The standard filtration on cohomology with compact supports
with an appendix on the base change map and the Lefschetz hyperplane theorem

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Dedicated to Andrew J. Sommese on his 60th birthday, with admiration and respect.

Abstract. We describe the standard and Leray filtrations on the cohomology groups with compact supports of a quasi projective variety with coefficients in a constructible complex using flags of hyperplane sections on a partial compactification of a related variety. One of the key ingredients of the proof is the Lefschetz hyperplane theorem for perverse sheaves and, in an appendix, we discuss the base change maps for constructible sheaves on algebraic varieties and their role in a proof, due to Beilinson, of the Lefschetz hyperplane theorem.

Contents

1. Introduction 1
2. The geometry of the standard and Leray filtrations 4
3. Appendix: Base change and Lefschetz hyperplane theorem 10
References 22

1. Introduction

Let \( f : X \to Y \) be a map of algebraic varieties. The Leray filtration on the (hyper)cohomology groups \( H(X, \mathbb{Z}) = H(Y, Rf_* \mathbb{Z}_X) \) is defined to be the standard filtration on \( H(Y, Rf_* \mathbb{Z}_X) \), i.e. the one given by the images in cohomology of the truncation maps \( \tau_{\leq i} Rf_* \mathbb{Z} \to Rf_* \mathbb{Z} \). Similarly, for the cohomology groups with compact supports \( H_c(X, \mathbb{Z}) = H_c(Y, Rf_* \mathbb{Z}_X) \).

D. Arapura’s paper [1] contains a geometric description of the Leray filtration on the cohomology groups \( H(X, \mathbb{Z}) \) for a proper map of quasi projective varieties \( f : X \to Y \). For example, if \( Y \) is affine, then the Leray filtration is given, up to a suitable re-numbering, by the kernels of the restriction maps \( H(X, \mathbb{Z}) \to H(X_i, \mathbb{Z}) \).
to a suitable collection of subvarieties \( X_i \subseteq X \). This description implies at once that
the Leray filtration, in fact the whole Leray spectral sequence, is in the category of
mixed Hodge structures.

The same proof works if we replace the sheaf \( \mathbb{Z}_X \) with any bounded complex
\( C \) of sheaves of abelian groups on \( X \) with constructible cohomology sheaves. Such
complexes are simply called constructible.

Since the key constructions take place on \( Y \), given a constructible complex
\( K \) on \( Y \), one obtains an analogous geometric description for the standard filtration
on \( H(Y, K) \). For example, if \( Y \) is affine, then there is a collection of subvarieties
\( Y_i \subseteq Y \), obtained as complete intersections of suitably high degree hypersurfaces
in special position, such that the standard filtration is given by the kernels of the
restriction maps \( H(Y, K) \to H(Y_i, K_{|Y_i}) \).

The case of the Leray filtration for a proper map mentioned above is then the
special case \( K = Rf_* C \), and the varieties \( X_i = f^{-1}(Y_i) \). The properness of the map
is used to ensure, via the proper base change theorem, that the natural base change
maps are isomorphisms, so that, in view of the fact that \( H(X, C) = H(Y, Rf_* C) \),
we can identify the two maps

\[
H(X, C) \to H(X_i, C_{|X_i}), \quad H(Y, Rf_* C) \to H(Y_i, Rf_* C_{|Y_i}),
\]

and hence their kernels.

We do not know of an analogous description of the Leray filtration on the
cohomology groups \( H(X, C) \) for non proper maps \( f : X \to Y \).

In [1], D. Arapura also gives a geometric description of the Leray filtration
on the cohomology groups with compact supports \( H_c(X, C) \) for a proper map \( f : X \to Y \) of quasi projective varieties by first “embedding” the given morphism into
a morphism \( \overline{f} : \overline{X} \to \overline{Y} \) of projective varieties, by identifying cohomology groups
with compact supports on \( Y \) with cohomology groups on \( \overline{Y} \), and then by applying
his aforementioned result for cohomology groups and proper maps. In his approach,
it is important that \( f \) is proper, and the identity \( f_! = f_* \) is used in an essential way.

The purpose of this paper is to prove that, given a quasi projective variety \( Y \)
and a constructible complex \( K \) on \( Y \) and, given a (not necessarily proper) map
\( f : X \to Y \) of algebraic varieties and a constructible complex \( C \) on \( X \), one obtains a geometric description of the standard filtration on the cohomology groups
with compact supports \( H_c(Y, K) \) (Theorem 2.8), and of the Leray filtration on the cohomology groups with compact supports \( H_c(X, C) \) (Theorem 2.9).

The proof still relies on the geometric description of the cohomology groups
\( H(X, C) \) for proper maps \( f : X \to Y \). In fact, we utilize a completion \( \overline{f} : \overline{X} \to \overline{Y} \)
of the varieties and of the map; see diagram (2.7).

For completeness, we include a new proof of the main result of [1], i.e. of the
geometric description of the Leray filtration on the cohomology groups \( H(X, C) \)
for proper maps \( f : X \to Y \); see Corollary 2.3. In fact, we point out that we can
extend the result to cover the case of the standard filtration on the cohomology groups \( H(Y, K) \); see Theorem 2.2. Theorem 2.2 implies Corollary 2.3.

The proof of Theorem 2.2 is based on the techniques introduced in [9], which
deals with perverse filtrations. In the perverse case there is no formal difference
in the treatment of cohomology and of cohomology with compact supports. This
contrasts sharply with the standard case.
Even though the methods in this paper and in [9, 7] are quite different from the ones in [1], the idea of describing filtrations geometrically by using hyperplane sections comes from [1].

In either approach, the Lefschetz hyperplane Theorem 3.14 for constructible sheaves on varieties with arbitrary singularities plays a central role. This result is due to several authors, Beilinson [3], Deligne (unpublished) and Goresky and MacPherson [14]. Beilinson’s proof works in the étale context and is a beautiful application of the generic base change theorem.

In the Appendix §3, I discuss the base change maps for constructible sheaves on algebraic varieties and the role played by them in Beilinson’s proof of the Lefschetz hyperplane theorem. This is merely an attempt to make these techniques more accessible to non-experts and hopefully justifies the length of this section and the fact that it contains facts well-known to experts. For a more complete panoramic of these techniques, see the textbook [19].

The notation employed in this paper is explained in some detail, especially for non-experts, in §3.1. Here is a summary. A variety is a separated scheme of finite type over $\mathbb{C}$. We employ the classical Euclidean topology. We work with bounded complex of sheaves of abelian groups on $Y$ with constructible cohomology sheaves (constructible complexes, for short) and denote the corresponding derived-type category by $D_Y$. The results hold, with essentially the same proofs, in the context of étale cohomology for varieties over algebraically closed fields; we do not discuss this variant. For $K \in D_Y$, we have the (hyper)cohomology groups $H(Y, K)$ and $H_c(Y, K)$, the truncated complexes $\tau_{\leq i} K$ and the cohomology sheaves $H^i(K)$ which fit into the exact sequences (or distinguished triangles)

$$
0 \longrightarrow \tau_{\leq i-1} K \longrightarrow \tau_{\leq i} K \longrightarrow H^i(K)[-i] \longrightarrow 0.
$$

Filtrations on abelian groups, complexes, etc., are taken to be decreasing, $F^i K \supseteq F^{i+1} K$. The quotients (graded pieces) are denoted $\text{Gr}_p^i K := F^i K / F^{i+1} K$.

The standard (or Grothendieck) filtration on $K$ is defined by setting $\tau^p K := \tau_{\leq -p} K$. The graded complexes satisfy $\text{Gr}_p^i K = H^{-p}(K)[p]$. The corresponding decreasing and finite filtration $\tau$ on the cohomology groups $H(Y, K)$ and $H_c(Y, K)$ are called the standard (or Grothendieck) filtrations. Given a map $f : X \to Y$ and a complex $C \in D_X$, the derived image complex $Rf_* C$ and the derived image with proper supports complex $Rf! C$ are in $D_Y$ and the standard filtrations on $H(Y, Rf_* C) = H(X, C)$ and $H_c(Y, Rf! C) = H_c(X, C)$ are called the Leray filtrations.

A word of caution. A key fact used in [9] in the case of the perverse filtration is that exceptional restriction functors $i_p^!$ to general linear sections $i_p : Y_p \to Y$ preserve perversity (up to a shift). This fails in the case of the standard filtration where we must work with hypersurfaces in special position. In particular, $i_p^!$ of a sheaf is not a sheaf, even after a suitable shift, and this prohibits the extension of our inductive approach in Theorem 2.2 and in Corollary 2.3 from cohomology to cohomology with compact supports. As a consequence, the statements we prove in cohomology for the standard and and for the Leray filtrations do not have a direct counterpart in cohomology with compact supports, by, say, a reversal of the arrows. The remedy to this offered in this paper passe through completions of varieties and maps.
All the results of this paper are stated in terms of filtrations on cohomology groups and on cohomology groups with compact supports, but hold more generally, and with the same proofs, for the associated filtered complexes and spectral sequences. However, for simplicity of exposition, we only state and prove these results for filtrations.

