THE EXISTENCE OF (STABLE) ALMOST COMPLEX STRUCTURES ON LOW-DIMENSIONAL MANIFOLDS

Let us consider the obstruction theory in determining whether an even-dimensional oriented manifold M^{2n} admits an almost complex structure (inducing the given orientation). A choice of almost complex structure J is equivalent to a reduction of the structure group of the tangent bundle of the manifold from $SO(2n)$ to $U(n)$, which is equivalent to a section of the associated $SO(2n)/U(n)$ bundle over M. So, the obstructions to finding a section (i.e. an almost complex structure) are in $H^*(M, \pi_{*-1}(SO(2n)/U(n)))$, and the obstructions to uniqueness are in $H^*(M, \pi_*(SO(2n)/U(n))$.

In dimensions 2, 4, and 6, these spaces $SO(2n)/U(n)$ have very explicit descriptions.

Dimension 2. The space $SO(2)/U(1)$ is a point, and so every oriented 2-manifold has a (homotopically unique) almost complex structure.

Dimension 4. The space $SO(4)/U(2)$ is diffeomorphic to S^2. The obstructions to the existence of an almost complex structure in this situation are in $H^3(M, \pi_2(S^2)) = H^3(M, \mathbb{Z})$ and $H^4(M, \pi_3(S^2)) = H^4(M, \mathbb{Z})$, A theorem of Wu tells us that these obstructions are precisely the third integral Stiefel-Whitney class $W_3 \in H^3(M, \mathbb{Z})$, which is made zero by choosing an integral lift $c \in H^2(M, \mathbb{Z})$ of $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ (which will be the first Chern class $c_1(M, J)$ of the almost complex structure obtained if the next obstruction vanishes) followed by the obstruction $c^2 - 3\sigma(M) - 2\chi(M) \in H^4(M, \mathbb{Z})$ depending on the chosen integral lift. Here σ and χ denote the signature and Euler characteristic, respectively. The necessity of these obstructions vanishing follows from

$$3 \cdot \sigma(M) = p_1(TM) = c_1(M, J)^2 - 2c_2(M, J) = c_1(M, J)^2 - \chi(M),$$

which holds for any almost complex structure J on TM. Here we are identifying $H^4(M, \mathbb{Z}) = \mathbb{Z}$ via integration over the fundamental cycle, and we are using the Hirzebruch signature formula $p_1 = 3\sigma$ along with the fact that the top Chern class of any almost complex structure on TM evaluates to the Euler characteristic.

Remark. The spaces $SO(2n)/U(n)$ fiber over each other in a nice way. Think of $J(2n) = SO(2n)/U(n)$ as the space of almost complex structures on $\mathbb{R}^{2n}$. Fix a unit vector e in $\mathbb{R}^{2n}$. An almost complex structure J on $\mathbb{R}^{2n}$ must take e to a unit vector in the plane $\mathbb{R}^{2n-1}$ orthogonal to e. So, $J(e)$ must be something in the unit sphere S^{2n-2}. Once that is chosen, any choice of J on the $2n-2$–plane orthogonal to e and $J(e)$ will give an almost complex structure on the total space $\mathbb{R}^{2n}$. So, $J(2n-2)$ fibers over S^{2n-2}, and the total space of this fibration is $J(2n)$. That is, we have the fibration

$$SO(2n-2)/U(n-1) \to SO(2n)/U(n) \to S^{2n-2}.$$

In the case of $n = 3$, we have a fibration $S^2 \to SO(6)/U(3) \to S^4$. This is the same fibration as the one considered above, $S^2 \to Z \to S^4$, where sections of the second map correspond to almost complex structures on S^4.

Dimension 6. In dimension 6, the space $SO(6)/U(3)$ turns out to be $\mathbb{CP}^3$. The homotopy groups of $\mathbb{CP}^3$ relevant to doing obstruction theory on a six–manifold are, starting at π_1,

$$0, \mathbb{Z}, 0, 0, 0.$$

This can be seen from the fibration $S^1 \to S^7 \to \mathbb{CP}^3$. So, the only obstruction to the existence of a J on M^6 is in $H^3(M, \mathbb{Z})$. It is equal, again, to the third integral Stiefel-Whitney class W_3 (see Massey [2], Remark 1). Peculiar to dimension 6 is that the requirements for an almost complex structure are less demanding than in dimension 4. This is related to the fact that $4k$–manifolds have a signature, which imposes an additional relation on its Pontryagin classes and hence on the Chern classes of any almost complex structure.

The almost complex structures on M^6 are in bijective correspondence with integral lifts of w_2. A given integral lift will be the first Chern class $c_1(M, J)$ of the corresponding almost complex structure J. We can then determine $c_2(M, J)$ from $p_1(TM) = c_1(M, J)^2 - 2c_2(M, J)$, so

$$c_2(M, J) = \frac{1}{2} \cdot (c_1(M, J)^2 - p_1(TM)).$$

The top Chern class $c_3(M, J)$ is pre-determined by the requirement that it be the Euler class.

