The Betti numbers of the free loop space of a simply connected closed manifold are bounded if and only if the cohomology ring of the manifold is singly generated

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Following a paper of Sullivan and Vigué-Poirrier, we show that a simply connected closed manifold has non-singly generated cohomology ring if and only if the sequence of Betti numbers of its free loop space is unbounded. Invoking a theorem of Gromoll-Meyer, we can conclude that such a manifold has infinitely many geometrically distinct closed geodesics for any metric.

We begin with several lemmas, and then move on to the main result. Throughout, x will denote even-degree generators in a given minimal dga, and y will denote odd-degree generators.

Lemma 1. The cohomology ring $H^*(\Lambda, d)$ of a minimal dga is singly-generated if and only if $\Lambda = \Lambda(y)$ or $\Lambda = \Lambda(x, y)$ with $dx = 0, dy = x^k$ for some $k \geq 2$.

Proof. A direct computation shows that $H^*(\Lambda(y))$ and $H^*(\Lambda(x, y))$ are singly-generated (by $[y]$ and $[x]$, respectively). Conversely, assume $H^*(\Lambda, d)$ is singly-generated. Take a generator of lowest degree in Λ. If this is an odd generator, call it y, observe that $dy = 0$ and any other generator of the same or next lowest degree would have to be closed as well, since the algebra is minimal and $y^2 = 0$. So, in this case, $\Lambda = \Lambda(y)$. If this generator of lowest degree is instead even, call it x, observe $dx = 0$ and consider any other generator of the same or next lowest degree. (If there is no such other generator, then $\Lambda = \Lambda(x)$, a case which we are not too interested in since this algebra has infinite cohomology and so cannot correspond to a closed manifold). If this other generator is also even, then it is closed as well, giving us two generators for $H^*(\Lambda, d)$, contrary to assumption. So, if another generator exists, it is odd. Call it y, and note that it cannot be closed, so we have $dy = cx^k$ for some $c \neq 0$. A change of variable lets us say instead $dy = x^k$. Note $k \geq 2$ by minimality. Now suppose there is yet another generator in the algebra. Take one of the next lowest degree. If it is even, call it x', then (again changing variables to remove constants) $dx' = x^l y$, which after differentiating gives us $x^{l+k} = 0$, which cannot be. If this third generator is odd instead, call it y', then $dy' = x^l$. Now $yx^{l-k} - y'$ gives us another generator in cohomology, contrary to assumption. So, there could have only been the original x and the y such that $dy = x^k$. Therefore, in this case $\Lambda = \Lambda(x, y)$. □

Lemma 2. In a free dga Λ, suppose we have an odd generator y such that dy contains a non-zero summand consisting only of even degree factors. Then

$$H^*(\Lambda/\text{ideal}(y, dy), d') = H^*(\Lambda, d),$$

where $d' : \Lambda/\text{ideal}(y, dy) \to \Lambda/\text{ideal}(y, dy)$ is the induced differential.

Proof. First observe that d' is well-defined. Indeed, for any $ay + bdy$ in the ideal generated by y and dy, we have $d(ay + bdy) = (da)y \pm ady + (db)(dy)$, which is also in the ideal. Now, $H^*(\Lambda/\text{ideal}(y, dy), d') = \frac{\ker d'}{\text{image } d'}$, so let us figure out what $\ker d'$ is first. If $[\xi] \in \ker d'$, then $\xi \in \Lambda$ is such that $d\xi = ay + bdy$. Note that we can rewrite $ay + bdy$ as $(a - db)y + d(by)$, so let us just write $d\xi = ay + d(by)$. From here we see that $ay \in y\Lambda$ is in the kernel of d. Any such element is 0. Indeed, $0 = d(ay) = (da)y \pm a(\text{product of even terms}) \pm a(\text{product of mixed terms})$. From freeness of Λ we can conclude that a itself must have a factor of y, and so since $y^2 = 0$ we have $ay = 0$. So, we have $d\xi = d(by)$, i.e. we can write $\xi = (\xi - by) + by \in \ker d + y\Lambda$. As we just saw, $\ker d \cap y\Lambda = 0$, so we conclude $\ker d' = \frac{\ker d \oplus y\Lambda}{\text{ideal}(y,dy)}$.