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2. The geometry of the standard and Leray filtrations

In this section we give a geometric description of standard and Leray filtrations on cohomology and on cohomology with compact supports in terms of flags of subvarieties.

2.1. Adapting [9] to the standard filtration in cohomology: affine base. While the paper [9] is concerned with the perverse filtration, its formal set-up is quite general and is easily adapted to the case of the standard filtration. In this section, we briefly go through the main steps of this adaptation and prove the key Theorem 2.2 and its Corollary 2.3. We refer the reader to [9] for more details and for the proofs of the various statements we discuss and/or list without proof.

The shifted filtration $Dec(F)$ associated with a filtered complex of abelian groups $(L, F)$ is the filtration on $L$ defined as follows:

$$Dec(F)^aL^i := \{ x \in F^{n+i}L^i \mid dx \in F^{n+i+1}L^{i+1} \}.$$ 

The resulting filtrations in cohomology satisfy

$$Dec(F)^aH^i(L) = F^{n+i}H^i(L).$$

PROPOSITION 2.1. Let $(L, P, F)$ be a bifiltered complex of abelian groups. Assume that

$$H^r(Gr^b_P Gr^a_L L) = 0 \quad \forall r \neq a - b. \tag{2.1}$$

Then $L = Dec(F)$ on $H(L)$.

Let $K \in D_Y$ be a constructible complex on a variety $Y$. By replacing $K$ with a suitable injective resolution, we may assume that $K$ is endowed with a filtration $\tau$ such that the complex $Gr^b_P K[\tau b]$ is an injective resolution of the sheaf $H^{-b}(K)$. We take global sections and obtain the filtered complex $(R\Gamma(Y, K), \tau)$ for which we have $Gr^b_P R\Gamma(Y, K) = R\Gamma(Y, H^{-b}(K)[\tau b])$. The filtration $\tau$ on $R\Gamma(Y, K)$ induces the standard filtration (denoted again by $\tau$) on the cohomology groups $H^*(Y, K)$.

An $n$-flag on $Y$ is an increasing sequence of closed subvarieties

$$Y_0 : \quad 0 = Y_{-1} \subseteq Y_0 \subseteq \ldots \subseteq Y_n = Y.$$ 

The flag $Y_\ast$ induces a filtration $F_{Y_\ast}$ on $K$ as follows: (recall that $j_l = Rj_l$ and $k_l = Rk_l$ are extension by zero) set $j_a : Y \setminus Y_{a-1} \to Y$ and define

$$F_{Y_\ast}^a K := j_{a,l} j_{l}^* K = K_{Y_a - Y_{a-1}}.$$ 

Setting $k_a : Y_a \setminus Y_{a-1} \to Y$, we have $Gr^a_P K = k_{a!}k^*_a K = K_{Y_a - Y_{a-1}}$. The corresponding filtration in cohomology is

$$F_{Y_\ast}^a H^r(Y, K) = Ker \{ H^r(Y, K) \to H^r(Y_{a-1}, K_{|Y_{a-1}}) \}.$$ 

Taking global sections, we get the filtered complex $(R\Gamma(Y, K), F_{Y_\ast})$ with the property that

$$Gr^b_P R\Gamma(Y, K) = R\Gamma(Y, K_{Y_a - Y_{a-1}}) = R\Gamma(Y_a, (K_{|Y_a})_{Y_a - Y_{a-1}}). \tag{2.2}$$
We have
\[ \text{Gr}^b_r \text{Gr}^b_{r+1} \Gamma(Y, K) = \Gamma(Y, H^{-b}(K)[b]_{Y-1}), \]
so that
\[ H^r(\text{Gr}^b_r \text{Gr}^b_{r+1} \Gamma(Y, K)) = H^{r+b}(Y, (H^{-b}(K)[b]_{Y-1})_{Y-1}). \]  
(2.3)

Note that the left-hand-side is the relative cohomology group
\[ H^{r+b}(Y, a_{Y-1}, H^{-b}(K)[b]_{Y-1}) = H^{r+b}(Y, a_{Y-1}, j_0^* H^{-b}(K)[b]_{Y-1}), \]  
(2.4)
where \( j_a : Y_a \to Y_a \). This is important in what follows as it points to the use we now make of the Lefschetz hyperplane theorem for Sheaves 3.14.

**Theorem 2.2.** Let \( Y \) be an affine variety of dimension \( n \) and \( K \in \mathcal{D}_Y \) be a constructible complex on \( Y \). There is an \( n \)-flag \( Y \) on \( Y \) such that
\[ \tau = \text{Dec}(F_{Y^*}) \quad \text{on} \quad H(Y, K). \]

**Proof.** The goal is to choose the flag \( Y \) so that (2.1) holds when \( (L, P, F) := (\Gamma(Y, K), \tau, F_{Y^*}) \). In view of (2.4), we need the flag to satisfy the condition
\[ H^r(Y, a_{Y-1}, (H^{-b}(K)[b]_{Y-1})_{Y-1}) = 0 \quad \forall r \neq a, \forall a \in [0, n], \forall \beta. \]  
(2.5)

Note that Theorem 3.14 applies to any finite collection of sheaves (in fact it applies to any collection of sheaves which are constructible with respect to a fixed stratification). The flag is constructed by descending induction on the dimension of \( Y \). By definition, \( Y_n = Y \). It is sufficient to choose \( Y_{n-1} \) as in Theorem 3.14. We repeat this process, replacing \( Y_n \) with \( Y_{n-1} \) and construct the wanted flag inductively. \( \Box \)

Let \( f : X \to Y \) be a map of algebraic varieties with \( Y \) affine and \( C \in \mathcal{D}_X \). The Leray filtration \( L^f \) on \( H(X, C) = H(Y, Rf_* C) \) is, by definition, the standard filtration \( \tau \) on \( H(Y, Rf_* C) \). Theorem 2.2 yields an \( n \)-flag \( Y \) on \( Y \) such that \( L^f = \text{Dec}(F_{Y^*}) \). In the applications though, it is more useful to have a description in terms of a flag on \( X \). Let \( X_\alpha := f^{-1} Y_a \) be the pull-back flag on \( X \), i.e. \( X_\alpha := f^{-1} Y_a \).

There is the commutative diagram
\[ \begin{array}{ccc} H(X, \pi^* C) & \longrightarrow & H(Y, Rf_* \pi^* C) \\ \downarrow r & & \downarrow r' \\ H(X_\alpha, i_\alpha^* \pi^* C) & = & H(Y_a, Rf_* i_\alpha^* \pi^* C) \end{array} \]
where \( b \) stems from the base change map (3.4) \( i_\alpha^* Rf_* C \to Rf_* i_\alpha^* C \). The kernels of the vertical restriction maps \( r \) and \( r' \) define the filtrations \( F_{X_\alpha} \) and \( F_{Y_\alpha} \). It is clear that \( \text{Ker} \ r \supseteq \text{Ker} \ r' \), i.e. that \( F_{X_\alpha} \supseteq F_{Y_\alpha} \), and that equality holds if the base change map is an isomorphism.

The following is now immediate.

**Corollary 2.3.** Let \( f : X \to Y \) be a proper map with \( Y \) affine of dimension \( n \) and \( C \in \mathcal{D}_X \). There is an \( n \)-flag \( X_\alpha \) on \( X \) such that
\[ L^f = \text{Dec}(F_{X_\alpha}) \quad \text{on} \quad H(X, C). \]

**Proof.** Since \( f \) is proper, the base change map \( i_\alpha^* Rf_* C \to Rf_* i_\alpha^* C \) is an isomorphism. \( \Box \)

In this section we have proved results for when \( Y \) is affine. In this case the statements and proofs are more transparent and the flags are on \( Y \) (pulled-back from \( Y \) in the Leray case). The case when \( Y \) quasi-projective case is easily reduced to the affine case in the next section.
2.2. Standard and Leray filtrations in cohomology: quasi projective base. In this section we extend the results of the previous section from the case when \( Y \) is affine, to the case when \( Y \) is quasi projective. The only difference is that, given a quasi projective variety \( Y \), we need to work with an auxiliary affine variety \( Y' \) which is a fiber bundle, \( \pi : Y' \to Y \), over \( Y \) with fibers affine spaces \( \mathbb{A}^d \) and we need the flag to be an \((n+d)\)-flag \( \mathcal{Y} \) on \( Y' \). This construction is due to Jouanolou.

Here is one way to prove this. In the case \( Y = \mathbb{P}^n \), take \( Y' := (\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta \) with \( \pi \) either projection. In general, take a projective completion \( Y' \) of \( Y \). Blow up the boundary \( Y' \setminus Y \) and obtain a projective completion \( \overline{Y} \) of \( Y \) such that \( Y \to \overline{Y} \) is affine. Embed \( \overline{Y} \) in some \( \mathbb{P}^N \). Take the restriction of the bundle projection \( (\mathbb{P}^N \times \mathbb{P}^N) \setminus \Delta \to \mathbb{P}^N \) over \( Y \) to obtain the desired result.

