VERIFYING THE HILALI CONJECTURE IN FORMAL DIMENSION ≤ 20

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Abstract. We prove that in formal dimension ≤ 20 the Hilali conjecture holds, i.e. that the total dimension of the rational homology bounds from above the total dimension of the rational homotopy for a simply connected rationally elliptic space.

1. Introduction

The Hilali conjecture [HM08a] in rational homotopy theory states that for a minimal commutative differential graded algebra over the rationals $(\Lambda V, d)$ with $V^1 = 0$ whose cohomology $H^*(\Lambda V, d) = \bigoplus_i H^i(\Lambda V, d)$ and space of indecomposables V are both finite–dimensional, we have $H^*(\Lambda V, d) \geq \dim V$. Translated into a geometric statement, this says that the total dimension of the rational cohomology of a simply connected space bounds the total dimension of the rational homotopy from above if the latter quantity is finite.

Simply connected spaces with such minimal models, called rationally elliptic spaces, are known to satisfy very restrictive topological conditions. For such a space X, the topological Euler characteristic is non-negative and the homotopy Euler characteristic $\sum_i (-1)^i \pi_i(X) \otimes \mathbb{Q}$ is non-positive; furthermore, one is non-zero if and only if the other is zero [FHT, Prop. 32.10]. Such spaces are akin to closed manifolds, as they satisfy a Poincaré duality on their rational cohomology [FHT Prop. 38.3]: $H^*(X;\mathbb{Q}) \cong H^{n-*}(X;\mathbb{Q})$, where n is the formal dimension $\mathrm{fd}(X)$ of X, i.e. the largest index for which the rational cohomology does not vanish. In fact, if the homotopy Euler characteristic of X is negative, one can find a simply connected closed smooth manifold M and a rational homotopy equivalence $M \to X$ by the Barge–Sullivan theorem [FrH79, p.124].

Friedlander and Halperin [FrH79] identified the condition under which a set of integers occurs as the degrees of a homogeneous basis of $\pi_*(X) \otimes \mathbb{Q}$ of a rationally elliptic space X. Namely, the sequence $(2a_1, \ldots, 2a_r, 2b_1 - 1, \ldots, 2b_q - 1)$ denotes the degrees of a homogeneous basis of $\pi_*(X) \otimes \mathbb{Q}$ of some elliptic X if and only if the following *strong arithmetic condition* is satisfied: for every subsequence A^* of $(a_1, \ldots, a_r)$ of length s, at least s many elements b_j in $(b_1, \ldots, b_q)$ can be written as $b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i$, where the γ_{ij} are non-negative integers whose sum for any fixed j is at least two. Call such a sequence a *homotopy rank type*; note that the homotopy rank type does not uniquely determine the space X up to rational homotopy equivalence, even amongst elliptic spaces.

Using this characterization, Nakamura and Yamaguchi [NaYa11] wrote a `C++` program to output all the homotopy rank types of simply connected elliptic spaces up to a given formal dimension. In the present paper, after establishing some preliminary results, we will verify the Hilali conjecture up to formal dimension 20 by employing our results into the code of [NaYa11] to significantly reduce the number of homotopy rank types that need to be considered manually. In [HM08b], the conjecture is claimed to be verified up to formal dimension 10; in [NaYa11] this claim was pushed to formal dimension 16. However, the tables of homotopy rank types in [HM08b] are slightly incomplete (for example the homotopy rank type $(2; 11)$ corresponding to $\mathbb{CP}^5$ is not present in Table 1 therein), and the current authors failed to understand how an inequality in the proof of the crucial Proposition 4.3 in the latter article was obtained. We hence reverify the conjecture in these dimensions carefully and extend the verification up to dimension 20. In the next section the reader may see how the number of homotopy rank types increases considerably with the formal dimension.

Throughout, $(\Lambda V, d)$ will denote a minimal commutative differential graded algebra modelling a given space X; V^k will denote the degree k elements of the space of indecomposables V, and $(\Lambda V)^k$ the degree k elements in the algebra. Likewise $\Lambda V^{\leq m}$ will denote the subalgebra of ΛV generated by the elements of degree at most m, and $(\Lambda V^{\leq m})^k$ will denote the vector space of degree k elements in this subalgebra. For ease of notation we will denote by H^* the total cohomology $\bigoplus_i H^i(\Lambda V, d)$.

Acknowledgments. The authors would like to acknowledge the support of the Directed Reading Program at Stony Brook University, under which this project was initiated. Several computations were carried out with the Commutative Differential Graded Algebras module for SageMath [Sage] written by Miguel Marco and John Palmieri, for which the authors are duly grateful.

2. Verification in dimension ≤ 20

We now collect some general statements and ad hoc arguments which we will implement into the code found in [NaYa11] in order to reduce the verification of the Hilali conjecture in formal dimension ≤ 20 to several cases, which we will then rule out by hand. Following the notation of [NaYa11], homotopy ranks types will be denoted by $(2a_1, \ldots, 2a_n : 2b_1 - 1, \ldots, 2b_{n+p} - 1)$, where the sequences a_i and b_i are (not necessarily strictly) increasing. Note that $-p$ equals the homotopy Euler characteristic of any space X realizing the given homotopy rank type.