As for image d', from $d'(a + \text{ideal}(y, dy)) = da + \text{ideal}(y, dy)$ we have $\text{image } d' = \frac{\text{image } d + y\Lambda}{\text{ideal}(y,dy)}$, but since image d' in inside $\ker d'$, the sum is a direct sum, and so $\text{image } d' = \frac{\text{image } d \oplus y\Lambda}{\text{ideal}(y,dy)}$. Taking the quotient we see that $\frac{\ker d'}{\text{image } d'} = \frac{\ker d}{\text{image } d}$. □

For conclusions relating to boundedness of Betti numbers, it is useful to consider the Poincaré series

$$S_{\Lambda,d}(t) = \sum_{n\geq 0} (\dim \Lambda_n) t^n, \qquad S_{H^*(\Lambda,d)}(t) = \sum_{n\geq 0} (\dim H^n(\Lambda,d)) t^n.$$

Here Λ_n is the vector space spanned by the generators of degree n.

Lemma 3. If y is a closed odd generator in a dga (Λ, d), then

$$S_{H^*(\Lambda/y\Lambda,d')}(t) \leq \frac{S_{H^*(\Lambda,d)}(t)}{1-t^{|y|-1}}.$$

Here d' is the induced differential $\Lambda/y\Lambda \to \Lambda/y\Lambda$, well-defined since $d(ya) = (dy)a - yda = yda$ because y is closed. Inequality between Poincar'e series means termwise inequality.

Proof. Consider the short exact sequence of dga's

$$0 \longrightarrow \Sigma^{|y|}(\Lambda/y\Lambda) \xrightarrow{f} \Lambda \xrightarrow{\pi} \Lambda/y\Lambda \longrightarrow 0.$$

Here $\Sigma^{|y|}$ is obtained by taking a graded vector space basis for the dga, raising all degrees by $|y|$ (let us denote this by moving ξ from the original dga up to $\hat{\xi}$ in the suspended dga), setting $d(\hat{\xi}) = \hat{d\xi}$, and declaring the multiplication to be trivial. The map f takes $[\hat{\xi}] \in \Sigma^{|y|}(\Lambda/y\Lambda)$ and sends it to ξy. Due to the suspension by an appropriate amount, this map preserves degree. It is also well defined, since if we take $\xi + y\gamma$ in the same class as ξ in $\Lambda/y\Lambda$, applying f to $\widehat{\xi + y\gamma}$ gives us $y(\xi + y\gamma) = y\xi$ since $y^2 = 0$. The map π is the canonical projection. It is immediate from the definitions of these two outer dga's that the maps respect the differential, and that the sequence is exact.

So, we obtain a long exact sequence in cohomology,

$$\cdots \longrightarrow H^n(\Sigma^{|y|}(\Lambda/y\Lambda)) \xrightarrow{f_n^*} H^n(\Lambda) \xrightarrow{\pi^*} H^n(\Lambda/y\Lambda) \longrightarrow H^{n+1}(\Sigma^{|y|}(\Lambda/y\Lambda)) \longrightarrow \cdots.$$

As for spaces, there is a suspension isomorphism, which lets us rewrite this long exact sequence as

$$\cdots \longrightarrow H^{n-|y|}(\Lambda/y\Lambda) \xrightarrow{f_n^*} H^n(\Lambda) \xrightarrow{\pi^*} H^n(\Lambda/y\Lambda) \longrightarrow H^{n+1-|y|}(\Lambda/y\Lambda) \longrightarrow \cdots.$$

From this long exact sequence we can make a sequence of long exact sequences by interpolating the images of f^*,

$$0 \longrightarrow \mathrm{image}(f_n^*) \hookrightarrow H^n(\Lambda) \xrightarrow{\pi^*} H^n(\Lambda/y\Lambda) \longrightarrow H^{n+1-|y|}(\Lambda/y\Lambda) \xrightarrow{f_{n+1}^*} \mathrm{image}\, f_{n+1}^* \longrightarrow 0.$$