Let \( Y \) be a quasi projective variety. We fix a “Jouanolou fibration” \( \pi : Y' \to Y \) as above. If \( Y \) is affine, then \( \mathcal{Y} = Y \) and the flag is an \( n \)-flag on \( Y \).

In order to generalize Corollary 2.3 about the Leray filtration, we form the Cartesian diagram (where maps of the “same” type are denoted with the same symbol)

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\downarrow \pi & & \downarrow \pi \\
X & \xrightarrow{f} & Y 
\end{array}
\]

and there are the identifications (the last one stems from Base change for Smooth Maps)

\[H(\mathcal{X}, \pi^* C) = H(X, C) = H(Y, Rf_* C) = H(\mathcal{Y}, \pi^* Rf_* C) = H(\mathcal{Y}, Rf_* \pi^* C)\]

and we have the identity of the corresponding filtrations

\[L^f \colon \mathcal{X} \to \mathcal{Y} = L^f \colon \mathcal{X} \to \mathcal{Y} = \tau \mathcal{Y} = \tau \mathcal{Y} = \tau \mathcal{Y}.
\]

Theorem 2.5. Let \( f : X \to Y \) be a proper map of algebraic varieties, let \( Y \) be quasi projective and \( C \in D_X \). There is an \((n+d)\)-flag \( \mathcal{X} \) on \( X \) such that

\[ (H(X, C), L^f) = (H(\mathcal{X}, \pi^* C), Dec(F_{\mathcal{X}^*})) \]

i.e.

\[ L^p H^\tau(X, C) = \text{Ker} \{ \pi_{p+r-1}^* : H^\tau(X, C) \to H^\tau(X_{p+r-1}, \pi_{p+r-1}^* C) \}. \]
Proof. We have $L^f : X \to Y = L^f : X \to Y$ so that we may replace $f : X \to Y$ with $f : X' \to Y$. Since $Y$ is affine, we can apply Corollary 2.3 and conclude.

Remark 2.6. If the map $f$ is not proper, then the relevant base change map is not an isomorphism. While it is possible to describe the Leray filtration using a flag on $Y$ (on $Y$ if $Y$ is affine), we do not know how to describe it using a flag on $X$ (on $X$ if $Y$ is affine). This latter description would be more desirable in view, for example, of the following Hodge-theoretic application due to Arapura [1]. We also do not know how to do so using compactifications; see Remark 2.12.

Corollary 2.7. Let $f : X \to Y$ be a proper morphism with $Y$ quasi projective. Then the Leray filtration on $H_*(X, \mathbb{Z})$ is by mixed Hodge substructures.

Proof. The Leray filtration satisfies $L^f = Dec(F_Y)$ for some flag on the auxiliary space $Y$. Since the base change maps are isomorphisms, $F_Y = F_X$. By the usual functoriality property of the canonical mixed Hodge structures on varieties, the latter filtration is given by mixed Hodge substructures of $H(X, \mathbb{Z}) = H(X, \mathbb{Z})$.

2.3. Standard and Leray filtrations in cohomology with compact supports. Since sheaves do not behave well with respect to Verdier Duality, it is not possible to dualize the results in cohomology to obtain analogous ones for cohomology with compact supports.

Given a map of varieties $f : X \to Y$ and $C \in \mathcal{D}_X$, the Leray filtration $L^f : X \to Y$ on $H_c(X, C) = H_c(Y, Rf_! C)$ is defined to be the standard filtration on the last group.

In this section we give a geometric description of the standard and of the Leray filtrations on cohomology with compact supports. The description of the Leray filtration on the cohomology groups with compact supports $H_c(X, C)$ is valid for any (not necessarily proper) map, and this is in contrast with the case of the cohomology groups $H(X, C)$ (see Remark 2.6).

Arapura’s [1] proves an analogous result for proper maps, but to our knowledge, that method does not extend to non proper maps. Nevertheless, the method presented here is close in spirit to Arapura’s.

The method consists of passing to completion of varieties and maps and then use the base change properties associated with these compactifications to reduce to the case of cohomology and proper maps. One main difference with cohomology is that, even if we start with $Y$ affine, the flag is on the auxiliary space $Y$ (see below).

2.3.1. Completions of varieties and maps. We use freely the fact, due to Nagata, that varieties and maps can be compactified, i.e. any variety $Y$ admits an open immersion into a complete variety with Zariski dense image, and given any map $f : X \to Y$, there are a proper map $f' : X' \to Y$ and an open immersion $X \to X'$ with Zariski-dense image such that $f'|_X = f$. For a modern account of this fundamental result, and for further references, see [6].

Since our result are valid without the quasi projectivity assumption on $X$, we invoke Nagata’s deep result. If $X$ and $Y$ are both quasi projective, then it is easy to get by taking projective completions of $X$ and $Y$ and by resolving the indeterminacies.

Let $f : X \to Y$ be a map with $Y$ quasi projective. Choose a projective completion $j : Y \to \overline{Y}$ such that $j$ is an affine open embedding. This can be achieved by first taking any projective completion and then by blowing up the boundary.
Choose an \(A^d\)-fibration \(\pi : \mathcal{Y} \to \mathcal{Y}\). Choose a completion \(j : X \to \overline{X}\) such that \(f\) extends to a (necessarily) proper \(\overline{f} : \overline{X} \to \overline{Y}\). Choose closed embeddings \(i : \mathcal{Y}_a \to \mathcal{Y}\), e.g. the constituents of a flag \(\mathcal{Y}_a\) on \(\mathcal{Y}\).

There is the following commutative diagram, where the squares and parallelograms labelled \(\textcircled{1}, \ldots, \textcircled{8}\) are Cartesian:

If \(f : X \to Y\) is not proper, then the commutative trapezoids are not Cartesian.

Note that since \(j : Y \to \overline{Y}\) and \(\mathcal{Y}\) are affine, the map \(\pi : \mathcal{Y} \to Y\) is an \(A^d\)-bundle with affine total space \(\mathcal{Y}\).

We shall use freely the facts that follow.

(1) Due to the smoothness of the maps \(\pi\) and the properness of the maps \(\overline{f}\), the base change Theorem holds for \(\textcircled{1}, \textcircled{3}, \textcircled{4}, \textcircled{6}\) and \(\textcircled{7}\).
(2) For the remaining Cartesian squares (2), (3), and (5) we have the base change maps:
\[ i^* R j_* \longrightarrow R j_* i^*, \quad i^* R f_* \longrightarrow R f_* i^*. \] (2.8)

(3) The exactness of \( j_! \) implies that \( R^0 j_! = j_! \) is simply extension by zero
and that it commutes with ordinary truncation, i.e. \( j_! \circ \tau_{\leq i} = \tau_{\leq i} \circ j_! \); similarly,
for the formation of cohomology sheaves. The compactness of \( \overline{Y} \) and \( \overline{X} \) implies that,
\( H(\overline{Y}, -) = H_c(\overline{Y}, -) \), etc. Recall that \( H_c(Y, -) = H_c(\overline{Y}, j_!(-)) \). It follows
that we have, for every \( K \in D_Y \), canonical identifications of filtered groups
\[ (H_c(Y, K), \tau_Y) = (H_c(\overline{Y}, j_! K), \tau_{\overline{Y}}) = (H(\overline{Y}, j_! K), \tau_{\overline{Y}}). \] (2.9)

If \( K \in D_{\overline{Y}} \) is any extension of \( K \in D_Y \) to \( \overline{Y} \), e.g. \( j_! K, j_* K \) etc., then we also have
\[ (H_c(Y, K), \tau_Y) = (H(\overline{Y}, j_! j^! K), \tau_{\overline{Y}}). \] (2.10)

Similarly, for the other open immersions \( j \) in diagram (2.7). In view of
the definition of relative cohomology as the hypercohomology of \( j_! j^! (-) \),
we also have \( H(\overline{Y}, j_! j^! K) = (H(\overline{Y}, \overline{Y} \setminus \overline{K}) \).

(4) There are canonical identifications:
\[ (H(Y, K), \tau_Y) = (H(\mathcal{Y}, \pi^* K), \tau_\mathcal{Y}). \] (2.11)

Similarly, for the other maps \( \pi \).

(5) There are canonical identifications:
\[ (H(X, C), L^j_Y) = (H(Y, f_* C), \tau_Y) = \] (2.12)
\[ = (H(\mathcal{Y}, \pi^* f_* C = f_* \pi^* C), \tau_\mathcal{Y}) = (H(X, \pi^* C), L^j_Y). \]

Similarly, for the \( \overline{Y} \) in (4).

(6) Recall that for a filtration \( F \), the filtration \( F(l)^{\overline{T}} \) is defined by setting
\( F(l)^{\overline{T}} := F^{l+i} \). Since \( \pi^* = \pi^! [-2d] \) is an exact functor, we have canonical
identifications of filtered groups
\[ (H_c(X, C), L^j_Y) = (H_c(Y, f_* C), \tau_Y) = (H_c(\mathcal{Y}, \pi^! f_* C = f_! \pi^* C), \tau_\mathcal{Y} (2d)). \] (2.13)

Similarly, for the other maps of type \( f \) and \( \overline{F} \).