Proposition 2.1. *If $p = 0$, then the Hilali conjecture holds.*

Proof. The vanishing of the homotopy Euler characteristic χ_π implies that the Euler characteristic of any such space is positive. This now implies the space admits a pure minimal model (the existence of a pure model is stated in [FHT Prop. 32.10], and minimality of this model can be seen from the proof therein), and so by [BFMM14 Section 3] the conjecture holds. □

Remark 2.2. In the lemmas to follow we will rely on the existence of elements of V in degree strictly smaller than half the formal dimension. We thus verify now that the Hilali conjecture holds for simply connected spaces X of formal dimension n for which $b_1, \ldots, b_{\lceil \frac{n}{2} \rceil - 1} = 0$. If the formal dimension is odd or if $b_{\frac{n}{2}} = 0$, then by Poincaré duality X is rationally homotopy equivalent to a sphere, for which the conjecture holds. If the formal dimension is even, $n = 2k$, and $\dim V_k = 1$, then X has minimal model $\Lambda(x_k, y_{3k-1})$ with $dx = 0$, $dy = x^3$, and so the conjecture holds. If $\dim V_k = 2$, the space X will admit a minimal model over the complex numbers of the form $\Lambda(x_k, x'_k, y_{2k-1}, y'_{2k-1})$ with $dx = dx' = 0$ and $dy = x^2 - x'^2$, $dy' = xx'$ (tensoring with the complex numbers has the advantage of making the nondegenerate pairing in the middle degree cohomology equivalent to the pairing represented by the identity matrix). We see that $\dim H^*(X; \mathbb{C}) = 4$ and $\dim \pi_*(X) \otimes \mathbb{C} = 4$; since these dimensions are independent of the choice of coefficient field of characteristic zero, the conjecture is verified. In the case of $\dim V_k \geq 3$, one can build the minimal model over the complex numbers (again to simplify the intersection pairing) and see that one must introduce at least two generators in degrees $> n$, showing that this space is not elliptic [FHT p.441] (cf. with the rational hyperbolicity of $\#_{i=1}^k \mathbb{CP}^2$ for $k \geq 3$). Alternatively, any rational Poincaré duality space with $b_1, \ldots, b_{\lceil \frac{n}{2} \rceil - 1} = 0$ is formal by [Mi79] and hence satisfies the Hilali conjecture by [HiMa08a, Theorem 2].

Lemma 2.3. *Let X be a simply connected rationally elliptic space with $p > 0$. Suppose the smallest degree in which $\pi_*(X) \otimes \mathbb{Q}$ is nonzero is strictly less than $\frac{\mathrm{fd}(X)}{2}$, and denote the dimension of this space by k. If $\mathrm{fd}(X)$ is odd, then $\dim H^*(X; \mathbb{Q}) \geq 2k+2$. If $\mathrm{fd}(X)$ is even, and the smallest degree in which $\pi_*(X) \otimes \mathbb{Q}$ is nonzero is odd, then $\dim H^*(X; \mathbb{Q}) \geq 4k$. Otherwise, if the smallest nonzero degree of $\pi_*(X) \otimes \mathbb{Q}$ is even, we have $\dim H^*(X; \mathbb{Q}) \geq 4k + 4$.*

Proof. Note that every element in the smallest nonzero degree of $\pi_*(X) \otimes \mathbb{Q}$ corresponds to a closed, non-exact element in the minimal model of X for degree reasons. The first statement now follows from $\dim H^0(X; \mathbb{Q}) = 1$ and Poincaré duality. If the formal dimension of X is even, and the smallest nonzero degree of $\pi_*(X) \otimes \mathbb{Q}$ is odd, Poincaré duality ensures $2k$ independent cohomology classes of odd degree in X. Since $p \neq 0$, the Euler characteristic of X is zero, providing us with another $2k$ independent cohomology classes, of even degree. If $\mathrm{fd}(X)$ is even and the smallest nonzero degree of $\pi_*(X) \otimes \mathbb{Q}$ is even, then Poincaré duality gives us at least $2k + 2$ independent cohomology classes in even degree, since $\dim H^0(X; \mathbb{Q}) = 1$. The vanishing of the Euler characteristic then provides another $2k + 2$ independent cohomology classes, now of odd degree. □

Lemma 2.4. *Let X be a simply connected rationally elliptic space with $p > 0$. Suppose the smallest degree d in which $\pi_*(X) \otimes \mathbb{Q}$ is nonzero is even, and denote the dimension of this space by k. Suppose further that the second smallest nonzero degree of $\pi_*(X) \otimes \mathbb{Q}$ is $2d-1$, of dimension l, with $2d-1 < \frac{\mathrm{fd}(X)}{2} - 1$. Then is $fd(X)$ is even and $\binom{k+1}{2} \leq l$, then $\dim H^* \geq 4(1+k+l-\binom{k+1}{2})$; if $\binom{k+1}{2} > l$, then $\dim H^* \geq 4(\binom{k+1}{2} - l)$. If $fd(X)$ is odd, then in either case $\dim H^* \geq 2(1 + k + \left|l - \binom{k+1}{2}\right|)$.*