Taking the Euler characteristic of this sequence, summing over all n, and discarding negative degree terms gives us

$$(1+\frac{1}{t})S_{\mathrm{image}(f^*)}(t) - S_{H^*(\Lambda,d)}(t) + (1-t^{|y|-1})S_{H^*(\Lambda,y\Lambda,d')}.$$

Now, observe that all the terms of $(1+\frac{1}{t})S_{\mathrm{image}(f^*)}(t)$ are non-negative, and so we obtain the desired inequality. □

Lemma 4. If x is a closed even generator in a dga (Λ, d), then

$$S_{H^*(\Lambda/x\Lambda,d')}(t) \leq (1+t^{|x|-1})S_{H^*(\Lambda,d)}(t).$$

Proof. The proof is analogous to that of the previous Lemma, starting from the short exact sequence

$$0 \longrightarrow \Sigma^{|x|}\Lambda \xrightarrow{f} \Lambda \xrightarrow{\pi} \Lambda/x\Lambda \longrightarrow 0$$

instead. □

Now we can prove the main result. Our space M will be a simply connected closed manifold, and we denote its minimal model by Λ. Denote the minimal model of the free loop space LM by (Λ', d').

Theorem. The (rational) cohomology ring H^*M requires at least two generators if and only if the sequence of Betti numbers $\{b_i(LM)\}_i$ is unbounded.

Proof. First let us suppose the cohomology ring H^*M is singly generated. By Lemma 1, $\Lambda = \Lambda(y)$ or $\Lambda = \Lambda(x, y)$ with $dy = x^k$. If $\Lambda = \Lambda(y)$, then $\Lambda' = \Lambda(y, \overline{y})$ with $d'y = d'\overline{y} = 0$. From here we see that the dimension of any $H^k(LM)$ is bounded by 1 (the cohomology group being generated by a power of $\overline{y}$ or by a power of $\overline{y}$ multiplied by y). If $\Lambda = \Lambda(x, y)$, then $\Lambda' = \Lambda(x, y, \overline{x}, \overline{y})$, with $d'x = d'\overline{x} = 0$, $d'y = x^k$, and $d'\overline{y} = -kx^{k-1}\overline{x}$. Now, apply Lemma 2 to conclude $H^*\Lambda' = H^*(\Lambda'/(y, x^k))$. From here we see that in a given cohomology group, a generator is obtained by choosing to include a factor of $\overline{x}$ or not, choosing a power of x less than k, and multiplying by and appropriate power of $\overline{y}$. So, the Betti numbers of Λ' are bounded by $2k$.

Now assume H^*M requires at least two generators. First let us show that this is equivalent to Λ having at least two odd generators. If Λ has two or more odd generators, then by Lemma 1 we conclude that H^*M is not singly generated. Conversely, assume Λ has one or zero odd generators. Since H^*M is finite, it cannot be the case that Λ has no odd generators. So suppose Λ has only one odd generator, call it y. Then $\Lambda = \Lambda(x_i, y)$ for some even generators x_i. If $dy \neq 0$, then dy is some polynomial P in the x_i, and we can apply Lemma 2 to conclude $H^*(\Lambda) = H^*(\Lambda(x_i)/P \cdot \Lambda(x_i))$. This can only be finite dimensional (as necessary) if there is but one x_i. So, $\Lambda = \Lambda(x, y)$, $dy = x^k$, and so H^*M is singly generated. If instead $dy = 0$, we can apply Lemma 3 to conclude

$$S_{H^*(\Lambda(x_i)),}(t) \leq \frac{S_{H^*(\Lambda)}(t)}{1 - t^{|y|-1}}.$$

Since H^*M is bounded, the right hand side is bounded as well, and so the left hand side is, as well. Again, this implies there is but one x_i. So, $\Lambda = \Lambda(x, y)$ with $dy = 0$, and so Λ has infinite cohomology contrary to assumption. (If there was more than one even generator, say x_1 and x_2, we could have $dx_2 = x_1 y$ and the desired contradiction would not follow).