2.3.2. Filtrations on \( H_c \) via compactifications. Let \( Y \) be a quasi projective variety of dimension \( n \) and \( K \in D_Y \) be a constructible complex on \( Y \). Consider any
diagram as in (2.7). (4) We can choose \( \overline{Y} \) to be of dimension \( n \).

**Theorem 2.8.** There is an \((n + d)\)-flag \( \overline{\mathcal{Y}}_* \) on \( \overline{Y} \) for which we have the following
identity of filtered groups
\[ (H_c(Y, K), \tau_Y) = H(\overline{Y}, \pi^* j_! K), Dec(F_{\overline{\mathcal{Y}}_*})). \]

**Proof.** In view of (2.9) and of (2.11) applied to \( \pi: \overline{\mathcal{Y}} \to \overline{Y} \), we have canonical
identifications of filtered groups
\[ (H_c(Y, K), \tau_Y) = (H(\overline{\mathcal{Y}}, j_! K), \tau_{\overline{\mathcal{Y}}}) = (H(\overline{Y}, \pi^* j_! K) = H(\overline{Y}, j_! \pi^* K), \tau_{\overline{Y}}). \] (2.14)
The conclusion follows from Theorem 2.4 applied to the pair \( (\overline{Y}, j_! K) \). \( \Box \)
Let $f : X \to Y$ be a map of algebraic varieties and $C \in D_X$. Consider any diagram as in (2.7). We can choose $Y$ to be of dimension $n$. There are natural identifications of groups

$$H_c(X, C) = H_c(X, j_! C) = H(X, j_C) = H(X, \pi^* j_C = j_! \pi^*(C)).$$

(2.15)

We endow these groups with the respective Leray filtrations. Note that since $\mathcal{F}$ is proper, the Leray filtration on $H_c(X, j_C)$ coincides with the ones for $H(X, j_C)$.

**Theorem 2.9.** There is an $(n+d)$-flag $\mathcal{X}_*$ on $\mathcal{X}$ for which we have the following identity of filtered groups

$$L^i_{\mathcal{F}}: X \to Y = L^j_{\mathcal{F}}: \mathcal{X} \to Y = Dec(F_{\mathcal{X}_*}) = Dec(F_{\mathcal{X}_*}), \quad \text{on } H_c(X, C).$$

(2.16)

**Proof.** The filtration $L^i_{\mathcal{F}}: X \to Y$ is the standard filtration on $H_c(Y, Rf_* C)$ which in turn, by the exactness of $j_!$ and the equality $H_c(Y, -) = H_c(\mathcal{Y}, j_!(-))$, coincides with the standard filtration on $H_c(\mathcal{Y}, j_! Rf_* C)$. Since $\mathcal{Y}$ is compact and $\mathcal{F}$ is proper (so that $Rf_! = Rf_*), by the commutativity of the base trapezoid diagram in (2.7), we have that $Rf_! j_! = j_* Rf_!$ so that $H_c(\mathcal{Y}, j_! Rf_* C) = H(\mathcal{Y}, Rf_* j_C)$. This implies the equality $L^i_{\mathcal{F}}: X \to Y = L^j_{\mathcal{F}}: \mathcal{X} \to Y$. We are now in the realm of cohomology and proper maps and the rest follows from Theorem 2.5 applied to $\mathcal{F}$ and to $j_C$.

**Corollary 2.10.** The Leray filtration on $H_c(X, \mathbb{Z})$ is by mixed Hodge substructures.

**Proof.** By Theorem 2.9 and (2.10), the filtration in question is the one induced by the flag $\mathcal{X}_*$ on the relative cohomology group $H(\mathcal{X}, j_! \mathbb{Z}) = H(\mathcal{X}, \mathcal{X}, \mathbb{Z})$. The result follows from Deligne’s mixed Hodge Theory [12].

**Remark 2.11.** The case when $f$ is proper is proved in [1].

**Remark 2.12.** If one tries to imitate the procedure we have followed in the case of the cohomology groups with compact supports $H_c(X, C)$ for an arbitrary map $f : X \to Y$, with the goal of obtaining an analogous result for the Leray filtration on the cohomology groups $H(X, C)$, then one hits the following obstacle: indeed, there are identifications $H(Y, Rf_* C) = H(\mathcal{Y}, Rf_* j_! C) = H(\mathcal{Y}, Rf_! j_C) = H(Y, Rf_* j_C)$, however, since $Rf_*$ does not commute with truncation, the Leray filtrations for $f$ and $\mathcal{F}$ do not coincide, and the imitation of the procedure would yield a geometric description only for the case of $\mathcal{F}$.

3. Appendix: Base change and Lefschetz hyperplane theorem

3.1. Notation and background results.

**Varieties and maps.** A variety is a separated scheme of finite type over the field of complex numbers $\mathbb{C}$. In particular, we do not assume that varieties are irreducible, reduced, or even pure dimensional. Since we work inductively with intersections of special hypersurfaces, we need this generality even if we start with a nonsingular irreducible variety. A map is a map of varieties, i.e. map of $\mathbb{C}$-schemes.

**Coefficients.** The results of this paper hold for sheaves of $R$-modules, where $R$ is a commutative ring with identity with finite global dimension, e.g. $R = \mathbb{Z}, \mathbb{R}$ a field, etc. For the sake of exposition we work with $R = \mathbb{Z},$ i.e. with sheaves of abelian groups.

**Variants.** The results of this paper hold, with routine adaptations of the proofs, in the case of varieties over an algebraically closed field and étale sheaves.
with the usual coefficients: \( \mathbb{Z}/l^n\mathbb{Z}, \mathbb{Q}_l, \mathbb{Z}_l[E], \mathbb{Q}_l[E] \) (\( E \supseteq \mathbb{Q}_l \) a finite extension) and \( \mathbb{Q}_l \). These variants are not discussed further (see [4], §3.2.2 and §6).

**Stratifications.** The term stratification refers to an algebraic Whitney stratification [5, 13, 14]. Recall that any two stratifications admit a common refinement and that maps of varieties can be stratified. See also §3.2.1.

The **constructible derived category** \( \mathcal{D}_Y \). Let \( Y \) be a variety, \( Sh_Y \) be the abelian category of sheaves of abelian groups on \( Y \) and \( D(Sh_Y) \) be the associated derived category. A sheaf \( F \in Sh_Y \) is **constructible** if there is a stratification of \( Y \) such that the restriction of \( F \) to each stratum is locally constant with stalk a finitely generated abelian group. A complex is **bounded** if the cohomology sheaves \( H^i(K) = 0 \) for \( |i| \gg 0 \). A complex \( K \in D(Sh_Y) \) with constructible cohomology sheaves is said to be **constructible**. The category \( \mathcal{D}_Y = \mathcal{D}_Y(\mathbb{Z}) \) is the full subcategory of the derived category \( D(Sh_Y) \) whose objects are the bounded constructible complexes. For a given stratification \( \Sigma \) of \( Y \), a complex with this property is called \( \Sigma \)-constructible. Given a stratification \( \Sigma \) of \( Y \), there is the full subcategory \( \mathcal{D}_Y^\Sigma \subseteq \mathcal{D}_Y \) of complexes which are \( \Sigma \)-constructible. Hypercohomology groups are denoted \( H(Y, K) \) and \( H_*(Y, K) \). If \( K \in \mathcal{D}_Y \) and \( n \in \mathbb{Z} \), then \( K[n] \in \mathcal{D}_Y \) is the \( (n \text{-shifted}) \) complex with \( (K[n])^i = K^{i+n} \). One has, for example, \( H^i(Y, K[n]) = H^{i+n}(Y, K) \).

The **four functors associated with a map**. Given a map \( f : X \to Y \), there are the usual four functors \( (f^*, Rf_*, Rf^!, f^!) \). By abuse of notation, denote \( Rf_* \) and \( Rf^! \) simply by \( f_* \) and \( f^! \). The four functors preserve stratifications, i.e. if \( f : (X, \Sigma') \to (Y, \Sigma) \) is stratified, then \( f_* : D_X^{\Sigma'} \to D_Y^{\Sigma} \) and \( f^! : D_Y^{\Sigma} \to D_X^{\Sigma'} \).

**Verdier Duality.** The Verdier Duality functor \( \mathbb{D} = \mathbb{D}_Y : \mathcal{D}_Y \to \mathcal{D}_Y \) is an autoequivalence with \( \mathbb{D} \circ \mathbb{D} = \text{Id}_{\mathcal{D}_Y} \) and it preserves stratifications. We have \( \mathbb{D}_Y f^! = f^! \mathbb{D}_X \) and \( \mathbb{D}_X f^! = f^* \mathbb{D}_Y \).

