COMPLEX, ALMOST COMPLEX, AND STABLE ALMOST COMPLEX STRUCTURES

Consider the following table depicting the relationship between the sets of complex, almost complex, and stably almost complex manifolds of even dimension. (For a manifold to be C, AC, or SAC means that it admits such a corresponding structure.)

dim	C	$\subset$	AC	$\subset$	SAC	non-SAC
2		$=$		$=$		$\emptyset$
4		$\subsetneq$ 6		$\subsetneq$ 1		$\emptyset$ 4
6		?		$=$ 3		$\neq \emptyset$ 5
8		?		$\subsetneq$ 2		$\neq \emptyset$ 5
10		?		$\subsetneq$ 2		$\neq \emptyset$ 5
...		...		...		...

The circled numbers indicate the order in which we will demonstrate the claimed equality or inequality of sets. We will address 1 – 5. For 6 and some of the necessary background, you can check my page.

An even-dimensional manifold has a *complex structure* if there exists an atlas on it with holomorphic transition maps. An *almost complex structure* on a manifold M is an endomorphism J of the tangent bundle TM such that $J^2 = -\mathrm{Id}$. A stable almost complex structure is an endomorphism squaring to $-\mathrm{Id}$ on $TM \oplus \varepsilon_{\mathbb{R}}^k$, where $\varepsilon_{\mathbb{R}}^k$ is some trivial real line bundle. If a manifold admits one of the above structures, we will say it is C, AC, or SAC, respectively. Note that being C implies being AC, and being AC implies being SAC.

1 Let us show that there are SAC manifolds which are not AC. Concretely, S^4 admits a stable almost complex structure, but no almost complex structure. Namely, embed S^4 into $\mathbb{R}^5$ in the usual way, and observe that the normal line bundle is trivial. So, $TS^4 \oplus \varepsilon^1 = \varepsilon^5$. We add another trivial line bundle to obtain $TS^4 \oplus \varepsilon^2 = \varepsilon^5 \oplus \varepsilon^1 = \varepsilon^6$, on which we can put an almost complex structure by choosing a frame $\{e_1, \ldots, e_6\}$ and defining $Je_1 = e_2, Je_2 = -e_1, Je_3 = e_4, Je_4 = -Je_3, Je_5 = e_6, Je_6 = -e_5$.

Now let us show that S^4 admits no almost complex structure. If it did, we could consider the Chern classes corresponding to some J. From the general relation

$$(1 - p_1 + p_2 - \cdots) = (1 - c_1 + c_2 - \cdots) \cdot (1 + c_1 + c_2 + \cdots)$$

between the Pontryagin and Chern classes of an AC manifold, we conclude

$$p_1(TS^4) = c_1(TS^4, J)^2 - 2c_2(TS^4, J).$$

(The Pontryagin classes do not depend on the almost complex structure like the Chern classes do.) From the Hirzebruch signature formula, we know that $\int_{S^4} p_1(TS^4) = 3{\cdot}\sigma(S^4)$, where $\sigma(S^4)$ is the signature. Since S^4 has no middle cohomology, $\sigma(S^4) = 0$. Combining all this, we conclude

$$\int_{S^4} c_1(TS^4, J)^2 - 2 \int_{S^4} c_2(TS^4, J) = 0.$$

From the construction of the Chern classes, it holds in general that the integral of the top Chern class is equal to the Euler characteristic of the manifold considered. So, $\int_{S^4} c_2(TS^4, J) = 2$. So we conclude $\int_{S^4} c_1(TS^4, J)^2 = 4$. However, c_1 lives in $H^2(S^4, \mathbb{Z})$, which is empty, so this equation cannot be satisfied. So, S^4 cannot be AC.

② We show that for dimensions $8, 10, 12, \ldots$ there are SAC manifolds that are not AC. Again, these will be spheres S^n as above. The same consideration shows that all the spheres are SAC. To show that these spheres are not AC using the above method gets messy (and is only applicable in dimensions $4n$) though, so we take another approach.