Proof. We note that $(\Lambda V^{\leq d})^{2d}$ has dimension $k + \binom{k}{2}$ (spanned by squares of a basis of generators in degree d and products of two distinct generators). These elements are closed, and the dimension of the image of d in this space is bounded by l. Now the lemma follows by combining this with $\dim H^0(X;\mathbb{Q}) = 1$, Poincaré duality, and $\chi(X) = 0$ as in Lemma 2.4. □

Remark 2.5. In the above Lemma 2.5, if the smallest nonzero degree of odd rational homotopy is strictly less than $2d-1$, then the corresponding elements in the minimal model are closed and non-exact, and so by Poincaré duality we have $\dim H^*(X;\mathbb{Q}) \geq 2(1+k+l)$.

Lemma 2.6. *Let X be a simply connected rationally elliptic space. Suppose the smallest degree d in which $\pi_*(X)\otimes\mathbb{Q}$ is nonzero is even, and denote the dimension of this space by k. Suppose further that the second smallest nonzero degree of $\pi_*(X)\otimes\mathbb{Q}$ is $2d-1$. Denote $l = \dim \pi_{2d-1}(X)\otimes\mathbb{Q}$ and $m = \dim \pi_{3d-2}(X)\otimes\mathbb{Q}$. If $3d-1 < \frac{\mathrm{fd}(X)}{2}$, then*

$$\dim H^*(X;\mathbb{Q}) \geq 2\left(k+1+\left|l-\binom{k+1}{2}\right| + \max\left(0, kl-k^2-\binom{k}{3}-m\right)\right).$$

Proof. Note that $\dim(\Lambda V^{\leq d})^{3d} = k^2 + \binom{k}{3}$. In $(\Lambda V)^{3d-1}$, there is a kl dimensional subspace W spanned by products of degree d generators and degree $2d-1$ generators. The image of d applied to this subspace W lies in $(\Lambda V^{\leq d})^{3d}$. Since W is spanned by quadratic elements, an element in it is exact only if it is in the image of the differential applied to the m–dimensional V^{3d-2}. Hence we have at least $\max(0, kl-k^2-\binom{k}{3}-m)$ independent cohomology classes in degree $3d-1$. Combining this with the degree 0 class, the k-dimensional cohomology we obtain in degree d, and the $\left|l-\binom{k+1}{2}\right|$-dimensional cohomology in degree $2d-1$ or $2d$ as in Lemma 2.5, along with Poincaré duality, we obtain the desired bound. □

Remark 2.7. Note that if the smallest nonzero degree d of $\pi_*(X)\otimes\mathbb{Q}$ is odd, of dimension l, and the smallest nonzero even degree d' of $\pi_*(X)\otimes\mathbb{Q}$ is strictly less than $3d-1$, these two vector spaces must correspond to closed, non-exact elements in the minimal model of X for degree reasons. Indeed, the differential applied to a generator in the smallest even degree would have to land in the subalgebra of odd degree elements, producing a polynomial all of whose monomials are at least cubic and hence of degree at least $3d$. If furthermore we denote $m = \dim \pi_{2d-1}(X)\otimes\mathbb{Q}$, we have an additional $\left|\binom{l}{2}-m\right|$ independent cohomology classes in degree $2d-1$ or $2d$. Indeed, the differential applied to a degree $2d-1$ generator must land in the subspace of quadratic polynomials in the degree d generators for degree reasons, which is of dimension $\binom{l}{2}$. If $2d$ and d' are both strictly less than $\frac{\mathrm{fd}(X)}{2}$, then Poincaré duality gives us $\dim H^*(X;\mathbb{Q}) \geq 2(1+l+\dim\pi_{d'}(X)\otimes\mathbb{Q}+\left|\binom{l}{2}-m\right|)$.

Remark 2.8. Two more quick observations that will rule out several homotopy rank types are the following:

(1) Every even generator whose degree is smaller than than the lowest degree among odd generators is closed and non-exact; likewise all products of such

generators (for our purposes we will only need squares) whose total degree is smaller than the lowest odd degree are closed and non-exact.

