p = 0$ and $dx_k = \sum_{i<j<k} c_{ij}^k x_i x_j$. That we can order the basis so follows from nilpotency of the Lie algebra of $G$ (which in turn follows from the nilpotency of $G$ by considering the second-order term of the Taylor expansion of the derivative $Dm_{(e,e)}$ of multiplication $G \times G \xrightarrow{m} G$). Namely, we can order the basis so that the differential $d$ applied to a given basis element is a sum of products of basis elements preceding it. Since $dx_i$ is of degree 2 for any basis element, all of the products involved in $dx_i$ are two-fold products. Observe that this condition implies that there at least $x_1$ and $x_2$ are closed. (Otherwise we are considering a one-dimensional Lie group, which is not of interest here.) From $x_k$ onward, the basis elements are not closed, but rather non-trivial sums of products of two basis elements coming before it.

So, Model($M$) $= \Lambda(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n; d)$, where the differential $d$ is described above. Let us consider how we can detect if $M$ is formal from this real minimal model.

*Theorem.* The nilmanifold $M^n$ is formal if and only if it is a torus $(S^1)^n$.

*Proof.* First let us argue that $M = G/\Gamma$ is a torus if and only if $G$ is abelian. If $M$ is a torus, then $G = (\mathbb{R}^n, +)$ by the uniqueness of a simply connected Lie group with given Lie algebra. In this case $\Gamma$ is a lattice (a free discrete subgroup of rank $n$), as all discrete subgroups of $\mathbb{R}^n$ are free. Conversely, if $G$ is abelian, by uniqueness it is $(\mathbb{R}^n, +)$, and so $\Gamma$ must be a lattice and hence $M$ is a torus.

The Lie group $G$ is abelian if and only if its Lie algebra $\mathfrak{g}$ is abelian (i.e. has vanishing bracket). This follows from the Baker-Campbell-Hausdorff formula and the surjectivity of the exponential map in this situation.

So, $M$ is a torus if and only if $\mathfrak{g}$ is abelian, i.e. $[X_i, X_j] = 0$ for all vector fields $X_i, X_j$ dual to the basis one-forms $x_1, \dots, x_n$. Dualizing this equality (in general, the relations $[X_i, X_j] = -\sum_k c_{ij}^k X_k$ for $[-,-]$ dualize to the relations $dx_k = \sum_{i,j} c_{ij}^k x_i x_j$ for the differential) gives us $dx_k = 0$ for all basis one-forms $x_k$. So, in our ordering of the basis one-forms, we can take $p = n$, i.e. all of them are closed. Therefore $M$ is a torus if and only if all the basis one-forms are closed.

Now we show that $p = n$ (i.e. all the basis one-forms are closed) if and only if $M$ is formal. If $p = n$, then the model of $M$ is a free exterior algebra with trivial differential, thus formal. Conversely, suppose $p < n$. Then $dx_l = \sum_{i<j<l} c_{ij}^l x_i x_j$, for all $l > p$. Now suppose $M$ is formal, i.e. there is a quasi-isomorphism $\Lambda(x_1, \dots, x_n) \xrightarrow{\Phi} (H^*(M, \mathbb{R}), d \equiv 0)$. Then $\Phi$ sends the closed one-forms to their cohomology classes, $\Phi(x_1) = [x_1], \dots, \Phi(x_p) = [x_p]$, and on the rest we have $\Phi(x_l) = \sum_i \leq pc_i[x_i]$. Now, consider $x_1 x_2 \cdots x_n$. We have that $\Phi(x_1 x_2 \cdots x_n)$ is some non-zero (since $H^n(M, RR)$ is one-dimensional) multiple of the cohomology class of the volume form. On the other hand, $\Phi(x_1 \cdots x_n) = [x_1] \dots [x_p] \cdot \Phi(x_{p+1}) \cdot \cdots \cdots \Phi(x_n) = 0$ since every summand in this expression will contain the square of some element $[x_i]$, and all squares are 0 (since the $x_i$ are of degree 1).

Therefore $M$ is formal if and only if $n = p$, that is, if and only if $M$ is a torus. □

Observe that we only proved formality over $\mathbb{R}$, but since formality descends to subfields, we also have the analogous statement for formality over $\mathbb{Q}$.

## 2. An explicit example

Consider the real Heisenberg group of $3 \times 3$ upper triangular matrices with one's on the diagonal, with matrix multiplication,

$$H = \{ \left( \begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix} \right), \ x, y, z \in \mathbb{R} \}.$$

Consider the cocompact discrete subgroup $\Gamma$ consisting of integer-valued matrices in $H$, and denote the three manifold obtained as the quotient $H/\Gamma$ by $N$. To compute the real minimal model $\mathrm{Model}(N)$, we will figure out the Lie algebra structure on $G$ and dualize to obtain a basis of left-invariant one-forms with corresponding differential.