Denote by $BU(n)$ the classifying space of the unitary group $U(n)$. (Isomorphism classes of complex rank n vector bundles over a closed manifold M are in bijective correspondence with $[M, BU(n)]$, i.e. homotopy classes of maps from M to $BU(n)$. The space $BU(n)$ can be realized concretely as the Grassmannian of n-planes in $\mathbb{C}^\infty$.) To say that a manifold M^{2n} is AC is to say that the map $M \xrightarrow{f} BSO(2n)$ classifying the tangent bundle TM lifts to a map to $BU(n)$. That is, we have the commutative diagram

$$\begin{tikzcd} & BU(n) \arrow[d] \\ M \arrow[ur] \arrow[r] & BSO(2n) \end{tikzcd}$$

We can consider the union of all such Grassmannians $BU(k)$, and obtain the classifying space of the stable unitary group, BU. The inclusion $U(n) \hookrightarrow U$ induces a map on classifying spaces $BU(n) \to BU$. Bott showed that the sequence of homotopy groups π_* of BU is $0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \ldots$ (starting at $\pi_1(BU)$). So, for even spheres, $[S^{2k}, BU] = \mathbb{Z}$. Take a map ϕ that corresponds to $1 \in \mathbb{Z}$ in this homotopy group $\pi_{2k}(BU)$. Bott also showed that $\phi^*(c_k) = (k-1)! \cdot \iota$, where $c_k \in H^{2k}(BU, \mathbb{Z})$ is the universal k-th Chern class, and ι denotes the generator for $H^{2k}(S^{2k}, \mathbb{Z})$ such that $\int_{S^4} \iota = 1$.

Now suppose S^{2k} had an almost complex structure, for $k \geq 4$. We would then have a classifying map f for its tangent bundle, from S^{2k} to $BU(k)$. Following this map by the map $BU(k) \to BU$ induced by inclusion, we obtain a map $S^{2k} \to BU$ (let us denote this map by f as well). The map f represents some multiple of the generator $\phi \in [S^{2k}, BU]$, so we obtain $f^*(c_k) = m \cdot (k-1)! \cdot \iota$, where m denotes the integer that f corresponds to in the homotopy group. Now, since f is the classifying map for the tangent bundle, $f^*(c_k)$ is the top Chern class of this almost complex structure on S^{2k}. Therefore integrating it over the fundamental class $[S^4]$ should give us 2. On the other hand, since $\int_{S^4} \iota = 1$, and $f^*(c_k) = m \cdot (k-1)! \cdot \iota$, we conclude $m \cdot (k-1)! = 2$, which cannot be, since $(k-1)!$ is at least 6.

③ Note that the argument used above does not work to conclude that S^6 is not AC. The six-sphere is in fact one of only two spheres (the other being S^2) to admit an almost complex structure. Let us describe one such structure.

Think of S^6 as sphere of imaginary octonions of unit length, sitting inside the sphere S^7 of unit octonions. We can think of S^6 as being those unit octonions that are orthogonal to the element 1 (with respect to the standard inner product on $\mathbb{O} = \mathbb{R}^8$). Now, take a unit imaginary octonion $q \in S^6$, and consider the tangent space to S^6 at this point. We can identify the tangent space with those quaternions that are orthogonal to both 1 and q. On this tangent space, define J_q to be multiplication from the left by q. Since multiplication by q is an orthogonal transformation, $(qx, 1) = 0$ and $(qx, q) = (x, 1) = 0$, so this is a well-defined operation on the tangent space. This defines a global endomorphism squaring to $-\mathrm{Id}$.

It is an open problem to determine whether S^6 is C. In fact, there are no known examples of manifolds in dimension 6 or greater that are AC but not C.

It turns out that all SAC 6-manifolds are AC. A standard result tells us that an orientable 6-manifold M is AC if and only if the second Stiefel-Whitney class $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ has an integral lift (i.e. there is a class $c \in H^2(M\mathbb{Z})$ such that its mod 2 reduction is w_2). Suppose M is only SAC. Then $TM \oplus \varepsilon^k$ has an almost complex structure, so we can consider $c_1(TM \oplus \varepsilon^k)$. The mod 2 reduction of a Chern class is the corresponding Stiefel-Whitney class in that degree. So, $w_2(TM \oplus \varepsilon^k) = c_1(TM \oplus \varepsilon^k)$. The Stiefel-Whitney classes are unchanged under sums with trivial bundles, so we have $w_2(TM \oplus \varepsilon^k) = w_2(TM)$. So, we have found an integral lift of $w_2(TM)$, namely $c_1(TM \oplus \varepsilon^k)$, and therefore M is AC.

