Homework 1

Exercise 1 This exercise will use a similar idea to that developed in class to compute the integral of $f(x) = x^3$ on the interval $[0, L]$.

(a) Use the equality $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to compute the following sum in terms of "n":

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + (n-1)^3 + (n)^3$$

(b) Sketch a graph of f between 0 and x. In this graph, indicate one of the approximation steps used to compute the integral.

(c) When using "n" rectangles to approximate the area, compute the length of the base of each rectangle and the **right** endpoints of all subintervals.

(d) Use the information obtained in the previous 3 parts to compute

$$\int_0^L x^3 dx$$

Exercise 2 This exercise is intended to compute the area of a circle of radius R, using the ideas developed in the first lecture.

(a) Convince yourself that it is enough to compute the area of a semicircle.

(b) The well-known formula for the area of a triangle involves computing the length of the base and heights of the triangle. Show that this formula can be rephrased as follows: the area of a triangle ΔABC can be computed by

$$\text{Area} = \frac{AB \cdot AC \sin(B\hat{A}C)}{2}.$$

Here, AB and AC are the measures of two adjacent sides, and $B\hat{A}C$ is the angle between them.

(c) The n-th step of the interior approximation can be described as follows. Divide the semicircle into $(n+1)$ arcs of equal length. Join the adjacent endpoints of these arcs by a line segment, and trace the radii from these endpoints to the center of the semicircle. This will create $(n+1)$ triangles inside the semicircle. Convince yourself that all these triangles are congruent. Using the formula on part(b), compute the area of the region formed by the union of these triangles (in terms of "n").

(d) The n-th step of the exterior approximation can be described as follows. Divide the semicircle into $(n+1)$ arcs of equal length. Draw the tangents to the endpoints of each of these arcs, marking their intersections. The region R_n, bounded by all of these tangents and the diameter of the semicircle, is the region used to approximate the area (draw a few cases to familiarize yourself with this procedure). By tracing the line segments from the origin to each of the endpoints of the arcs drawn, and to the intersection points of the tangents, you decompose the region R_n into $(2n+2)$ right triangles. Use this decomposition to compute the area of the region R_n.

(e) Show that the area of the interior (resp. exterior) approximations increases (resp. decreases) as n increases. Also, argue that the area of the interior (resp. exterior) approximations is bounded above (resp. below).

(f) Compute the limits of areas obtained in parts (c) and (d), and show that they are the same.

Exercise 3 Use geometric reasoning to compute the following integral:

$$\int_0^3 \sqrt{9-x^2}dx.$$

Exercise 4 Use geometric reasoning and the properties of integration to compute the following integral:

$$\int_{-\pi}^{\pi} \sin(x).$$

Exercise 5 Use an example to show that the following formula is, in general, false:

$$\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$$

Exercise 6 Using the Fundamental Theorem of Calculus and your knowledge of derivatives acquired in Calculus I, compute the following:

(a)

$$\int_1^4 (e^x + \ln x)dx$$

(b)

$$\int_{-\pi}^{0} sin(2x)dx$$

(c)

$$\int_1^4 x^2 e^{x^3} dx$$

(d)

$$\int_0^{x^2} (s^3 + 1) ds$$

(e)

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1 - x^2}} dx$$

(f)

$$\int_1^2 \frac{2x^2 + 4x + 2}{x^3 + 3x^2 + 3x + 1} dx$$