(2) A homotopy rank type in fd ≥ 9 of the form $(2, a; 3, b, c)$, where fd $-2 > a \geq 4$ and $fd - 2 > b \geq 5$, satisfies the conjecture. Indeed, let us denote the generators by their degree for simplicity of notation. Note that 2 is closed and non-exact, and so by Poincaré duality, since $\dim H^0 = 1$, we have $\dim H^* \geq 4$. If we find one more independent cohomology class the conjecture is verified. If $a < b$, then $da = \alpha(3 \cdot 2^k)$ for some $\alpha \in \mathbb{Q}$, $k \geq 1$. Now, either 3 is closed and we are done, or $d3 = \beta 2^2$ for some $\beta \neq 0$; however, this would mean da is not closed, which cannot be. If $b < a$, then $db = \alpha 2^k$ for some $\alpha \in \mathbb{Q}$, $k \geq 3$. Either 3 or b is closed and we are done, or b plus a multiple of $3 \cdot 2^{k-2}$ is closed and necessarily non-exact by minimality.

We now list the homotopy rank types remaining upon implementation of the above observations into the code of [NaYa11], and for illustration include the total number of homotopy rank types in a given formal dimension. Recall that we adopt the convention that we list the subsequences of even and odd numbers in ascending order in a given homotopy rank type.

fd $= 3$: number of homotopy rank types $= 1$, all ruled out

fd $= 4$: number of homotopy rank types $= 3$, all ruled out

fd $= 5$: number of homotopy rank types $= 2$, all ruled out

fd $= 6$: number of homotopy rank types $= 6$, all ruled out

fd $= 7$: number of homotopy rank types $= 4$, all ruled out

fd $= 8$: number of homotopy rank types $= 13$, all ruled out

fd $= 9$: number of homotopy rank types $= 9$, all ruled out

fd $= 10$: number of homotopy rank types $= 22$, all ruled out

fd $= 11$: number of homotopy rank types $= 17$, all ruled out

fd $= 12$: number of homotopy rank types $= 45$, all ruled out

fd $= 13$: number of homotopy rank types $= 32$

$p = 1 : (2, 4, 4 : 3, 3, 7, 7)$,

fd $= 14$: number of homotopy rank types $= 73$, all ruled out

fd $= 15$: number of homotopy rank types $= 58$

$p = 1 : (2, 4, 4 : 3, 5, 7, 7), (2, 4, 6 : 3, 3, 7, 11), (2, 2, 4, 4 : 3, 3, 3, 7, 7)$,

fd $= 16$: number of homotopy rank types $= 134$, all ruled out

fd $= 17$: number of homotopy rank types $= 103$

$p = 1 : (2, 4, 4 : 3, 3, 7, 11), (2, 4, 4 : 3, 7, 7, 7), (2, 4, 6 : 3, 3, 9, 11), (2, 4, 6 : 3, 5, 7, 11)$,
$(2, 4, 8 : 3, 3, 7, 15), (2, 2, 4, 4 : 3, 3, 5, 7, 7), (2, 2, 4, 6 : 3, 3, 3, 7, 11), (2, 4, 4, 4 : 3, 3, 7, 7, 7)$,

$p = 3 : (2 : 3, 5, 5, 5)$,

fd $= 18$: number of homotopy rank types $= 217$, all ruled out

fd $= 19$: number of homotopy rank types $= 173$

$p = 1 : (8, 8 : 3, 15, 15), (2, 4, 4 : 3, 5, 7, 11), (2, 4, 4 : 3, 7, 7, 9), (2, 4, 6 : 3, 3, 11, 11), (2, 4, 6 : 3, 5, 9, 11)$,
$(2, 4, 6 : 3, 7, 7, 11), (2, 4, 8 : 3, 5, 7, 15), (2, 4, 10 : 3, 3, 7, 19), (2, 6, 6 : 3, 5, 11, 11)$,
$(4, 6, 6 : 3, 7, 11, 11), (2, 2, 4, 4 : 3, 3, 3, 7, 11), (2, 2, 4, 4 : 3, 3, 7, 7, 7), (2, 2, 4, 6 : 3, 3, 3, 9, 11)$,
$(2, 2, 4, 6 : 3, 3, 5, 7, 11), (2, 2, 4, 8 : 3, 3, 3, 7, 15), (2, 4, 4, 4 : 3, 5, 7, 7, 7), (2, 4, 4, 6 : 3, 3, 7, 7, 11)$,
$(2, 2, 4, 4, 4 : 3, 3, 3, 7, 7, 7)$,

$$p = 3 : (2 : 3, 5, 5, 7), (2, 4 : 3, 3, 5, 5, 7),$$

$$\text{fd} = 20 : \text{ number of homotopy rank types } = 373$$

$$p = 2 : (2, 4, 4, 4 : 3, 3, 3, 7, 7, 7).$$

Theorem 2.9. *The Hilali conjecture holds in formal dimension* ≤ 20*.*

Proof. We now deal with the remaining cases listed above. When counting arguments fail to rule out a given case, we instead detect Massey products to obtain the sought after amount of cohomology.

In formal dimension 13, it only remains to verify the conjecture for spaces with homotopy rank type $(2, 4, 4 : 3, 3, 7, 7)$. Denote the generator in degree 2 by x, and choose a basis $\{y, y'\}$ of V^3. If $dy = dy' = 0$, then we are done as $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3) = 8$. Otherwise, after a change of basis for V^3 we have $dy = x^2$ and $dy' = 0$. Then $\ker d \cap (\Lambda^{\leq 3})^5$ is spanned by xy', and so $\dim \ker d \cap V^4 \geq 1$, giving $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3 + \dim H^4) \geq 8$.

