This homework is due on Wednesday, 8/16, in class.

Homework 4

Exercise 1 Determine wheter the sequences below converge or diverge. Explain your reasoning.

(a) $a_n = \frac{n^3}{n^3+1}$.

(b) $a_n = \frac{3^{n+1}}{5^n}$.

(c) $a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}$.

Exercise 2 A sequence is given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. The goal of this exercise is to decide whether this sequence converges or not, and if it does, to compute its limit.

(a) Show that this sequence is increasing.

(b) Show that this sequence is bounded above.

(c) Does the limit $\lim_{n\to\infty} a_n$ exist? Explain your reasoning.

(d) If your answer to part (c) was yes, compute the limit.

Exercise 3 For each of the power series below: determine their radius of convergence; check for convergence at the endpoints of the interval of convergence. (Hint: read examples 4 and 5 on section 8.5 of the textbook).

(a)
$$\sum_{n=1}^{\infty} \sqrt{n} x^n$$

(b)
$$\sum_{n=1}^{\infty} \frac{(2n\,!)}{2^n} x^n$$

(c)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} (x-3)^n$$

(d)
$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$$

(Hint: for part (d), rewrite the expression so that the factor involving x becomes $(x-a)^n$, for some number $a \in \mathbb{R}$.)

Exercise 4 Find power series representations of the following functions nearby the given points. Compute the radius of convergence of these power series. You may use any technique studied in class (operations with known power series or direct application of the Taylor theorem). You do not need to prove that the power series represents the function on its interval of convergence.

(a) The function $f(x) = \frac{1}{x+1}$, at the point $a = 1$.

(b) The function $f(x) = e^x + e^{-x}$, at the point $a = 0$.

(c) The function $f(x) = \ln(1+x^2)$, at the point $a = 1$.

(d) The function $f(x) = x^2 \sin(x)$, at the point $a = 0$.

Exercise 5 This exercise is to show one can use the Taylor theorem to compute the values of certain series. Consider the series

$$\sum_{i=1}^{\infty} \frac{(-1)^n}{n}.$$

In a previous section, we saw that this series was convergent. To compute its value, do the following items.

(a) Compute the Taylor series of the function $f(x) = ln(x)$ at the point $a = 1$. (Do not simply write down the series - I want to see the computations of the coefficients).

(b) Compute the radius of convergence of this Taylor series, and check if 1 is in its interval of convergence.

(c) Use the results obtained in parts (a) and (b) to compute the value of the sum.

Remark: You may assume that the function is represented by its Taylor series in its interval of convergence.

Exercise 6

(a) Find the Maclaurin series for $f(x) = x\sin(x)$. Compute its radius of convergence.

(b) Use Taylor's inequality to estimate how many terms are necessary to approximate $f(x)$ for $0 \leq x \leq \frac{\pi}{3}$ to within 0.0001.

Exercise 7 This exercise introduces a smooth function whose values cannot be computed from its Maclaurin series, even though the series has infinite radius of convergence.

Consider the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, given by $f(x) = 0$ if $x \leq 0$, and $f(x) = e^{-\frac{1}{x^2}}$ if $x > 0$.

(a) Using induction, show that this function has derivatives of all orders.

(b) Compute the Maclaurin series of f, and verify that its radius of convergence is infinite.

(c) Show that the Maclaurin series at x is never equal to $f(x)$, if $x > 0$.