④ All orientable 4-manifolds are SAC. Indeed, to be SAC means that there is a lift to BU of the stabilized classifying map from the manifold X to BSO.

$$\begin{tikzcd}
 & & BU \arrow[d] & SO/U \arrow[l] \\
X \arrow[r] \arrow[urr] & BSO(4) \arrow[r] & BSO &
\end{tikzcd}$$

The (homotopy) fiber of the map $BU \to BSO$ (which is induced by inclusion) is the quotient SO/U. The obstructions to lifting the classifying map $X \xrightarrow{f} BSO$ to a map $X \xrightarrow{\tilde{f}} BU$ are in $H^*(X, \pi_{*-1}(SO/U))$. To compute $\pi_{*-1}(SO/U)$, we use the fact that, by Bott, O/U is homotopy equivalent to ΩO. By Bott periodicity,

$$\pi_*(O) = \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots$$

(starting at $\pi_0(O)$. So, ΩO (meaning in fact the loop space of a connected component of O) has homotopy groups $\mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots$ (just a shift left by one from those of O). Now, SO/U is connected since SO is, so we have, starting at $\pi_0(SO/U)$,

$$\pi_*(SO/U) = 0, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots$$

Since we are taking X to be a 4-manifold, all cohomology above degree 4 vanishes. So we see that in fact the only obstruction to lifting the classifying map to BU lies in $H^3(X, \pi_2(SO/U)) = H^3(X, \mathbb{Z})$. This obstruction is the third integral Stiefel-Whitney class W_3. This class vanishes (and so X is SAC) if and only if $w_2(TX)$ has an integral lift.

Indeed, from the short exact sequence of groups $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, we obtain a long exact sequence in cohomology, part of which is

$$\cdots \to H^1(X, \mathbb{Z}_2) \xrightarrow{\beta} H^2(X, \mathbb{Z}) \xrightarrow{2} H^2(X, \mathbb{Z}) \xrightarrow{\text{mod } 2} H^2(X, \mathbb{Z}_2) \xrightarrow{\beta} H^3(X, \mathbb{Z}) \xrightarrow{2} \cdots$$

The integral third Stiefel-Whitney class $W_3 \in H^3(X, \mathbb{Z})$ can be defined to be $\beta(w_2)$. By exactness of the sequence, this is zero if and only if w_2 is in the image of the mod 2 map, i.e. if and only if w_2 has an integral lift. Requiring that w_2 has an integral lift is one way of saying that the manifold we are considering is *spin*^{c}.

Lemma 0.1. *An orientable closed 4-manifold is spin*^{c} *(i.e.* w_2 *has an integral lift).*

Proof. Denote by r the mod 2 reduction map $H^2(M, \mathbb{Z}) \xrightarrow{r} H^2(M, \mathbb{Z}_2)$, and denote by T^i the torsion in $H^i(M, \mathbb{Z})$. Note that on the $\mathbb{Z}_2$ -vector space $H^2(M, \mathbb{Z}_2)$, every element x determines a functional $\hat{x}$ by setting $\hat{x}(y) = x \cup y$ (where we interpret $x \cup y \in H^4(M, \mathbb{Z}_2)$ as 0 or 1). Making no distinction between an element x and its associated functional, we will show that the annihilator of $r(T^2)$ is precisely $r(H^2(M, \mathbb{Z}))$, i.e. those elements with integral lifts. With the observation that w_2 annihilates all of $r(T^2)$, we will conclude that w_2 has an integral lift.

