TOPOLOGY. – *Complex vector bundles with discrete structure group.* A note by Messrs. **Pierre Deligne** and **Dennis Sullivan**, communicated by Mr. Jean-Pierre Serre.

Any complex vector bundle whose structure group can be reduced to a discrete group, and whose base is a compact polyhedron, becomes trivial over some finite cover of the base.

THEOREM. – *Let V be a complex local system of finite dimension n over a compact polyhedron X, and let $\mathcal{V}$ be the corresponding complex vector bundle. There exists a surjective finite cover $\pi : \widetilde{X} \to X$ of X such that the pullback of $\mathcal{V}$ over $\widetilde{X}$ is trivial (as a vector bundle).*

We can suppose X is connected, equipped with basepoint p. Once a basis of V_p is chosen, the local system V corresponds to a homomorphism $\rho : \pi_1(X,p) \to \mathrm{GL}(n,\mathbb{C})$. Since the group $\pi_1(X,p)$ is of finite type, there exists a subring A in $\mathbb{C}$ of finite type over $\mathbb{Z}$ such that ρ factors through $\mathrm{GL}(n,A)$ [take a finite generating family (g_i) of $\pi_1(X,p)$, and generate A by the coefficients of the g_i]. Let m_1 and m_2 be two maximal ideals in A such that the finite fields A/m_1 and A/m_2 have distinct characteristic. We will take $\widetilde{X}$ to be the cover of X corresponding to the subgroup of $\pi_1(X,p)$ consisting of g such that $\rho(g) \equiv 1$ (mod m_1 and mod m_2). Its index divides the order of $\mathrm{GL}(n,A/m_1) \times \mathrm{GL}(n,A/m_2)$. To show that this $\widetilde{X}$ is appropriate is equivalent to proving the

PROPOSITION – *Suppose V is defined by $\rho : \pi_1(X,p) \to \mathrm{GL}(n,A)$, or that A is a subring of finite type over $\mathbb{Z}$ inside $\mathbb{C}$. If there exist two maximal ideals m_1 and m_2 of A with residue fields of distinct characteristic such that ρ is trivial mod m_1 and mod m_2, then $\mathcal{V}$ is trivial.*

Let d and N be two integers, with $\dim(X) \leq d$ and N large enough ($N \geq \frac{d}{2}$). The Grassmannian $G_{n,N}$ of n-planes in $\mathbb{C}^{n+N}$ is a good enough approximation to $\mathrm{BGL}(n,\mathbb{C})$ so that $[X, G_{n,N}] = [X, \mathrm{BGL}(n,\mathbb{C})]$, with $\mathcal{V}$ corresponding to a homotopy class of maps $v : X \to G_{n,N}$. Let X' be the fiber bundle over X whose fiber at $x \in X$ is the ($(2N+1)$-connected, hence d-connected) space of injective linear maps from $\mathcal{V}_x$ into $\mathbb{C}^{n+N}$, and v' the map $\iota \mapsto$ image of ι from X' to $G_{n,N}$. The projection $\pi : X' \to X$ admits a section s which is unique up to homotopy, and $v = v's$.

$$\begin{tikzcd} X' \arrow[r, "v'"] \arrow[d, shift left, "\pi"] & G_{n,N} \\ X \arrow[u, shift left, "s"] \arrow[ur, "v"'] & \end{tikzcd}$$

The following conditions are equivalent: (a) $\mathcal{V}$ is trivial; (b) $v \sim 0$ (i.e. v is homotopic to a constant map); (c) v' is ~ 0 on the d-skeleton of X', i.e. for all $w : K \to X'$ with $\dim K \leq d$ (or just for K being the d-skeleton of some triangulation of X') we have $v'w \sim 0$; (d) the composite map $v'' : X' \xrightarrow{v'} G_{n,N} \to \mathrm{cosq}(G_{n,N})$, where $\mathrm{cosq}_d(G_{n,N})$ is the dth level of the Postnikov tower of $G_{n,N}$ (same π_i for $i \leq d$), is ~ 0. The equivalence of (c) and (d) is a general fact for all maps v'; that of (c) and (d) is a consequence of the fact that for all complexes K of dimension $\leq d$ we have $[K, X'] \xrightarrow{\sim} [K, X]$.