Dimension 8. The space $SO(8)/U(4)$ does not turn out to have an even simpler description, but its homotopy groups relevant to obstruction theory on an 8–manifold are known. In general, the first $2n - 2$ homotopy groups of $SO(2n)/U(n)$ are stable, i.e. they coincide with those of the stable space SO/U, which has homotopy groups corresponding to a shift of those found in BSO,

$$\pi_* SO/U = 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots.$$

The first unstable group of $SO(2n)/U(n)$, that is $\pi_{2n-1} SO(2n)/U(n)$, depends on n mod 4. In dimension 8, we have $n = 0$ mod 4, and $\pi_7(SO(8)/U(4)) = \mathbb{Z} \oplus \mathbb{Z}_2$ (see [2]).

In summary, the relevant homotopy groups of $SO(8)/U(4)$ are

$$\pi_* SO(8)/U(4) = 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2.$$

The first obstruction to the existence of an almost complex structure on an 8–manifold M is, as before, $W_3 \in H^3(M, \mathbb{Z})$. Next up, we have an obstruction in $H^7(M, \mathbb{Z})$. In ([2], Remark 1) it is observed that this obstruction is in fact the seventh integral Stiefel-Whitney class W_7 of the tangent bundle. In ([3], Theorem 2) it is shown that the second-to-last integral Stiefel-Whitney class of an orientable even-dimensional manifold always vanishes (i.e. the third-to-last Stiefel-Whitney class has an integral lift), so this obstruction vanishes. The next and final obstruction is some class $o \in H^8(M, \mathbb{Z} \oplus \mathbb{Z}_2) = H^8(M, \mathbb{Z}) \oplus H^8(M, \mathbb{Z}_2)$. This obstruction class splits as the sum $o = o_s + o_u$ of a stable obstruction class $o_2 \in H^8(M, \mathbb{Z}_2)$ and an unstable class $o_u \in H^8(M, \mathbb{Z})$.

Remark. The stable obstruction class is what we would meet if we were just looking for a *stable* almost complex structure. (Since the pair $(SO/U, SO(2n)/U(n))$ is $(2n-2)$–connected, the obstructions to the existence of an almost complex structure coincide with those for the existence of an almost complex structure, up to *but not including* the top skeleton.)

In [4] we find the following descriptions of o_s and o_u (relying on results of Massey and Heaps),

$$o_s = r_2(\chi(M) + \tfrac{1}{2}c_1c_3),$$
$$o_u = \tfrac{1}{4}(2\chi(M) - 2c_1c_3 + c_2^2 - p_2(TM)).$$

Here c_1, c_2, c_3 denote the Chern classes of the almost complex structure that has been built up to (including) the 7–skeleton of M (assuming the previous obstruction vanished), and r_2 denotes the mod 2 reduction map.

Example. Let us see which of the connected sums of quaternionic projective planes $\mathbb{HP}^{2\#k}\#\overline{\mathbb{HP}^2}^{\#l}$ admit almost complex structures (here $\overline{\mathbb{HP}}^2$ denotes $\mathbb{HP}^2$ with the reversed orientation). First, consider just $\mathbb{HP}^2$. The first potentially non-trivial obstruction to finding an almost complex structure is in $H^4(\mathbb{HP}^2, \pi_3(SO(8)/U(4)))$. However, $\pi_3(SO(8)/U(4)) = 0$, so there is no obstruction. Note that this gives us the existence of an almost complex structure on the 4–skeleton of $\mathbb{HP}^2$ (which is S^4). The obstructions to its uniqueness lie in $H^4(\mathbb{HP}^2, \pi_4(SO(8)/U(4)))$, which is also trivial. So, there is a unique (up to homotopy, as always) almost complex structure on $T\mathbb{HP}^2|_{S^4}$. The next obstruction we meet is the o at the top, $o \in H^8(\mathbb{HP}^2, \mathbb{Z} \oplus \mathbb{Z}_2)$, which splits as $o = o_s + o_u$. By the above formulas, $o_s = \chi(\mathbb{HP}^2) \bmod 2 = 1$, since $c_1 \in H^2(\mathbb{HP}^2, \mathbb{Z}) = 0$ and $c_3 \in H^6(\mathbb{HP}^2, \mathbb{Z}) = 0$. We also have

$$o_u = \tfrac{1}{4}(6 + c_2^2 - p_2(T\mathbb{HP}^2)).$$

The total Pontryagin class of $\mathbb{HP}^2$ is given by

$$p(T\mathbb{HP}^2) = \frac{(1+a)^6}{1+4a} = 1 + 2a + 7a^2,$$

where $a \in H^4(\mathbb{HP}^2, \mathbb{Z})$ is a generator such that $\int_{\mathbb{HP}^2} a^2 = 1$. From the relation $p_1(T\mathbb{HP}^2) = c_1^2 - 2c_2$ for any contending almost complex structure, and $c_1 \in H^2(\mathbb{HP}^2, \mathbb{Z}) = 0$, we conclude $2a = -2c_2$. This is an equation in $H^4(\mathbb{HP}^2, \mathbb{Z}) = \mathbb{Z}$, so we have $c_2 = -a$. Therefore,