First, let us observe that w_2 annihilates $r(T^2)$. Take $x \in r(T^2)$ and consider $x \cup w_2$. By considerations involing the Wu formula and Steenrod squares, it follows that on an orientable 4-manifold, $x \cup w_2 = x^2$. Since x is the mod 2 reduction of a torsion integral class $\tilde{x}$, we can obtain x^2 by taking the mod 2 reduction of $\tilde{x}^2$. But $\tilde{x}^2$ is a torsion element in the free group $H^4(M, \mathbb{Z})$, therefore $\tilde{x}^2 = 0$ and so $x^2 = 0$.

Now we show that the annihilator of $r(T^2)$ is $r(H^2(M, \mathbb{Z}))$. The group $r(H^2(M, \mathbb{Z}))$ is certainly contained in the annihilator of $r(T^2)$, since for any $x \in r(H^2(M, \mathbb{Z}))$ and $y \in r(T^2)$, we can take integral lifts $\tilde{x}$ and $\tilde{y}$, observe that $\tilde{x} \cup \tilde{y}$ is torsion in $H^4(M, \mathbb{Z})$ and hence 0, and reduce mod 2 to conclude $x \cup y = 0$. To show that the annihilator of $r(T^2)$ is in fact equal to $r(H^2(M, \mathbb{Z}))$, we show that it has the right dimension, i.e.

$$\text{we show that } \dim H^2(M, \mathbb{Z}_2) - \dim r(T^2) = \dim r(H^2(M, \mathbb{Z})).$$

Denote by b_i the rank of $H^i(M, \mathbb{Z})$, i.e. the number of $\mathbb{Z}$ summands, and denote by c_i the number of summands of one of the types $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8, \ldots$ in T^i.

Consider the following stretch of long exact sequence corresponding to the short exact sequence of coefficients,

$$\cdots \to H^2(X, \mathbb{Z}) \xrightarrow{r} H^2(X, \mathbb{Z}_2) \xrightarrow{\beta} H^3(X, \mathbb{Z}) \xrightarrow{2} H^3(M, \mathbb{Z}) \to \cdots$$

Observe that r will convert the $\mathbb{Z}$ factors in H^2 into $\mathbb{Z}_2$'s, it will convert $\mathbb{Z}_{2^k}$'s into $\mathbb{Z}_2$'s, and it will convert everything else into 0. So, $\dim r(H^2) = b_2 + c_2$, and $\dim r(T^2) = c_2$. To figure out the total dimension of $H^2(M,\mathbb{Z}_2)$, we consider $H^2(M,\mathbb{Z}_2)/\mathrm{image}(r)$, which is by exactness equal to $H^2(M,\mathbb{Z}_2)/\ker(\beta) = \mathrm{image}(\beta) = \ker(\cdot 2)$. This last kernel obtains one dimension for every $\mathbb{Z}_{2^k}$ factor in $H^3(M,\mathbb{Z})$ (since $\mathbb{Z}_{2^{k-1}} \subset \mathbb{Z}_{2^k}$ will be sent to zero under multiplication by two), so we conclude that $\ker(\cdot 2)$ has dimension c_3. Therefore $H^2(M,\mathbb{Z}_2)$ has dimension

$$\dim H^2(M,\mathbb{Z}_2) = \dim \mathrm{image}(r) + \dim H^2(M,\mathbb{Z}_2)/\mathrm{image}(r) = b_2 + c_2 + c_3.$$

So, $\dim H^2(M,\mathbb{Z}_2) - \dim r(T^2) = b_2 + c_2 + c_3 - c_2 = b_2 + c_3$. On the other hand, $\dim r(H^2(M,\mathbb{Z})) = b_2 + c_2$. Since by Poincaré duality we have that the torsion in $H^2(M,\mathbb{Z})$ is the torsion in $H_2(M,\mathbb{Z})$, which by the universal coefficient theorem is the torsion in $H^3(M,\mathbb{Z})$, we conclude that $c_2 = c_3$ and we have the desired equality of dimensions. □

So, every 4-manifold is spin^c, and so the only obstruction to being SAC (i.e. having an integral lift for w_2) vanishes. Therefore, every orientable 4-manifold is SAC.