**Perverse sheaves.** We consider only the middle perversity \( t \)-structure on \( \mathcal{D}_Y \) [4]. There is the full subcategory \( \mathcal{P}_Y \subseteq \mathcal{D}_Y \) of perverse sheaves on \( Y \). The elements are special complexes in \( \mathcal{P}_Y \). An important example is the intersection complex of an irreducible variety [13, 5]. Let \( j : U \to Y \) be an open immersion; then \( j^! = j^*: \mathcal{P}_Y \to \mathcal{P}_U \), i.e. they preserve perverse sheaves. Let \( j : U \to Y \) be an **affine** open immersion; then \( j_!, j_* : \mathcal{P}_U \to \mathcal{P}_Y \). The Verdier Duality functor \( \mathbb{D} : \mathcal{P}_Y \to \mathcal{P}_Y \) is an autoequivalence.

**Distinguished triangles for a locally closed embedding.** There is the notion of distinguished triangle in \( \mathcal{D}_Y \): it is a sequence of maps \( X \to Y \to Z \to X[1] \) which is isomorphic in \( \mathcal{D}_Y \) to the analogous sequence of maps arising from the cone construction associated with a map of complexes \( X' \to Y' \). Let \( j : U \to Y \) be a locally closed embedding with associated “complementary” embedding and \( i : Y \setminus U \to Y \). For every \( K \in \mathcal{D}_Y \), we have distinguished triangles

\[
\begin{align*}
\xymatrix{ j_* j^! K & K & i_* i^! K \ar[r]^-{[1]} & K \ar[r]^-{j_* j^* K} & j_* j^* K \ar[r]^-{[1]} }
\end{align*}
\]

**Various base change maps.** Given two maps \( Y' \xrightarrow{g} Y \xleftarrow{f} X \), there is the Cartesian diagram

\[
\begin{align*}
\xymatrix{ X' \ar[r]^g & X \\
Y' \ar[r]^g \ar[u]^f & Y \ar[u]_f }
\end{align*}
\]
The ambiguity of the notation (clearly the two maps \( g \) are different from each other, etc.) does not generate ambiguous statements in what follows, and it simplifies the notation.

There are the natural maps

\[
g f \rightarrow f g, \quad g f \rightarrow f g.
\]

There are the base change maps

\[
g f \rightarrow f g, \quad g f \leftarrow f g.
\]

and the base change isomorphisms

\[
g f \simeq f g, \quad g f \simeq f g.
\]

Similarly, for the higher direct images \( R^i f \) and \( R^i f \).

**Example 3.1.** Let \( Y' \rightarrow Y \) be the closed embedding of a point \( y \rightarrow Y \). The first base change map in (3.4) yields a map \((R^i f y, Z) \rightarrow H^i((f^{-1}(y), Z))\). Given a sufficiently small, contractible neighborhood of \( y \) in \( Y \), we have \( H^i((f^{-1}(U), Z)) = (R^i f y, Z)\). This base change map is seldom an isomorphism, e.g. the open immersion \( X = \mathbb{C}^* \rightarrow \mathbb{C} = Y, \ y = 0 \). This failure is corrected in (3.5) by taking the direct image with proper supports.

**Base change theorems.** The base change maps (3.4) are isomorphisms if either one of the following conditions is met: \( f \) is proper, \( f \) is locally topologically trivial over \( Y \), or \( g \) is smooth.

**The octahedron axiom.** This is one of the axioms for a triangulated category and can be found in [4], 1.1.6. Here is a convenient way to display it (see [4], 1.1.7.1). Given a composition \( X \rightarrow Y \rightarrow Z \) of morphisms one has the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
Z' & \rightarrow & Y' \\
\end{array}
\]

where \((X, Y, Z'), (Y, Z, X'), (X, Z, Y')\) and \((Z', Y', X')\) are distinguished triangles.

**Remark 3.2.** It is clear that \( g \) is an isomorphism if and only if \( Y' \simeq 0 \) if and only if \( X' \rightarrow Z'[1] \) is an isomorphism.

**The term “general.”** Let \( \mathcal{P} \) be a property expressed in terms of the hyperplanes of a projective space \( \mathbb{P} \), i.e. the elements of \( \mathbb{P}^\vee \). We say that \( \text{property } \mathcal{P} \text{ holds for a general hyperplane} \) if there is a Zariski-dense open subset \( V \subseteq \mathbb{P}^\vee \) such that property \( \mathcal{P} \) holds for every hyperplane in \( V \). Of course this terminology applies to propositions “parameterized” by irreducible varieties and one can talk about a general pair of hyperplanes, in which case the variety is \( \mathbb{P}^\vee \times \mathbb{P}^\vee \), etc.

As it is customary, we often denote a canonical isomorphism with the symbol \( \simeq \).
3.2. Base change with respect to subvarieties. Let us discuss the following two special cases of (3.2). Even though (3.7) is a special case of (3.8), it is convenient to distinguish between the two (see Propositions 3.4, 3.5). Let $i : H \to Y$ be a closed embedding. Let $j : U \to Y$ be an open embedding and $f : X \to Y$ be a map. We obtain the following two Cartesian diagrams

$$
\begin{array}{ccc}
U \cap H & \xrightarrow{i} & U \\
\downarrow j & & \downarrow j \\
H & \xrightarrow{i} & Y,
\end{array}
$$

(3.7)

$$
\begin{array}{ccc}
X_H & \xrightarrow{i} & X \\
\downarrow f & & \downarrow f \\
H & \xrightarrow{i} & Y.
\end{array}
$$

(3.8)

**Question 3.3.** Let $K \in D_Y, C \in D_X$. Which conditions on the closed embedding $i : H \to Y$ ensure that the base change maps

$$
j_* i^* K \longrightarrow i^* j_* K, \quad j_! i^! K \longrightarrow i^! j_! K
$$

are isomorphisms?

Which conditions on the closed embedding $i : X_H \to X$ ensure that the base change maps

$$
f_* i^* K \longrightarrow i^* f_* K, \quad f_! i^! K \longrightarrow i^! f_! K
$$

are isomorphisms?

Answers to these questions are given in Propositions 3.4, 3.5. Since these results involve the notion of stratifications, we discuss briefly stratifications in the next section.

3.2.1. Some background on stratifications.

**Stratifications.** For background, see [5, 14]; see also [16, 19]. The datum of a stratification $\Sigma$ of the variety $Z$ includes a disjoint union decomposition $Z = \bigsqcup_{i} \Sigma_i$ into locally closed nonsingular irreducible subvarieties $\Sigma_i$ called *strata*. One requires that the closure of a stratum is a union of strata. These data are subject to the Whitney Conditions A and B, which we do not discuss here. Every variety admits a stratification. Any two stratifications of the same variety admit a common refinement. Given a stratification $\Sigma$ of $Z$, every point $z \in Z$ admits a fundamental system of standard neighborhoods homeomorphic, in a stratum-preserving way, to $C^l \times C(L)$, where $C$ denotes the real cone (with vertex $v$), $\mathcal{L}$ is the link of $z$ in $Z$ (relative to $\Sigma$) (it is a stratified space obtained by embedding $Z$ in some manifold, intersecting $Z$ with a submanifold meeting the stratum transversally at $z$ and then intersecting the result with a small ball centered at $z$), and $C^l \times v$ is the intersection of the stratum $\Sigma_i$ to which $z$ belongs with a small ball (in the big manifold containing $Z$) centered at $z$.

**Constructible complexes.** Let $\Sigma$ be a stratification of the variety $Z$. The bounded complexes constructible with respect to $\Sigma$ form the category $D_{\Sigma}^b$ which is a full subcategory of the constructible derived category $D^b_Z$. If $K \in D^b_{\Sigma} Z$, $z \in Z$, $U := C^l \times C(L)$ is a standard neighborhood of $z$ with second projection $\pi$, then $K_{|U} \cong \pi^* \pi_* K_{|U}$, i.e. $K$ is locally a pull-back from the cone over the link.
Stratified maps. Algebraic maps can be stratified: given a map $f : X \to Y$, there are stratifications $\Sigma_X$ for $X$ and $\Sigma_Y$ for $Y$ such that (i) for every stratum $S$ on $X$, the space $f^{-1}S$ is a union of strata on $X$ and (ii) for $y \in S$ there exists a neighborhood $U$ of $y$ in $S$, a stratified space $F$ and a stratification-preserving homeomorphism $F \times U \simeq f^{-1}U$ which transforms the projection onto $U$ into $f$.

If $f$ is a closed embedding, then each stratum in $X$ is the intersection of $X$ with a stratum of $Y$ of the same dimension. If $f$ is an open immersion, using standard neighborhoods, the local model at $y \in Y \setminus X$ is $f : C^l \times (C(L) \setminus C(L')) \to C^l \times C(L)$, where $L'$ is the link at $z$ of $Y \setminus X$.

Normally nonsingular inclusions. A closed embedding $i : H \to Y$ is normally nonsingular with respect to a stratification $\Sigma$ of $Y$ if $H$ is obtained locally on $Y$ by the following procedure: embed $Y$ into a manifold $M$ and $H$ is the intersection $H' \cap Y$, where $H' \subseteq M$ is a submanifold meeting transversally all the strata of $\Sigma$. See [13, 8]. Note that a normally nonsingular inclusion is locally of pure codimension. If $Y$ is embedded into some projective space, then by the Bertini Theorem a general hyperplane section yields a normally nonsingular inclusion. More generally, the general element of a finite dimensional base-point-free linear system of $Y$ yields a normally nonsingular inclusion ([15]). Let $K \in D^b_{\Sigma U}$ and $i : H \to Y$ be a normally nonsingular inclusion of complex codimension $r$ with respect to $\Sigma$. Then $i^* K = i^! K[-2r]$ (cf. [8]).