So, assuming H^*M requires at least two generators is equivalent to Λ having at least two odd generators. Denote the odd generators of lowest degree by y_1 and y_2. We make use of them to show $b_i(\Lambda')$ is unbounded.

Since M is simply connected, the model Λ of M has finitely many generators in each degree. So, we can order the generators close to y_1 and y_2 as

$$x_1, x_2, \ldots, x_n,\ y_1,\ x_{n+1}, \ldots, x_r,\ y_2, \ldots.$$

Note that $x_1, \ldots, x_n$ are closed. Suppose now that $dy_1 \neq 0$. So, $dy = P(x_1, \ldots, x_n)$, some polynomial in the preceding even generators. From here we can show that $x_{n+1}, \ldots, x_r$ are closed as well. Indeed, consider $dx_{n+1} = Q(x_1, \ldots, x_n)y_1$. Differentiating, we obtain $Q(x_1, \ldots, x_n)P(x_1, \ldots, x_n) = 0$. Since $P \neq 0$, we conclude $Q = 0$ and so x_{n+1} is closed. Analogously we conclude $dx_{n+2} = \cdots = dx_r = 0$. Finally, moving over to Λ', we conclude that all elements of the form $\bar{x_1}\bar{x_2}\cdots\bar{x_r}\bar{y_1}^a\bar{y_2}^b$ are closed, for any choice of a, b. Moreover, since d' applied to anything results in an element containing a factor of some x_i, we conclude that all of these $\bar{x_1}\bar{x_2}\cdots\bar{x_r}\bar{y_1}^a\bar{y_2}^b$ are independent in cohomology. For sufficiently high multiples of the lowest common multiples of the degrees of $\overline{y_1}$ and $\overline{y_2}$, shifted up by the sum of the degrees of the $\overline{x_i}$, we conclude that we can obtain arbitrarily many such cohomologically independent elements. Therefore, $\{b_i(\Lambda')\}_i$ is unbounded.

If in fact $dy_1 = 0$, then we take a different approach. If $dy_2 = 0$ as well, we

consider the cohomologically independent elements $\bar{y_1}^a\bar{y_2}^b$ and obtain the desired conclusion of unboundedness of $\{b_i(\Lambda')\}_i$ again. If $dy_2 \neq 0$, then since dy_2 is of even degree, $dy_2 = P(x_1, \ldots, x_r)$. Since $d'x_1 = 0$, we can consider $\Lambda'_1 = \Lambda/x_1\Lambda$ with the induced differential d'_1, and applying Lemma 4 we obtain

$$S_{H^*(\Lambda'_1, d'_1)} \leq (1 + t^{|x_1|-1})S^*_H(\Lambda', d').$$

Inductively, suppose we have (Λ'_k, d'_k), and define $\Lambda'_{k+1} = \Lambda'_k/x_{k+1}\Lambda'_k$. Since $d'y_1 = 0$, we see that the induced differential d'_{k+1} is well-defined. Applying Lemma 4 to this situation we obtain

$$S_{H^*(\Lambda'_{k+1}, d'_{k+1})} \leq (1 + t^{|x_{k+1}|-1})S^*_H(\Lambda'_k, d'_k).$$

So, in the end, we obtain

$$S_{H^*(\Lambda'_r, d'_r)} \leq \prod_{i=1}^{r} (1 + t^{|x_i|-1}) S_H^*(\Lambda', d').$$

If the left hand side is unbounded, it follows that the right hand side is unbounded, from which we can conclude that $H^*(\Lambda', d')$ itself is unbounded. Observe that $d'_r \overline{y_2} = 0$, and so the cohomologically independent elements $\bar{y_1}^a \bar{y_2}^b$ give us unbounded cohomology for $H^*(\Lambda'_r, d'_r)$. Thus, $H^*(\Lambda', d')$ is unbounded. □

Reference

[1] Vigué-Poirrier, M. and Sullivan, D., 1976. The homology theory of the closed geodesic problem. Journal of Differential Geometry, 11(4), pp.633-644.