Since $G_{n,N}$ is simply connected, it follows from the Hasse principle for maps ((4), Thm. 3.1) that $v'' \sim 0$ if any only if for all l its l-adic completion $v''_l : X'_{l^\wedge} \to \mathrm{cosq}_d((G_{n,N})_{l^\wedge})$ is null-homotopic. The proposition is a consequence of the

LEMMA – *Suppose V is defined by $\rho : \pi_1(X,p) \to \mathrm{GL}(n,A)$, or that A is a subring of finite type over $\mathbb{Z}$ inside $\mathbb{C}$. If there exists a maximal ideal m of A with residue field of characteristic not equal to the prime number l, such that ρ is trivial mod m, then $v''_l \sim 0$.*

We can assume X is a union of faces of the simplex $\Delta \subset \mathbb{R}^M$ spanned by the basis vectors. For each face σ of Δ, let $\sigma_{\mathbb{R}}$ be the affine subspace spanned by σ, and let $\sigma_{\mathbb{Z}}$ be the affine subvariety of the affine space $A_{\mathbb{Z}}^M$ over $\mathbb{Z}$ whose $\sigma_{\mathbb{R}}$ is the set of real points. Let $X_{\mathbb{Z}}$ be the union of the $\sigma_{\mathbb{Z}}$ for $\sigma \subset X$. It is a scheme over $\mathbb{Z}$; for every ring B we denote by X_B the scheme over B obtained by extension of scalars. The inclusion of X into $X_{\mathbb{Z}}(\mathbb{C}) \subset \mathbb{C}^M$ is a homotopy equivalence, and for $l \neq p$ the cohomology mod l of X [or of $X_{\mathbb{Z}}(\mathbb{C})$] coincides with the mod l (étale) cohomology of $X_{\overline{\mathbb{F}}_p}$. This follows from Mayer-Vietoris by the same fact for affine space (SGA 4, XV 2.2).

The representation ρ gives us a vector bundle $\mathcal{V}_A$ over X_A. The bundle $\mathcal{V}$ is obtained from there by extension of scalars to $\mathbb{C}$, and its reduction mod m is trivial. The definition of X' has a purely algebraic meaning, that of a diagram of schemes over A:

$$X_A \xleftarrow{\pi_A} X'_A \xrightarrow{v'_A} (G_{n,N})_A. \tag{1}$$

Denote with subscript 1 the analogous objects for the unit representation of $\pi_1(X, p)$ in $\mathrm{GL}(n, A)$. The morphisms of schemes v'_A and v'_{1A} becomes isomorphisms upon reduction mod m. If k is an algebraic closure of A/m, and if B is the strict Henselization of A in k, then we have a chain of isomorphisms between morphisms of homotopy type completed in l:

$$v'_l \sim (v'_{\mathbb{C}})_l \sim (v'_B)_l \sim (v'_k)_l \sim (v'_{1k})_l \sim (v'_{1B})_l \sim (v'_{1\mathbb{C}})_l \sim v'_{1l}; \tag{2}$$

whence

$$v''_l \sim v''_{1l} \sim 0.$$

In the chain (2), the outer terms are the completions of morphisms between ordinary spaces ((4), §3), and the others are morphisms between étale homotopy types ((4), §3, (1)).

Remarks. – (a) If N is large, and if $2p \geq d$, the map induced by the Chern classes $\mathrm{cosq}_d(G_{n,N}) \to \mathrm{cosq}_d\big(\prod_1^n K(\mathbb{Z}, 2i)\big)$ is an isomorphism upon localizing at p. From here we can deduce that the hypothesis of the proposition can be replaced with "ρ is trivial mod m, for a maximal ideal m of residual characteristic p such that $H_*(X)$ does not have p-torsion and such that $2p \geq \dim(X)$".

(b) In (a) and in the proposition, we can replace "ρ is trivial mod m" with "mod m, ρ has image in the unipotent radical of a Borel subgroup".

(c)The theorem reinforces the theorem from differential geometry (3) according to which the real Chern classes of a vector bundle with integrable connection are trivial. The proof calls to mind the algebraic proof of this fact given by Grothendieck (2).

(d) The problem was suggested to one of us by (5): the theorem extends the domain of applicability of the computations there (Thm. C.2 and Thm. D.2).

(e) The analogue of the theorem for real vector bundles with discrete structure group in $\mathrm{SL}(n, \mathbb{R})$ is false: the real Euler class can be non-zero, already for $n = 2$ and X a Riemann surface.

PROBLEMS. Does the real analogue of the theorem hold for fiber bundles whose structure group can be reduced to $\mathrm{SL}(n, \mathbb{Z})$ [cf. (6)], or for n odd? Is the real Euler class the only obstruction?

(f) The technique of reducing modulo two maximal ideals, and a problem of Atiyah about vector bundles are described in the introduction of (1). The algebraic classifying map (diagram (1)) and the methods of (5) used above can solve the problem of Atiyah.

(*) A meeting on November 10, 1975.

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