3.2.2. Sufficient condition for the base change maps to be an isomorphism. Let $j : U \to Y$ be an open embedding and $\Sigma$ be a stratification of $Y$ such that, if $\Sigma_U$ is its trace on $U$, then the map $j : (U, \Sigma_U) \to (Y, \Sigma)$ is stratified. Such a stratification $\Sigma$ exists. Consider the situation (3.7).

**Proposition 3.4.** Assume that $i : H \to Y$ is normally nonsingular with respect to $\Sigma$. Then for every $K \in D^b_{\Sigma U}$ the base change maps $i^* j_* K \longrightarrow j_* i^* K, \quad j^! j^* K \leftarrow j_* i^* K$ are isomorphisms.

**Proof.** Here are two essentially equivalent proofs. While the first one seems shorter, it does rely on the formula $i^* = i^! [-2r]$, the second one is more direct.

1st proof. The assumptions imply that $i^* K = i^! K[-2r]$. The conclusion follows from the base change isomorphisms (3.5).

2nd proof. The complexes $j_! K, j_* K \in D^b_{\Sigma U}$. The assertion is local. The local model for (3.7) at a point $y \in H$ lying on a $l$-dimensional stratum with links $L$ for $y \in Y$ and $L' \subseteq L$ for $y \in Y \setminus U$ is, denoting by $C$ real cones and by $r$ the codimension of $H$ in $Y$:

$$
\begin{array}{ccc}
C^{l-r} \times (C(L) \setminus C(L')) & \longrightarrow & C^l \times (C(L) \setminus C(L')) \\
\downarrow j & & \downarrow j \\
C^{l-r} \times C(L) & \longrightarrow & C^l \times C(L)
\end{array}
$$

(3.9)

with $Id \simeq \pi^* \pi_* = \pi^! \pi_*$ for $\Sigma$-constructible complexes. One has, using the base change Theorem for the smooth map $\pi \circ i$:

$$
i^* j_* K \simeq i^* \pi_! \pi_* j_* K = i^* \pi^* j_* \pi_* K = j_* i^* \pi_* K \simeq j_* i^* K.$$

This proves the first assertion. The second one is proved in the same way:

\[ i^! j_! K \simeq i^! \pi^! j_! K = i^! \pi^! j_! \pi_! K = j_! i^! \pi^! \pi_! K \simeq j_! i^! K. \]

Note that once the first assertion is proved, one can prove the second one also as follows. Given \( K \in D^\Sigma_U \), we have that \( K \lor \in D^\Sigma_U \). We have proved that the first assertion holds for every \( K \in D^\Sigma_U \) so that it holds for \( K \lor \):

\[ i^! j^* C \rightarrow f^* i^* C, \quad i^! f^* C \leftarrow f^! i^* C \]

are isomorphisms.

**Proposition 3.5.** Let \( i : X_H \rightarrow X \) be a normally nonsingular inclusion with respect to \( \Sigma' \). Then for every \( C \in D^\Sigma_X \) the base change maps

\[ i^* f^* C \rightarrow f^* i^* C, \quad i^! f^! C \leftarrow f^! i^! C \]

are isomorphisms.

**Proof.** Let \( j : X \rightarrow \overline{X} \xrightarrow{\overline{f}} Y \) be a completion of the map \( f \), i.e. \( j \) is an open immersion with dense image and \( \overline{f} \) is proper. Such a completion exists by a fundamental result of Nagata. The Cartesian diagram (3.8) can be completed to a commutative diagram with Cartesian squares

\[
\begin{array}{ccc}
X_H & \xrightarrow{i} & X \\
\downarrow{j} & & \downarrow{j} \\
\overline{X}_H & \xrightarrow{i} & \overline{X} \\
\downarrow{f} & & \downarrow{f} \\
H & \xrightarrow{i} & Y \\
\end{array}
\] (3.10)

By virtue of Lemma 3.4, we have the base change isomorphism \( i^* j_* C \simeq j_* i^* C \) to which we apply the proper map \( \overline{f} : \overline{X}_H \rightarrow H \):

\[ \overline{f}^* i^* j_* C \simeq \overline{f}^* j_* i^* C = f^* i^* C. \]

The first assertion follows by applying the Base Change for Proper Maps to the first term:

\[ \overline{f}^* j_* i^* C = i^* \overline{f}^* j_* C = i^* f^* C. \]

The second one can be proved in either of two ways as in the proof of Proposition 3.4.

**Remark 3.6.** Note that \( X_H \) can be normally included in \( X \) with respect \( \Sigma' \) while \( H \) may fail to be so with respect to any stratification of \( Y \) for \( f_* C \) and \( f! C \). E.g. \( X \) is nonsingular, \( C = \mathbb{Z}_X \), but \( Y \) and/or \( f \) have singularities.
3.3. The Lefschetz hyperplane theorem. The classical Lefschetz hyperplane theorem states that if $Y$ is a projective manifold of dimension $n$ and $H$ is a smooth hyperplane section relative to an embedding into projective space, then the restriction map $H^i(Y, \mathbb{Z}) \rightarrow H^i(H, \mathbb{Z})$ is an isomorphism for $i < n - 1$ and injective for $i = n - 1$.

In fact, one can prove this (see [17], for example) by showing that the relative cohomology groups $H^i(Y, H, \mathbb{Z}) = 0$ for $i \leq n - 1$. If $j : Y \setminus H \rightarrow Y$ is the open embedding, then $j^* = j^!$ and, since $Y$ is compact, $H^i(Y, H, \mathbb{Z}) = H(Y, j^!j^*\mathbb{Z}_Y)$. We can thus reformulate the Lefschetz hyperplane theorem in terms of the following vanishing statement

$$H^i(Y, j^!j^*\mathbb{Z}[n]) = 0 \quad \forall \ i < 0.$$  

Note that $\mathbb{Z}[n]$ is a perverse sheaf on the nonsingular $Y$. Beilinson [3], Lemma 3.3, has given a proof of this important result which is valid in the étale case and for every perverse sheaf on a quasi projective variety $Y$. His proof is based on the natural map (3.16) being an isomorphism.

In this section, we discuss Beilinson’s proof, which boils down to an application of the base change Proposition 3.4.

3.3.1. The natural map $j^!J^* \rightarrow J^!j^!$. Let $Y$ be a quasi projective variety, $Y \subseteq \mathbb{P}^N$ be a fixed embedding in some projective space, $\overline{Y} \subseteq \mathbb{P}^N$ be the closure of $Y$, $\Lambda \subseteq \mathbb{P}^N$ be a hyperplane and $H \subseteq Y$ be the corresponding hyperplane sections. There is the Cartesian diagram

$$
\begin{array}{ccc}
H & \xrightarrow{i} & Y \\
\downarrow{j} & & \downarrow{j} \\
\overline{H} & \xrightarrow{i} & \overline{Y} \\
& \xrightarrow{j} & \downarrow{j} \\
& & \overline{U}.
\end{array}
$$

Let $K \in \mathcal{D}_Y$. Consider the composition

$$J_*K \rightarrow i_*i^*J_*K \xrightarrow{\phi} i_*J_*i^*K (= J_*i_*i^*K).$$

The octahedron axiom yields a distinguished triangle (the equality stems from (3.5))

$$j^!j^*J_*K[1] (= j_*J_*j^*K[1]) \rightarrow J_*j^!j^*K[1] \rightarrow \text{Cone}(\phi),$$

where the first map arises by applying (3.3) to $j^!K$.

Similarly, we have the composition

$$J_!K \leftarrow i_!i^!J_!K \xleftarrow{\varphi} i_!J_!i^!K (= J_!i_!i^!K)$$

and the octahedron axiom, yields a distinguished triangle

$$\leftarrow j_*j^*J_!K (= j_*J_*j^*K) \leftarrow J_*j_*j^*K \leftarrow \text{Cone}(\varphi),$$

where the second map arises by applying (3.3) to $j^*K$.

**Lemma 3.7.** The map

$$j_*J_*j^!K \rightarrow J_*j_*j^!K \quad (j_*J_*j^*K \leftarrow J_*j_*j^*K, \text{ resp.})$$

is an isomorphism if and only if the base change map $i^*J_*K \rightarrow J_*i^*K$ (i.e., $J_!K \leftarrow J_!i^!K$, resp.) is an isomorphism.
Proof. By Remark 3.2, the map (3.16) is an isomorphism if and only if the map $\phi$ is an isomorphism. The conclusion follows from the fact that since $i$ is a closed embedding, $i_*$ is fully faithful. The second assertion is proved using the same construction, with the arrows reversed.