⑤ Every orientable 2-manifold is C, so in particular it is SAC, and as we just saw, every orientable 4-manifold is SAC. In dimensions 6 and greater, there are manifolds that are not SAC. We will construct an example in each dimension by crossing one particular manifold with spheres of the appropriate dimension.

The particular manifold we consider is the *Wu manifold* $W = SU(3)/SO(3)$, obtained by embedding a 3×3 orthogonal matrix with determinant 1 into $SU(3)$ by just reinterpreting the coefficients to be complex instead of real, and taking the quotient. Our examples of non-SAC manifolds will be $W \times S^k$ for varying k. First let us consider some properties of the Wu manifold.

The inclusion $SO(3) \hookrightarrow SU(3)$ induces a map on classifying spaces $BSO(3) \to BU(3)$ whose (homotopy) fiber is $SU(3)/SO(3)$. That is, we have the fibration

$$\begin{tikzcd} SU(3)/SO(3) \arrow[r] & BSO(3) \arrow[d] \\ & BSU(3) \end{tikzcd}$$

Consider the long exact sequence in homotopy groups for this fibration,

$$\cdots \to \pi_3 BSU(3) \to \pi_2 W \to \pi_2 BSO(3) \to \pi_2 BSU(3) \to \pi_1 W \to \pi_1 BSO(3) \to \cdots$$

The homotopy groups of the classifying space BG of a group G are just the homotopy groups of G shifted to the right by one. So, for example, $\pi_3 BSU(3) = \pi_2 SU(3)$, and $\pi_2 SU(3)$ is already in the range where Bott periodicity applies, i.e. $\pi_2 SU(3) = \pi_2 SU = 0$ (the groups SU and U have the same higher homotopy groups, since the second is just an extension of the first by a circle group). We also have $\pi_2 BSU(3) = \pi_1 SU(3) = 0$, and $\pi_1 BSO(3) = \pi_0 SO(3) = 0$. So, our long exact sequence actually looks like this:

$$\cdots \to 0 \to \pi_2 W \to \pi_2 BSO(3) \to 0 \to \pi_1 W \to 0 \to \cdots$$

By exactness, we conclude $\pi_1 W = 0$ and $\pi_2 W = \pi_2 BSO(3) = \pi_1 SO(3) = \pi_1 \mathbb{R}P^3 = \mathbb{Z}_2$. By the Hurewicz theorem we conclude that $H_1(W,\mathbb{Z}) = 0$ and $H_2(W,\mathbb{Z}) = \mathbb{Z}_2$. From the universal coefficient theorem we obtain $H^1(W,\mathbb{Z}) = 0$ and $H^2(W,\mathbb{Z}) = 0$. Now

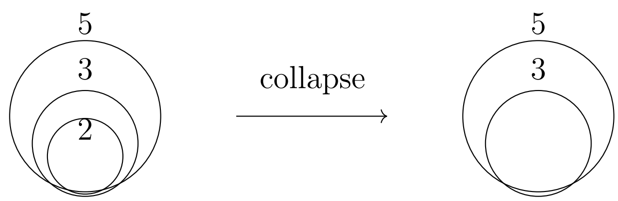
Poincaré duality gives us $H^3(W,\mathbb{Z})=\mathbb{Z}_2$, $H^4(W,\mathbb{Z})=0, H^5(W,\mathbb{Z})=\mathbb{Z}$, and again by the universal coefficient theorem we conclude $H_3(W,\mathbb{Z})=0, H_4(W,\mathbb{Z})=0, H_5(W,\mathbb{Z})=\mathbb{Z}$. Yet again by the universal coefficient theorem, now for $\mathbb{Z}_2$ coefficients, we conclude that $H^*(W,\mathbb{Z}_2)=\mathbb{Z}_2,0,\mathbb{Z}_2,\mathbb{Z}_2,0,\mathbb{Z}_2$.

Now, let us argue that W is not spin^c, and therefore not SAC. If w_2 had an integral lift, this integral lift would have to be 0, since $H^2(W,\mathbb{Z})=0$. So, if W was spin^c, we would have $w_2=0$, i.e. that W is *spin*. We can appeal to the fact that W is non-trivial in the fifth oriented cobordism group to conclude that W cannot be spin, but let us argue more directly.