$$o_u = \tfrac{1}{4}(6 + a^2 - 7a^2),$$

and so (since we are identifying a top cohomology class with its integral over $\mathbb{HP}^2$),

$$4 \cdot o_u = 6 - 6 \cdot \int_{\mathbb{HP}^2} a^2 = 0,$$

that is, $o_u = 0$. So our top obstruction is $o = (o_u, o_s) = (0, 1) \in \mathbb{Z} \oplus \mathbb{Z}_2$. In particular, $\mathbb{HP}^2$ does not admit an almost complex structure.

Now we consider how the top obstruction to an almost complex structure behaves under the operation of connect sum. Suppose we have two 8–manifolds M and N, and almost complex structures J_M, J_N on their respective 7–skeleta. Then there is a canonical almost complex structure J on the 7–skeleton of $M\#N$ such that the obstruction to extending it over all of $M\#N$ is given by

$$o(M\#N, J) = o(M, J_M) + o(N, J_N) - o(S^8),$$

where the terms involved are the mentioned obstructions, and $o(S^8)$ is the (only) obstruction to having an almost complex structure on S^8, which we interpret as the sphere obtained by collapsing both M and N in $M\#N$ to points.

We can also consider the situation of reversing orientation. If we have an almost complex structure J on the 7–skeleton of an 8–manifold M, then there is canonical almost complex structure $\overline{J}$ on the 7–skeleton of $\overline{M}$. The obstructions to extending over the respective 8–skeleta are related by

$$o(\overline{M}, \overline{J}) = -o(M, J) + \chi(M) o(S^8).$$

To apply these two results, all that remains is to compute the term $o(S^8)$. The first obstruction to an almost complex structure on S^8 is the one at the top, which is $o(S^8)$, and which splits as $o_s + o_u$. The above formulas for these terms give us

$$o_s = \chi(S^8) \bmod 2 = 0,$$
$$o_u = \tfrac{1}{4}(4 - p_2(TS^8)).$$

The Chern class terms c_1, c_2, c_3 vanish due to the absence of cohomology in the appropriate degrees. To compute $p_2(TS^8)$, we can use the Hirzebruch signature formula in this degree, which tells us $\sigma(S^8) = \frac{1}{45}(7p_2 - p_1^2)$. Since $p_1 \in H^4(S^8, \mathbb{Z}) = 0$ and $\sigma(S^8) = 0$, we conclude $p_2 = 0$. Therefore $o_u = 1$, and so $o = (o_s, o_u) = (1, 0)$ for the 8–sphere.

These two results (on connected sum and reversing orientation) are discussed in [1]. From the second result, we see that the only obstruction to putting an almost complex structure on $\overline{\mathbb{HP}^2}$ is given by

$$o(\overline{\mathbb{HP}^2}) = -(0, 1) + 3 \cdot (1, 0) = (3, 1) \in \mathbb{Z} \oplus \mathbb{Z}_2.$$

Combining all of this, we have that the top (and only) obstruction to an almost complex structure on $\mathbb{HP}^{2\#k} \# \overline{\mathbb{HP}^2}^{\#l}$ is given by

$$\begin{aligned} k \cdot o(\mathbb{HP}^2) + l \cdot o(\overline{\mathbb{HP}^2}) - (k + l - 1) \cdot o(S^8) &= k \cdot (0, 1) + l \cdot (3, 1) - (k + l - 1) \cdot (1, 0) \\ &= (2l - k + 1, k + l). \end{aligned}$$

In order for this to be zero in $\mathbb{Z} \oplus \mathbb{Z}_2$, we conclude that k and l have to have the same parity, and $k = 2l + 1$. In particular, k and l both have to be odd. So, $\mathbb{HP}^{2\#k} \# \overline{\mathbb{HP}^2}^{\#l}$ has an almost complex structure if and only if $(k, l) = (4n + 3, 2n + 1)$ for some $n \geq 0$.

References

[1] Kahn, Peter J. "Obstructions to extending almost X-structures." Illinois Journal of Mathematics 13, no. 2 (1969): 336-357.

[2] Massey, W. S. "Obstructions to the existence of almost complex structures." Bulletin of the American Mathematical Society 67, no. 6 (1961): 559-564.

[3] Massey, W. S. "On the Stiefel-Whitney Classes of a Manifold. II." Proceedings of the American Mathematical Society 13, no. 6 (1962): 938-42.

[4] Müller, S., and H. Geiges. "Almost complex structures on 8-manifolds." Enseignement Mathematique 46, no. 1/2 (2000): 95-108.