Remark 3.8. If in the set-up of diagram (3.11) the maps $J : Y \to \overline{Y}$ and $i : \overline{H} \to \overline{Y}$ are arbitrary locally closed embedding of varieties, then the proof of Lemma 3.7 shows that if the base change map $i^* J_! K \to J_* i^* K$ is an isomorphism, then the map (3.16) is also an isomorphism.

3.3.2. The Lefschetz hyperplane theorem for perverse sheaves. In this section, $Y$ is a quasi projective variety equipped with a fixed affine embedding $Y \subseteq \mathbb{P}^N$ is some projective space. Let us stress that we shall consider hyperplane sections with respect to this fixed affine embedding.

If $Y$ is affine, then every embedding into projective space is affine. Not every embedding of a quasi projective variety is affine, e.g. $\mathbb{A}^2 \setminus \{(0,0)\} \subseteq \mathbb{P}^2$. Affine embeddings always exist: take an arbitrary embedding (with associated closure) $Y \subseteq \tilde{Y} \subseteq \mathbb{P}^M$ into some projective space and blow up the boundary $\tilde{Y} \setminus Y$; the resulting projective variety $\overline{Y}$ contains $Y$ and the complement is a Cartier divisor, so that $Y \subseteq \overline{Y}$ is an affine embedding; finally embed $\overline{Y}$ into some projective space $\mathbb{P}^N$; this embedding is affine. If the embedding is not chosen to be affine, then the conclusion of Theorem 3.10 is false, as it is illustrated by the example of the punctured plane.

We need the following standard vanishing result due to M. Artin.

Theorem 3.9. Let $Y$ be an affine variety and $Q \in \mathcal{P}_Y$ be a perverse sheaf on $Y$. Then
\[ H^r(Y, Q) = 0, \quad \forall r > 0, \quad H^r_c(Y, Q) = 0, \quad \forall r < 0. \]

Proof. See [2] and [4].

Let $\Lambda \subseteq \mathbb{P}^N$ be a hyperplane, $H := Y \cap \Lambda \subseteq Y$ be the corresponding hyperplane section and consider the corresponding open and closed immersions.
\[ H \hookrightarrow Y \quad \text{and} \quad U \leftarrow Y \setminus H. \]

The following is Beilinson’s version of the Lefschetz Hyperplane Theorem. The proof is an application of the base change Proposition 3.4. One can also invoke (as in [3], Lemma 3.3) the generic base change theorem and reach the same conclusion (without specifying how one should choose the hyperplane).

Theorem 3.10. Let $Q \in \mathcal{P}_Y$ be a perverse sheaf on $Y$. If $\Lambda$ is a general hyperplane (for the given affine embedding $Y \subseteq \mathbb{P}^N$), then
\[ H^r(Y, j_! j^* Q) = 0, \quad \forall r < 0, \quad H^r_c(Y, j_! j^* Q) = 0, \quad \forall r > 0. \]

Moreover, if $Y$ is affine, then
\[ H^r(Y, j_! j^* Q) = 0, \quad \forall r \neq 0, \quad H^r_c(Y, j_! j^* Q) = 0, \quad \forall r \neq 0. \]

Proof. The idea of proof is to identify the cohomology groups in question (cohomology groups with compact supports, resp.) with cohomology groups with compact supports (cohomology groups, resp.) on an auxiliary affine variety, and then apply Artin vanishing Theorem 3.9.
Let $\overline{Y} \subseteq \mathbb{P}^N$ be the closure of $Y$. We have the following chain of equalities (see (3.11))

$$H^r(Y, j_! j^* Q) = H^r(\overline{Y}, J_- j_! j^* Q) \overset{\cong}{\longrightarrow} H^r(\overline{Y}, J_! j_* j^* Q) = H^r_c(\overline{Y}, J_! j_* j^* Q) = H^r_c(\overline{U}, J_! j_* j^* Q),$$

where we have applied Lemma 3.7 and Proposition 3.4 (applied to $i, J$) to obtain the second equality. Since $j$ is an open immersion, $j^! = j^*$ and $j^! Q$ is perverse. Since $J$ is an affine open immersion, $J_! j_* j^* Q$ is perverse. Since $\overline{U}$ is affine, the last group is zero for $r > 0$ by virtue of Theorem 3.9 and the assertion in cohomology is proved.

The assertion for $H_c(Y, j_* j^* Q)$, is proved in a similar way. The relevant sequence of identifications and maps is

$$H_c(Y, j_* j^* K) = H_c(\overline{Y}, J_! j_* j^* K) \overset{\cong}{\longrightarrow} H_c(\overline{Y}, J_!, J_* j^* K) = H(\overline{Y}, j_* j^* J_! J_* K).$$

### 3.3.3. A variant of Theorem 3.10 using two hyperplane sections

In this section $Y$ is a quasi projective variety and we fix an affine embedding $Y \subseteq \mathbb{P}^N$.

Let $\Lambda, \Lambda' \subseteq \mathbb{P}^N$ be two hyperplanes, $H := Y \cap \Lambda \subseteq Y$ and $j : Y \setminus H := U \rightarrow Y \leftarrow H : i$ be the corresponding open and closed immersions. Note that $j^! = j^*$. Similarly, we have $\Lambda', H', U', i', j'$.

We have the Cartesian diagram of open embeddings

$$
\begin{array}{ccc}
U & \rightarrow & j' U \\
\downarrow j & & \downarrow j' \\
Y & \leftarrow & j' U.
\end{array}
$$

Since the embedding $Y \subseteq \mathbb{P}^N$ is affine, these open embeddings are affine and so are the open sets $U, U', U \cap U'$. If the embedding were not affine, these open sets may fail to be affine and the conclusions on vanishing of Theorem 3.12 would not hold.

Using the natural maps and isomorphisms (3.3, 3.4, 3.5) and that $j^! = j^*$, $j^! = j^!$, we get the following maps

$$j_! j'_! j^* j^* \overset{\cong}{\longrightarrow} j_! j'_! j^* j^* \overset{\cong}{\longrightarrow} j_! j'_! j^* j^* \overset{\psi}{\longrightarrow} j'_! i_* i^* j^* = j'_! i_* i^* j^* \overset{\cong}{\longrightarrow} j'_! i_* i^* j^*$$

whose composition we denote by

$$c : j_! j'_! j^* j^* \longrightarrow j'_! i_* i^* j^*.$$

The octahedron axiom applied to the composition

$$j'_! j^* \longrightarrow i_* i^* j'_! j^* \overset{\psi}{\longrightarrow} i_* i^* j'_! j^* = j'_! i_* i^* j^*$$

yields a distinguished triangle

$$j_! j'_! j^* j^*[1] \rightarrow j'_! i_* i^* j^*[1] \rightarrow \text{Cone}(\psi) \quad (3.19)$$

**Lemma 3.11.** The map

$$c : j_! j'_! j^* j^* \longrightarrow j'_! i_* i^* j^*$$

is an isomorphism if and only if the base change map

$$i_* j'_! j^* \longrightarrow j'_! i_* j^*$$

is an isomorphism.

**Proof.** Same as Lemma 3.7. \qed
THEOREM 3.12. Let $Q \in \mathcal{P}_Y$. If $(\Lambda, \Lambda')$ is a general pair, then we have
\[ j_!j^!j'_*j'^*Q = j_!j^!j'^*j^!Q \]
and
\[ H^r(Y, j_!j^!j'_*j'^*Q) = H^r_c(Y, j_!j^!j'^*j^*Q) = 0, \quad \forall r \neq 0. \]

**Proof.** For a fixed and arbitrary $\Lambda'$, by virtue of Lemma 3.11 and Proposition 3.4 (applied to $i, j'$), the first equality holds for $\Lambda$ general. This implies that the first equality holds for a general pair.

We prove the statement in cohomology. The one in cohomology with compact supports is proved in a similar way, by switching the roles of the two hyperplanes $\Lambda$ and $\Lambda'$. Note that $j'_*j'^*Q$ is perverse. The vanishing of the groups for $r < 0$ and $\Lambda$ general follows from Theorem 3.10. The vanishing for $r > 0$ is obtained as follows:
\[ H^r(Y, j_!j^!j'_*j'^*Q) = H^r(Y, j_!j^*j'^*j'^*Q) = H^r(U', j'^*j^*Q); \quad U' \text{ is affine, } j'^*j^*Q \text{ is perverse and the last group is zero for } r < 0 \text{ by Theorem 3.9}. \]

**Remark 3.13.** Theorem 3.12 is due to Beilinson [3] and it is used in [9] to describe perverse filtrations on quasi projective varieties using general pairs of flags.

3.3.4. **The Lefschetz hyperplane theorem for constructible sheaves.** As it is observed in [18], Introduction, Theorem 3.10 admits a sheaf-theoretic version which we state and prove below. Let $Y \subseteq \mathbb{P}^N$ be a quasi projective variety of dimension $n$ embedded in some projective space in such a way that the embedding is affine. Let $V \subseteq \mathbb{P}^N$ be a hypersurface, $V = Y \cap \mathbb{P}^N$ and $j : Y \setminus V \to Y$.