In formal dimension 15, we rule out $(2, 4, 4 : 3, 5, 7, 7)$ by noting that if the generator in degree 3 is closed, we are done as the square of the degree 2 generator is then non-exact and so $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3 + \dim H^4) = 8$. Otherwise, upon rescaling the generator in degree 3 maps to the square of the degree 2 generator under the differential, in which case $\ker d \cap (\Lambda V^{\leq 3}) = \{0\}$, and so $\dim H^4 \geq 2$ and $\dim H^* \geq 8$.

For the case of $(2, 4, 6 : 3, 3, 7, 11)$, note that if d vanishes on V^3, we are done as $\dim H^* \geq 2(1 + \dim H^2 + \dim H^3) = 8$. Otherwise, we may choose bases $\{x\}$ and $\{y, y'\}$ of V^2 and V^3 such that $dy = 0$, $dy' = x^2$. Then either the degree 4 generator z is closed and we are done as $\dim H^3 + \dim H^4 = 2$, or upon scaling z we have $dz = xy$, in which case the Massey product $[yy' + xz]$ spans H^6 and we have $\dim H^3 + \dim H^6 \geq 2$.

The last remaining homotopy rank type in dimension 15 is $(2, 2, 4, 4 : 3, 3, 3, 7, 7)$; let $(\Lambda V, d)$ be a minimal cdga realizing it. The degree 2 generators and degree 0 give us three independent cohomology classes, so by Poincaré duality it remains to find three more independent cohomology classes; again by Poincaré duality, it will suffice to find two more independent classes in degree ≤ 7. If the kernel of $V^3 \xrightarrow{d} (\Lambda V)^4$ is non-trivial, then $\dim H^3 + \dim H^4 \geq 2$ (since $\dim V^3 = \dim(\Lambda V^{\leq 2})^4$) and we are done. If the kernel of $V^3 \xrightarrow{d} (\Lambda V)^4$ is trivial, we can choose a basis $\{y, y', y''\}$ of V^3 such that $dy = x^2$, $dy' = x'^2$, $dy'' = xx'$. Now $\ker d \cap (\Lambda V)^5$ is spanned by $xy' - x'y''$ and $x'y - xy''$. If $V^4 \xrightarrow{d} (\Lambda V)^5$ is not injective, then we are done as $\dim H^4 + \dim H^5 \geq 2$. If this d is injective, we can choose a basis $\{z, z'\}$ of V^4 such that $dz = xy' - x'y''$, $dz' = x'y - xy''$. We then have the Massey products $[y'y'' + x'z]$, $[yy'' + xz']$, $[yy' - xz + x'z']$ forming a basis for $H^6(\Lambda V, d)$.

In formal dimension 17, there are eight cases left to check for $p = 1$. To rule out the homotopy rank type $(2, 4, 4 : 3, 3, 7, 11)$, note that if d vanishes on V^3 we are done; otherwise $\ker d \cap V^3$ is one–dimensional, which implies that $\ker d \cap (\Lambda V^{\leq 5})$ is one–dimensional, yielding $\ker d \cap V^4 \geq 1$ since $\dim V^4 = 1$. In any case, we have $\dim H^* \geq 8$.

Next, $(2,4,4:3,7,7,7,)$ is ruled out by noting that if the generator in degree 3 is closed, we are done as $\dim H^4 \geq 1$. Otherwise the differential sends the degree 3 generator to a nonzero multiple of the square of the degree 2 generator, which implies $\ker d \cap (\Lambda V^{\leq 3})^5 = \{0\}$, giving $\dim H^4 = 2$ and thus $\dim H^* \geq 8$.

The types $(2,4,6:3,3,9,11)$ and $(2,4,8:3,3,7,15)$ are verified in the same way as $(2,4,6:3,3,7,11)$ in dimension 15. Due to the generator in degree 5, $(2,4,6:3,5,7,11)$ requires a slightly more involved argument: label the generator in degree i by x_i. If $dx_3 = 0$, we are done as x_2^2 is non-exact; so suppose that, upon rescaling, we have $dx_3 = x_2^2$. This implies there are no non-zero closed elements in $(\Lambda V^{\leq 3})^5$, and so $dx_4 = 0$. Now, $dx_5 = ax_2x_4 + bx_2^3$ for some $a, b \in \mathbb{Q}$. We see that $\ker d \cap (\Lambda V^{\leq 5})^7$ is spanned by $x_2x_5 - ax_3x_4 - bx_2^2x_3$. Now, either x_6 is closed and we have $\dim H^* \geq 8$, or upon rescaling $dx_6 = x_2x_5 - ax_3x_4 - bx_2^2x_3$. Since $d(x_2x_6) = d(x_3x_5) = x_2^2x_5 - ax_2x_3x_4 - bx_2^3x_3$, we have that $\ker d \cap (\Lambda V)^8$ is spanned by $\{x_2^4, x_2^2x_4, x_4^2, x_2x_6 - x_3x_5\}$. The vector space $(\Lambda V)^7$ is spanned by $\{x_2^2x_3, x_3x_4, x_2x_5, x_7\}$, with the image of the differential on the first three vectors being two dimensional. We conclude that $\dim H^7 + \dim H^8 \geq 1$ and thus $\dim H^* \geq 8$.