For computation's sake, let us think of $H$ rather as triples of real numbers, $(x, y, z)$, with multiplication given by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).$$

Take a basis $(1, 0, 0), (0, 0, 1), (0, 0, 1)$ for the tangent space at the identity $T_e H = T_{(0,0,0)} \mathbb{R}^3$. We will push these basis elements around by left multiplication to obtain the left-invariant vector fields on $H$. So, take an arbitrary $g = (a, b, c) \in H$. Left multiplication by $g$ on any $(x, y, z) \in H$ gives us

$$L_g(x, y, z) = (x + a, y + b, z + c + ay).$$

This is a smooth map from $\mathbb{R}^3$ to $\mathbb{R}^3$. We compute the differential $L_{g*}$ at $(x, y, z) = (0, 0, 0)$ and obtain

$$L_{g*(0,0,0)} = \begin{pmatrix} \frac{\partial(x+a)}{\partial x} & \frac{\partial(x+a)}{\partial y} & \frac{\partial(x+a)}{\partial z} \\ \frac{\partial(y+b)}{\partial x} & \frac{\partial(y+b)}{\partial y} & \frac{\partial(y+b)}{\partial z} \\ \frac{\partial(z+c+ay)}{\partial x} & \frac{\partial(z+c+ay)}{\partial y} & \frac{\partial(z+c+ay)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix}.$$

Applying this matrix to our basis of tangent vectors at the origin, we obtain

$$\begin{aligned} L_{g*(0,0,0)} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x}, \\ L_{g*(0,0,0)} \frac{\partial}{\partial y} &= \frac{\partial}{\partial y} + a\frac{\partial}{\partial z}, \\ L_{g*(0,0,0)} \frac{\partial}{\partial z} &= \frac{\partial}{\partial z}. \end{aligned}$$

Denoting $g = (x, y, z)$, this means we have

$$\begin{aligned} L_{g*(0,0,0)} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x}, \\ L_{g*(0,0,0)} \frac{\partial}{\partial y} &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \\ L_{g*(0,0,0)} \frac{\partial}{\partial z} &= \frac{\partial}{\partial z} \end{aligned}$$

So, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\}$ is a basis for our Lie algebra of left-invariant vector fields. Now, we can see just by inspection that $\{dx, dy, dz - xdy\}$ constitutes a dual basis. We can compute the differential on these one-forms either directly, or by dualizing the bracket on the left-invariant vector fields. For illustration let us take this latter approach. Denote from now on $\frac{\partial}{\partial x}$ by $\partial_x$ (and similarly for $y$ and $z$). Observe

$$\begin{aligned} [\partial_x, \partial_y + x\partial_z] &= \partial_z, \\ [\partial_x, \partial_z] &= 0, \\ [\partial_y + x\partial_z, \partial_z] &= 0. \end{aligned}$$

Reading off the coefficients, this tells us that the differential applied to the dual of $\partial_x$ is 0, applied to the dual of $\partial_y + x\partial_z$ is 0, and applied to the dual of $\partial_z$ is the product of the dual of $\partial_x$ and the dual of $\partial_y + x\partial_z$. On the dual basis we already found, we can verify that is true: $d(dx) = 0$, $d(dy) = 0$, and $d(dz - xdy) = -dxdy$ (note the sign).

Denoting $\alpha = dx$, $\beta = -dy$, $\gamma = dz - xdy$, we have that a model of $N$ is given by $\Lambda(\alpha, \beta, \gamma)$ with differential prescribed by $d\alpha = 0, d\beta = 0, d\gamma = \alpha\beta$. We can compute $H^2(N)$ from this model to be two-dimensional, generated by $[\alpha\gamma]$ and $[\beta\gamma]$, and so $N$ is not a torus (of the appropriate dimension three), and therefore $N$ is not formal. □

## 3. Detection of non-formality

Now let us focus on the non-formal case, and note that the **non-formality of a nilmanifold is always detected by a non-vanishing triple Massey product**. Let $M$ be a nilmanifold and $\Lambda(x_i)$ be its minimal model. We can order the variables so that the closed generators come first, $\text{Model}(M) = \Lambda(x_1, x_2, \ldots, x_k, u, \ldots)$. Here we are denoting by $u$ the first non-closed generator. In general, $du = \sum_{i,j} c_{ij} x_i x_j$. For simplicity, let us reorder and rescale the closed variables $x_i$ so that we have

$$du = x_1 x_2 + \sum_{i,j>2} c_{ij} x_i x_j.$$

Observe

$$x_3 x_4 \cdots x_k du = d(x_3 x_4 \cdots x_k u) = x_1 x_2 x_3 x_4 \cdots x_k,$$

and consider the Massey product

$$M([x_2 x_3 \cdots x_k], [x_1], [x_1]).$$

A generic element in this set looks like $(x_3 \cdots x_k u + c_{k-1}) x_1 \pm x_2 \cdots x_k c_1$, where $c_{k-1}$ and $c_1$ are closed elements of degree $k-1$ and 1 respectively. So, our generic element is of the form

$$x_3 \cdots x_k u x_1 + c_{k-1} x_1 - x_2 \cdots x_k c_1.$$

Observe that $c_1$ is in the span of $x_1, \ldots, x_k$. So, if this was to be zero for some choice of $c_1$ and $c_{k-1}$ (which would mean by definition that the Massey product vanishes), we would need $c_{k-1}$ to be of the form $c_{k-1} = u x_3 \cdots x_k$. But, $d(u x_3 \cdots x_k) = x_1 x_2 x_3 \cdots x_k \neq 0$. So, this triple Massey product is non-trivial.

## References

[1] Hasegawa, K., 1989. Minimal models of nilmanifolds. Proceedings of the American Mathematical Society, 106(1), pp.65-71.