A result of Smale tells us that we can write a cellular decomposition of W with as many k-cells as the rank of k-chains in a chain complex which computes the same integral homology as that of W. So, in our example, since $H_*(W,\mathbb{Z})=\mathbb{Z},0,\mathbb{Z}_2,0,0,\mathbb{Z}$, we conclude that we can write W as a 0-cell, 2-cell, 3-cell, and 5-cell, corresponding to the chain complex

$$\mathbb{Z}\to 0\to\mathbb{Z}\xrightarrow{2}\mathbb{Z}\to 0\to\mathbb{Z}.$$

It can be shown that by collapsing the 2-cell to a point,



the resulting space is the suspension $\Sigma\mathbb{CP}^2$ of $\mathbb{CP}^2$. On $\mathbb{CP}^2$, we have $w_2(T\mathbb{CP}^2)\cdot x=x^2$ for all $x\in H^2(\mathbb{CP}^2,\mathbb{Z}_2)$, as we saw before for 4-manifolds. What is really happening here is that the second *Steenrod square* $\mathrm{Sq}^2: H^2(-,\mathbb{Z}_2)\to H^4(-,\mathbb{Z}_2)$ is realized by multiplication by w_2 on an orientable 4-manifold. The Steenrod squares are *stable under suspension*, here meaning $\widetilde{\mathrm{Sq}^2(x)}=\mathrm{Sq}^2(\tilde{x})$,
where $\tilde{}$ denotes the image of the considered class under the suspension isomorphism $H^*(-)\to H^{*+1}(\Sigma -)$. So, $\mathrm{Sq}^2: H^3(\Sigma\mathbb{CP}^2,\mathbb{Z}_2)\to H^5(\Sigma\mathbb{CP}^2,\mathbb{Z}_2)$ is non-trivial, and so after pulling back by the collapse map, which induces isomorphisms on $H^3(-,\mathbb{Z}_2)$ and $H^5(-,\mathbb{Z}_2)$, and using the naturality of Sq^2, we conclude that $\mathrm{Sq}^2: H^3(W,\mathbb{Z}_2)\to H^5(W,\mathbb{Z}_2)$ is non-trivial as well.

$$\begin{tikzcd}
H^5(W,\mathbb{Z}_2) & H^5(\Sigma\mathbb{CP}^2,\mathbb{Z}_2) \arrow[l, "\sim"'] \\
H^3(W,\mathbb{Z}_2) \arrow[u, "\mathrm{Sq}^2"] & H^3(\Sigma\mathbb{CP}^2,\mathbb{Z}_2) \arrow[l, "\sim"'] \arrow[u, "\mathrm{Sq}^2"]
\end{tikzcd}$$

On an orientable manifold X of any dimension n, $\mathrm{Sq}^2: H^{n-2}(X,\mathbb{Z}_2)\to H^n(X,\mathbb{Z}_2)$ is realized by multiplication by w_2. Since we saw that $\mathrm{Sq}^2: H^3(W,\mathbb{Z}_2)\to H^5(W,\mathbb{Z}_2)$ is a non-trivial operation, we conclude by the non-degeneracy part of Poincaré duality that $w_2(TW)\neq 0$.

Therefore W is not spin, and therefore (since $H^2(W,\mathbb{Z})=0$) it is not spin^c. In any dimension $2n$ for $n\geq 3$, we can consider the manifold $W\times S^{2n-5}$. From the Künneth theorem over $\mathbb{Z}$ and the universal coefficient theorem, we can argue that a product of manifolds is spin^c if and only if each factor is spin^c. Therefore these manifolds $W \times S^{2n-5}$ are not spin^c, and hence not SAC.

Question. In dimensions 2 and 4, there are no non-SAC manifolds. In dimension 6, being non-SAC is equivalent to being non-spin^c. The constructed examples of non-SAC manifolds in dimensions 8 and greater were all non-spin^c. Construct examples of spin^c non-SAC manifolds in dimensions ≥ 8. Even better, construct examples of manifolds all of whose even Stiefel-Whitney classes w_{2i} have integral lifts, but such that the manifold is not SAC.