For the homotopy rank type $(2,2,4,4:3,3,5,7,7)$, note that if d is not injective on V^3, we are done as $\dim H^2 = 2$ and $\dim H^4 \geq 2$. If d is injective on V^3, then inspection of a matrix for $(\Lambda V^{\leq 3})^5 \xrightarrow{d} (\Lambda V^{\leq 2})^6$ yields $\dim \ker d \cap (\Lambda V^{\leq 3})^5 \leq 1$, and so $\dim \ker d \cap V^4 \geq 1$, giving us $\dim H^4 \geq 2$.

For $(2,2,4,6:3,3,3,7,11)$, note that if d is not injective on V^3, we are done since we have $\dim H^3 \geq 1$ and $\dim H^4 \geq 1$. Suppose therefore that for some bases $\{x, x'\}$, $\{y, y', y''\}$ of V^2 and V^3 respectively we have $dy = x^2$, $dy' = x'^2$, $dy'' = xx'$. Then $\ker d \cap (\Lambda V)^5$ is spanned by $xy' - x'y''$ and $xy'' - x'y$. Now, if the degree 4 generator z is closed, we are done. Otherwise, if $dz = xy'' - x'y$, then $[y'y'' - xz] \neq 0$ gives $\dim H^6 \geq 1$; if $dz = p(xy' - x'y'') + q(xy'' - x'y)$ for some non-zero $p \in \mathbb{Q}$, then $[-\frac{q}{p}yy' - \frac{q^2}{p^2}yy'' + y'y'' + \frac{q}{p^2}xz + \frac{1}{p}x'z] \neq 0$. In any case, $\dim H^* \geq 2(3 + \dim H^3 + \dim H^4 + \dim H^5 + \dim H^6) \geq 10$.

For the homotopy rank type $(2,4,4,4:3,3,7,7,7)$, note that $\dim \ker d \cap V^3 \in \{1,2\}$. Since $\dim V^2 = 1$ we have $\dim \ker d \cap (\Lambda V)^5 = \dim \ker d \cap V^3$, and so $\dim H^4 \geq 3 - \dim \ker d \cap (\Lambda V)^5$, giving $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3 + \dim H^4) \geq 2(2 + \dim \ker d \cap (\Lambda V)^5 + (3 - \dim \ker d \cap (\Lambda V)^5)) = 10$.

For $p = 3$ in formal dimension 17, it only remains to verify $(2:3,5,5,5)$. Since $\dim(\Lambda V)^6 = 1$, there is a two–dimensional space of closed degree 5 indecomposables. Further, every element in degree 4 is closed (since $(\Lambda V)^4$ is spanned by the square of the generator in degree 2), so $\dim H^5 \geq 2$ and we have $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^5) \geq 8$.

In formal dimension 19, there are eighteen cases left to check for $p = 1$. The homotopy rank type $(8,8,3,15,15)$ is ruled out by noting that the degree 8 generators must be closed. The cases of $(2,4,4:3,5,7,11)$ and $(2,4,4,3,7,7,9)$ are ruled out as $(2,4,4,3,7,7,7)$ in dimension 17. Next, $(2,4,6:3,3,11,11)$ is verified as $(2,4,6:3,3,9,11)$ in dimension 17.

The case of $(2,2,4,4 : 3,3,3,7,11)$ is covered in the same way as $(2,2,4,4 : 3,3,3,7,7)$ in formal dimension 15. The homotopy rank type $(2,2,4,4 : 3,3,7,7,7)$ is ruled out in the same way as $(2,2,4,4 : 3,3,5,7,7)$ in formal dimension 17. Further, $(2,2,4,6 : 3,3,3,9,11)$ is ruled out as $(2,2,4,6 : 3,3,3,7,11)$ in formal dimension 17, and $(2,4,6 : 3,5,9,11)$ is ruled out as $(2,4,6 : 3,5,7,11)$.

For $(2,4,6 : 3,7,7,11)$, note that if the degree 3 generator is closed, we have that the square of the degree 2 generator is non-exact. Otherwise, the degree 4 generator must be closed (since the kernel is trivial in degree 5), and its product with the degree 2 generator closed and non-exact. In either case, $\dim H^* \geq 8$.