**Theorem 3.14.** Let $T$ be a constructible sheaf on $Y$. There is a hypersurface $\mathcal{V}$ such that

1. $H^r(Y, j_!j^!T) = 0$, for every $r < n$ (for every $r \neq n$ if $Y$ is affine),
2. $\dim \mathcal{V} = \dim Y - 1$.

**Proof.** Let $\Sigma$ be a stratification of $Y$ with respect to which $T$ is constructible. The union $S_n$ of all $n$-dimensional strata is a non-empty, Zariski open subvariety of $Y$ with the property that $T|_{S_n}$ is locally constant and $\dim (Y \setminus S_n) \leq n - 1$.

Let $\mathcal{V} \subseteq \mathbb{P}^N$ be a hypersurface containing $Y \setminus S_n$ but not containing any of the irreducible components of $S_n$. Since the open embedding $j' : Y \setminus V' \to Y$ is affine, $j'_!j^!T[n]$ is a perverse sheaf on $Y$. We apply Theorem 3.10 to this perverse sheaf and conclude that the desired hypersurface is of the form $\mathcal{V} := \mathcal{V}' \cup \Lambda$ for some general hyperplane $\Lambda$.

**Remark 3.15.** The hypersurface $\mathcal{V}'$ must contain the “bad locus” of the sheaf $T$. In particular, it is a “special” hypersurface of sufficiently high degree. As the proof shows, it is not necessary to achieve 2. in order to achieve 1. However, 2. is useful in procedures where one uses induction on the dimension. I do not know of a version of Theorem 3.14 for cohomology groups with compact supports.

3.4. **The generic base change theorem.** The Generic Base Change Theorem was proved in [11] as an essential ingredient, in the étale context, towards the constructibility for direct images of complexes with constructible cohomology sheaves for morphisms of finite type over a field. These kinds of constructibility results are fundamental and permeate the whole theory of étale cohomology.

In this section we want to state the Generic Base Change Theorem and show how it can be applied in practice when one has a base change issues with “parameters,” e.g. elements of a linear system. For example, in the proof of the Lefschetz
hyperplane theorem 3.10 if one can afford to work with general linear sections, then Proposition 3.21 can be used in place of Proposition 3.4.

3.4.1. **Statement of the generic base change theorem.** Let $X \xrightarrow{f} Y \xrightarrow{p} S$ and $S' \to S$ be maps. Denote by $X' \xrightarrow{f'} Y' \xrightarrow{p'} S'$ the varieties and maps obtained by base change via the given $S' \to S$.

Let $C \in \mathcal{D}_X$. One says that the formation of $f_* C$ commutes with arbitrary base change if, for every $S' \to S$, the resulting first base change map (3.4) $g^* f_* C \to f_* g^* C$ is an isomorphism.

The issue does not arise for $f_1$, in fact, the base change isomorphism (3.5) $g^* f_1 = f_1 g^*$ implies that for every $C \in \mathcal{D}_X$ the formation of $f_1 C$ commutes with arbitrary base change.

Given a stratification $\Sigma$ of $X$ and a map $f : X \to Y$, one can refine $\Sigma$ so that the refinement is part of a stratification of the map $f$. It follows at once that, given $f : X \to Y$ and $C \in \mathcal{D}_X$, there is a Zariski-dense open subset $U \subseteq Y$ with the property that, given $f^{-1}(U) \to U = U$, the formation of $f_*(C_{|f^{-1}_1 U})$ commutes with arbitrary base change. It is sufficient to take for $U$ the dense open stratum on $Y$ of the stratification for $f$ refining the one for $C$. In fact, $f$ is then topologically locally trivial over $U$ and the base change maps are then isomorphisms.

However, what above is insufficient to prove the, for example, the vanishing Theorem 3.10. Moreover, it cannot be used for example to work with constructible sheaves for the étale topology for varieties over a field, where one cannot achieve the local triviality of $f : X \to Y$ over $U \subseteq Y$ (in fact, the generic base change theorem is a tool that effectively fixes this problem at the level of sheaves).

The Generic Base Change Theorem is a tool apt to deal with these and other situations.

**Theorem 3.16.** Let $X \xrightarrow{f} Y \xrightarrow{p} S$ be maps and $C \in \mathcal{D}_X$. There exists a Zariski open and dense subset $V \subseteq S$ such that, if one takes $(p f)^{-1}(V) \to p^{-1}V \to V$, then the formation of $f_*(C_{|(p f)^{-1} V})$ commutes with arbitrary base change $T \to V$.

**Proof.** For the étale case see [11], [Th. finitude], Th. 1.9. The proof in the case of complex varieties and $C \in \mathcal{D}_X$ is similar and, in fact, simpler. \qed

**Remark 3.17.** Note that the open set $V$ depends on $C$. However, an inspection of the proof reveals that given a stratification $\Sigma$ of $X$, one can choose the Zariski open and dense subset $V \subseteq S$ so that the conclusion of the Generic Base Change Theorem holds for every $C' \in \mathcal{D}_X^Z$.

**Remark 3.18.** For $\Sigma$ and $V$ as above, the formation of $f_1$ commutes with the formation of $g'$ over $V$ for every $K \in \mathcal{D}_X^Z$. In fact, to prove that $g^f f_1 K \to f_1 g' K$ is an isomorphism for $K \in \mathcal{D}_X^Z$, it is sufficient to observe that $K^V \in \mathcal{D}_X^Z$ and dualize the isomorphism $g^* f_* K^V \to f_* g^* K^V$ which holds over $V$ by Theorem 3.16.

3.4.2. **Generic base change theorem and families of hyperplane sections.** The following standard lemma is an illustration of the use of Generic Base Change. It is essentially a special case, formulated in a way that directly applies to the situation dealt-with in the Lefschetz Hyperplane Theorem.
Lemma 3.19. Let $f : X \to Y$ be a map, $C \in D_X$ and

be a commutative diagram with Cartesian squares satisfying:

1. $\pi$ smooth; in particular, $\pi^* f_* \simeq f_{1 *} \pi'^*$;
2. $g$ smooth; in particular, $g^* f_* \simeq f_{2 *} g'^*$;
3. $V \subseteq S$ is a Zariski-dense open subset such that the formation of

$$f_{2,*} u'^* v'^* \pi'^* C$$

commutes with arbitrary base change (on $V$).

For every $t : T \to V$, the natural base change map is an isomorphism:

$$i^*_T f_* C \xrightarrow{\sim} f_{2,T} i'^*_T C.$$

**Proof.** The natural map

$$i^*_T f_* C \to f_{2,T} i'^*_T C$$

factors as follows

$$i^*_T f_* C = \tau^* u'^* g^* f_* C \to \tau^* u'^* f_{2,*} g'^* C \to \tau^* f_{2,V,*} u'^* g'^* C \to f_{2,T,*} \tau'^* u'^* g'^* C = f_{2,T,*} i'^*_T C.$$

Since $g$ and $u$ are smooth, the first and second arrows are isomorphisms. The third one is an isomorphism by the choice of $V$. \hfill $\square$

**Remark 3.20.** Fix a stratification $\Sigma$ for $X$. As in Remark 3.17, we can choose $V$ so that 3. above holds for every $C' \in D^\Sigma_X$ and conclude (see Remark 3.18) that we have the base change isomorphisms

$$i^*_T f_* C \xrightarrow{\sim} f_{2,T} i'^*_T C', \quad i^*_T f_1 C \xrightarrow{\sim} f_{2,T} i'^*_T C', \quad \forall T \to V, \ \forall C' \in D^\Sigma_X.$$

We now apply Lemma 3.19 and Remark 3.20 to the following situation: let $f : X \to Y$ be a map of varieties, $|\delta|$ be a finite dimensional and base-point-free linear system on $Y$, e.g. the very ample linear system associated with an embedding on $Y$ into projective space. Given $H \in |\delta|$, we have the Cartesian diagram (3.8).

Proposition 3.21. Let $\Sigma$ be a stratification of $X$. If $H \in |\delta|$ is general, then for every $C \in D^\Sigma_X$ the base change maps

$$i^* f_* C \longrightarrow f_* i'^* C, \quad i^1 f_! C \longleftarrow f_! i'^* C$$

are isomorphisms.
1st proof (it uses the generic base change theorem and it does not single out a specific $H$). We only need to apply Lemma 3.19 and Remark 3.20 to the following situation: $Y_1 := Y \times |\delta|$, $Y_2 \subseteq Y \times |\delta|$ the universal hyperplane section, $S := |\delta|$ and $t : T \to V$ is the embedding of a closed point.

2nd proof (it uses Proposition 3.4 and it identifies precisely which conditions on $H$ must be satisfied). A general hyperplane $\Lambda$ is transverse to all the strata of a fixed stratification of $f^*C$. This means that $i : H \to Y$ is a normally nonsingular inclusion with respect to the given stratification. For such a $\Lambda, i^*f_*C = i^![2]f_*C = f_*i^!C[2]$. For $\Lambda$ general, $i : X_H \to X$ is transverse to all the strata of a fixed stratification of $C$ and we have $i^!C = i^!C[-2]$. This establishes that the first base change map in question is an isomorphism. The proof for the second one is similar.

References


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