Homework 2

Exercise 1 Using the Fundamental Theorem of Calculus, describe all continuously differentiable functions $f(x)$, defined on the interval $[0,1]$, so that the following properties hold:

- $f(0) = 0$
- $f(1) = 1$
- $f'(x) \geq 1$, for all $x \in (0,1)$.

Explain your answer.

Exercise 2 As part of the fundamental theorem of Calculus, we described in class how to compute the rate of change of functions defined by integrals, i.e.

$$\frac{d}{dx}\left[\int_a^x f(s)ds\right] = f(x).$$

Notice that this was done only in the case where the lower limit of integration is constant, and the upper limit is the same as the variable with respect to which we differentiate (i.e., x). In this exercise we will use this to compute the derivatives functions defined by integrals in ways that are more complicated than the above, such as cases in which both the integration bounds vary, or when one of the integration bounds varies according to a function of x (see examples below).

(a) Consider the function $g(x)$, defined by the integral

$$g(x) = \int_0^{x^2} 2sds.$$

If we wish to compute $\frac{dg}{dx}$, the above method will not work on its own, since the variable of differentiation, x, is not the same as that in the upper limit of integration, x^2. Using a change of variables $y = h(x)$, we can take care of this issue. Choose h appropriately to reduce the integral defining g to one we know how to differentiate. Once this is done, use the chain rule

$$\frac{dg}{dx} = \frac{dg}{dy}\frac{dy}{dx}$$

, to compute the derivative $\frac{dg}{dx}$.

(b) Using the same procedure as described above, compute the following:

$$\frac{d}{dx}\left[\int_0^{\cos(x)} sin(s)ds\right].$$

(c) Finally, we can also vary the bottom limit of integration. In this case, derivatives can be computed using the fact that the integral is additive under domains of integration. Using this idea, compute the following

$$\frac{d}{dx}\left[\int_{x^2}^{x^3} \ln(s)ds\right], \qquad x > 0.$$

Exercise 3 This exercise aims to familiarize you with the substitution method. Compute the indefinite integrals below. Clearly describe the steps in the solutions (as done in class).

(a)

$$\int \sqrt{9-x^2}dx$$

(b)

$$\int \frac{x}{4+x^2}dx$$

(c)

$$\int \frac{x^2+4x}{x^3+6x^2+2}dx$$

(d)

$$\int (\sec x)^3 \tan x dx$$

Exercise 4 Prove that for any continuous function f, defined on the interval $[0,1]$, the following is true:

$$\int_0^{\frac{\pi}{2}} f(\sin(x))dx = \int_0^{\frac{\pi}{2}} f(\cos(x))dx.$$

Exercise 5 Use integration by parts on the following problems. You may need to use integration by parts more than once on the same problem. Clearly describe the steps in the solutions.

(a)

$$\int_1^2 \frac{\ln x}{x^2}dx$$

(b)

$$\int_0^{\pi} t\sin(3t)dt$$

(c)

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$$

(d)

$$\int_0^{\pi} \sin(2x)\sin(x)dx$$

Exercise 6 Use partial fractions to solve the following problems. Clearly describe the steps in the solutions

(a)

$$\int \frac{x-4}{x^2-5x+6}dx$$

(b)

$$\int \frac{x^2+2x-1}{x^3-x}dx$$

(c)

$$\int \frac{2x^2+5}{(x^2+1)(x^2+4)}dx$$

(d)

$$\int \frac{x^2-x+6}{x^3+3x}dx$$

Exercise 7 Use the Trapezoidal Rule, the Midpoint Rule and Simpson's rule to approximate the following integrals, with the specified value of n. You may use a calculator. Approximate your answers to six decimal places. Clearly write the steps involved in the approximation (not only the result).

(a)

$$\int_0^3 \frac{1}{1+x^2+x^4}dx, \quad n=4.$$

(b)

$$\int_0^4 \sqrt{1+\sqrt{x}}dx, \quad n=8.$$

Exercise 8 This exercise is meant for you to verify how effective the approximation methods described in class are. We will compare how many approximation steps are necessary for the error in the approximation of

$$\int_0^1 e^{x^2}\,dx$$

to be accurate within 0.00001. Use the formulas for the error bounds given in section 5.9 of the textbook.

(a) How large should n be if approximating with the Trapezoidal rule?

(b) How large should n be if approximating with the Midpoint rule?

(c) How large should n be if approximating with Simpson's rule?

Exercise 9 In each of the following problems, verify if the integral is improper or not. If it is improper, check if it is convergent or divergent. If it is convergent, compute its value.

(a)

$$\int_0^{2\pi} \frac{\cos(\theta) - \frac{1}{\cos(\theta)}}{\tan^2(\theta)}\,d\theta$$

(b)

$$\int_0^2 \ln|x-1|\,dx$$

(c)

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$$

(d)

$$\int_{-\infty}^\infty \frac{e^x}{e^{2x}+4}\,dx$$

Exercise 10 Sketch the region enclosed by the following curves. Find the intersection points of the curves. Use integrals to compute the area of the region.

(a) $y = 12 - x^2$ and $y = x^2 - 6$

(b) $y = \cos x$, $y = xe^x$ and $x = 0$.

(c) $y = \sin(x)$, $y = \cos(x)$, $x = 0$ and $x = \frac{\pi}{2}$ (Notice that the region enclosed by these 4 curves has two separate parts.)