The homotopy rank type $(2,4,8 : 3,5,7,15)$ is covered similarly to $(2,4,6 : 3,5,7,11)$ (let us denote the generator in a given degree i by x_i as we did in that case): if x_3 is closed, we are done, so suppose $dx_3 = x_2^2$. Then $dx_4 = 0$ and $dx_5 = ax_2x_4 + bx_2^3$ for some $a, b \in \mathbb{Q}$, yielding the closed element $x_2x_5 - ax_3x_4 - bx_2^2x_4$ which is non-exact due to the absence of a generator in degree 6.

We rule out $(2,4,10 : 3,3,7,19)$ as we did $(2,4,6 : 3,3,7,11)$ in dimension 15. For $(2,6,6 : 3,5,11,11)$, label by x, y, u the generators of degree 2,3,5 respectively. Note that y is closed, it and xy provide two independent cohomology classes and we have $\dim H^* \geq 8$. Suppose then that $dy = x^2$. If u is closed we are done as x^3 is non-exact; otherwise we may assume $du = x^3$, in which case $\ker d \cap (\Lambda V^{\leq 5})^7$ is spanned by $x^2y - xu$. It follows from here that $\dim \ker d \cap V^6 \geq 1$, and the product of a non-zero class in this kernel with x must be closed and non-exact since there are no degree 7 generators to make this quadratic element exact; thus $\dim H^* \geq 8$.

Next, $(4,6,6 : 3,7,11,11)$ is verified by noting that the degree 3 and 4 generators must be closed and non-exact, along with at least one non-zero element in V^6.

For the case of $(2,2,4,6 : 3,3,5,7,11)$, note that if d is not injective on V^3, we have $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^4) \geq 10$ as $\dim(\Lambda V^{\leq 2})^4 = 3$. Suppose then that d is injective on V^3; from here it follows that $(\Lambda V^{\leq 3})^5 \xrightarrow{d} (\Lambda V)^6$ has at most 3–dimensional image (note $\dim(\Lambda V^{\leq 3})^5) = 4$) as in the case of $(2,2,4,4 : 3,3,5,7,7)$. Denote by $\{x, x'\}$, $\{y, y'\}$, $\{z\}$ bases of V^2, V^3, V^4 respectively. The injectivity of d on V^3 tells us $dy = ax^2 + bx'^2 + cxx'$ and $dy' = a'x^2 + b'x'^2 + c'xx'$ for independent $(a, b, c), (a', b', c') \in \mathbb{Q}^3$. If the image of $(\Lambda V^{\leq 3})^5 \xrightarrow{d} (\Lambda V)^6$ is 4–dimensional, i.e. the kernel is trivial, the generator in degree 4 must be closed and hence we are done. So suppose the kernel is one–dimensional and that the generator z in degree 4 maps to a non-zero element in this kernel (otherwise it is closed and we are done). We will show that the one–dimensionality of $\ker((\Lambda V^{\leq 3})^5 \xrightarrow{d} (\Lambda V)^6)$ and $dz \neq 0$ implies the existence of a closed element in the span of $\{yy', xz, x'z\}$; combined with the fact that every element in the 4–dimensional space $(\Lambda V^{\leq 2})^6$ is closed, and $\dim \mathrm{im}((\Lambda V)^5 \xrightarrow{d} (\Lambda V)^6) \leq 4$, we will have $\dim H^6 \geq 1$ and hence $\dim H^* \geq 10$. Now, $\dim \ker(\Lambda V^{\leq 3})^5 \xrightarrow{d} (\Lambda V)^6 = 1$ tells us that some non-trivial linear combination $kxy + lxy' + mx'y + nx'y'$ is closed, $k, l, m, n \in \mathbb{Q}$. This yields the equations

$$ka + la' = 0$$
$$kc + lc' + ma + na' = 0$$

$$kb + lb' + mc + nc' = 0$$
$$mb + nb' = 0$$

If $a \neq 0$, we can rearrange our basis for V^3 so that $dy = x^2 + bx'^2 + cxx'$, $dy' = b'x'^2 + c'xx'$. If furthermore $b' \neq 0$, we may take $b' = 1$ and $b = 0$, yielding $d(yy' - xz - cx'z) = 0$. If $a \neq 0$ and $b' = 0$, then upon change of basis for V^3 we have (using the above four equations to conclude $b = 0$) $dy = x^2$, $dy' = xx'$. Then $d(yy' - xz) = 0$. The case of $b \neq 0$ is analogous to the case of $a \neq 0$. Suppose now that $c \neq 0$ and $a, b = 0$; after change of basis we have $dy = xx'$, $dy' = a'x^2 + b'x'^2$. If $b' \neq 0$, upon change of basis we have $dy = xx'$ and $dy' = a'x^2 + x'^2$. Note however that the above four equations yield $n = 0$ and hence $ma' = 0$. Since $m = 0$ implies $k, l, m, n = 0$ (which we are assuming is not the case), we have $a' = 0$, and $d(yy' + x'z) = 0$. If $b' = 0$, we may assume $dy = xx'$, $dy = x^2$, giving $d(yy' + x'z) = 0$.

The homotopy rank type $(2, 2, 4, 8 : 3, 3, 3, 7, 15)$ is covered by the argument for $(2, 2, 4, 6 : 3, 3, 3, 7, 11)$ in formal dimension 17.

For $(2, 4, 4, 4 : 3, 5, 7, 7, 7)$, note that if the generator in degree 3 is closed, we have $\dim H^4 \geq 3$; if it is not closed, then d is injective on $(\Lambda V^{\leq 3})^5$ and so $\dim H^4 = 3$. In either case, we obtain $\dim H^* \geq 10$.

In the case of $(2, 4, 4, 6 : 3, 3, 7, 7, 11)$, note that if d vanishes on V^3 we have $\dim H^3 = 2$ and $\dim H^4 \geq 1$, and so $\dim H^* \geq 10$. Otherwise, denoting by x the generator in degree 2, we can choose a basis $\{y, y'\}$ of V^3 such that $dy = x^2$ and $dy' = 0$. Then $\ker d \cap (\Lambda V^{\leq 3})^5$ is spanned by xy', and so $\dim \ker d \cap V^4 \geq 1$. If d vanishes on V^4, we are done, so choose a basis $\{z, z'\}$ of V^4 such that $dz = xy'$ and $dz' = 0$. Then $[xz'] \neq 0$ gives us $\dim H^* \geq 2(2 + \dim H^3 + \dim H^4 + \dim H^6) \geq 10$.

Now we consider the homotopy rank type $(2, 2, 4, 4, 4 : 3, 3, 3, 7, 7, 7)$. Suppose first that d is injective on V^3. Then, as in the case of $(2, 2, 4, 6 : 3, 3, 3, 7, 11)$ in formal dimension 17, we see that $\ker d \cap (\Lambda V)^5$ is two–dimensional. Since V^4 is three–dimensional, there is a closed degree four element, so $\dim H^4 \geq 1$, and $\dim H^6 \geq 2$ since the product of this closed element with any degree two element is closed and non-exact (because there are no degree five generators to make such a quadratic element exact). Therefore, $\dim H^* \geq 12$. If $V^3 \xrightarrow{d} (\Lambda V)^4$ has trivial or one–dimensional image, then we see $\dim H^* \geq 12$ by considering only $\Lambda V^{\leq 3}$ up to degree 4. Now suppose that the image of $V^3 \xrightarrow{d} (\Lambda V^{\leq 2})^4$ is two–dimensional. We can choose bases $\{x, x'\}$ and $\{y, y', y''\}$ of V^2 and V^3 respectively such that $dy = ax^2 + bx'^2 + cxx'$, $dy' = a'x^2 + b'x'^2 + c'xx'$, $dy'' = 0$, where (a, b, c) and (a', b', c') are linearly independent. This implies the kernel of d on the six–dimensional space $(\Lambda V^{\leq 3})^5$ has dimension two or three. If the dimension is two, then $\dim \ker d \cap V^4 \geq 1$ and so $\dim H^* \geq 12$ since $\dim H^3 = 1$ and $\dim H^4 \geq 2$. If the dimension is three, then either d is not injective on V^4 in which case we are done, or we can choose a degree four generator z so that $dz = xy''$. Then $[y''z]$ is a non-zero class in H^7, and we have $\dim H^* \geq 2(3 + \dim H^3 + \dim H^4 + \dim H^7) \geq 12$.

In formal dimension 19 when $p = 3$, the case $(2 : 3, 5, 5, 7)$ is verified by the same reasoning as for $(2 : 3, 5, 5, 5)$ in formal dimension 17, with the modification that we only have $\dim H^5 \geq 1$. For the case of $(2,4 : 3,3,5,5,7)$, note that the degree two generator and at least one element of degree three contribute to the cohomology; if d vanishes on V^3 then we are done, so assume that for some basis $\{y, y'\}$ of V^3 we have that $dy = x^2$ is the square of the degree two generator x, and $dy' = 0$. Now, if the fourth degree generator z is closed, we are done as $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3 + \dim H^4) \geq 8$. If z is not closed, then $d^2 = 0$ tells us $dz = xy'$. Note that $\ker d \cap (\Lambda V)^6$ is spanned by x^3 and $yy' - xz$, and since $d(xy) = x^3$ we conclude that there is a closed element in $(\Lambda V)^5$ with a non-zero term in V^5 (and so by minimality it is not exact), yielding $\dim H^* \geq 8$.

In formal dimension 20, the only remaining homotopy rank type is $(2,4,4,4 : 3,3,3,7,7,7)$. If d vanishes on V^3, we are done as $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3) = 10$. Otherwise, we can choose a basis $\{y, y', y''\}$ of V^3 such that $dy = x^2$, $dy' = dy'' = 0$, where x denotes a degree two generator. We see now that $\dim \ker d \cap (\Lambda V)^5 = 2$, and so $\dim \ker d \cap V^4 \geq 1$. Therefore $\dim H^* \geq 2(\dim H^0 + \dim H^2 + \dim H^3 + \dim H^4) \geq 10$.

□

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