In the problems below, $V$ is a finite-dimensional vector space of dimension $n$, $e_1, \ldots, e_n$ is a basis in $V$. As always, $S_d$ is the group of all permutations of $\{1, \ldots, d\}$, acting on $V^\otimes d$ by permutation of components, and

$$\text{Sym} = \frac{1}{d!} \sum_{s \in S_d} s, \quad \text{Alt} = \frac{1}{d!} \sum_{s \in S_d} \text{sgn}(s)s$$

are projectors on the subspaces of symmetric (respectively, antisymmetric) tensors.

6. Let $\xi \in \Lambda^2 V$.

(a) Prove that it is possible to choose a basis $e_1, \ldots, e_n$ in $V$ such that $\xi = e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{k-1} \wedge e_k$ for some even $k \leq n$. (Hint: we have already proved this before, in different language...)

Solution: Suffices to note that $\Lambda^2 V$ is the space of skew-symmetric bilinear forms on $V^*$, and apply the classification theorem for skew-symmetric bilinear forms.

(b) Show that $\xi$ can be written in the form $v \wedge w$ for some $v, w \in V$ iff $\xi \wedge \xi = 0$.

Solution: One direction is obvious: if $\xi = v \wedge w$, then $\xi \wedge \xi = v \wedge w \wedge v \wedge w = -v \wedge v \wedge w \wedge w = 0$. Conversely, assume $\xi \wedge \xi = 0$. Write $\xi$ as in part a. Explicit calculation shows that if $k \geq 2$, then $\xi \wedge \xi = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 + \text{other monomials}$; thus, $\xi \wedge \xi = 0$ is only possible if $k = 2$, $\xi = e_1 \wedge e_2$.

7. Let $W \subset V^\otimes d$ be the subspace spanned by vectors of the form $t - s_i(t)$, where $s_i$ are elementary transpositions. Also, denote by $\varphi$ the natural surjection $V^\otimes d \to V^\otimes d/W$.

(a) Prove that $\text{Sym} |_W = 0$.

Solution: Obvious from $\text{Sym} s_i = \text{Sym}$.

(b) Prove that $S^d V \cap W = \{0\}$ and thus

$$\varphi |_{S^d V} : S^d V \to V^\otimes d/W$$

is injective.

Solution: If $t \in S^d V \cap W$, then $\text{Sym}(t) = t = 0$ (by part a).

(c) Prove that $\varphi(t) = \varphi(s_i(t)) = \varphi(s(t))$ for any $s \in S_d, t \in V^\otimes d$, and thus $\varphi(\text{Sym}(t)) = \varphi(t)$.

Solution: $\varphi(s_i(t)) - \varphi(t) = \varphi(s_i(t) - t) = 0$ since $s_i(t) - t \in W$. Applying this repeatedly, we get

$$\varphi(s_{i_1} \ldots s_{i_k}(t)) = \varphi(s_{i_2} \ldots s_{i_k}(t)) = \cdots = \varphi(t)$$

(d) Prove that the map (1) is an isomorphism.

Solution: This map is injective (by part b). It is also surjective: for any $t \in V^\otimes d$, the class of $t$ in $V^\otimes d/W$ can be written as $\varphi(\text{Sym}(t))$ by part c.
Define operators $\varepsilon_i, i_j : \Lambda V \to \Lambda V$ by

$$
\varepsilon_i w = e_i \wedge w \\
i_j(e_j \wedge w) = w \\
i_j(e_{i_1} \wedge \cdots \wedge e_{i_k}) = 0 \quad \text{if none of } i_i = j
$$

(a) Show that these conditions uniquely define $i_j$. Show that $i_j$ satisfies the skew-symmetric Leibniz identity:

$$i_j(\xi \wedge \eta) = (i_j \xi) \wedge \eta + (-1)^d \xi \wedge (i_j \eta)$$

if $\xi \in \Lambda^d V$. Thus, $i_j$ is the skew-symmetric analogue of the operator $\frac{d}{dx_j}$ on $SV = \mathbb{C}[e_1, \ldots, e_n]$.

**Solution:** It follows from the definition that

$$i_i \varepsilon_i \wedge \cdots \wedge e_{i_k} \wedge \ldots = (-1)^{k-1} i_i(e_{i_k} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \ldots)$$

$$= (-1)^{k-1} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \ldots$$

(we assume all $i_i$ are distinct.) Leibniz identity is straightforward from this.

(b) Prove that the operators $\varepsilon_i, i_j$ satisfy the Clifford algebra relations:

$$\varepsilon_i \varepsilon_j + e_j \varepsilon_i = 0$$

$$i_j i_k + i_k i_j = 0$$

$$i_j \varepsilon_i + \varepsilon_i i_j = \delta_{ij}$$

(compare with: $x_i \frac{d}{dx_j} - \frac{d}{dx_j} x_i = \delta_{ij}$, where $x_i$ is considered as the operator of multiplication by $x_i$ on $\mathbb{C}[x_1, \ldots, x_n]$).

**Solution:** The first two are straightforward. As for the last one, let us apply the right-hand side to a monomial in $\Lambda^d V$. Such a monomial can always be written in the form $x \wedge w$, where $x$ is one of $1, e_i, e_j, e_i \wedge e_j$ and $w$ does not contain $e_i, e_j$. For each of these cases the identity is easily checked by direct calculation.

9. Show that the wedge product is associative:

$$(x_1 \wedge \cdots \wedge x_p) \wedge (y_1 \wedge \cdots \wedge y_q) = x_1 \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \wedge y_q$$

i.e.

$$\text{Alt}_{p+q} \left( \text{Alt}_p(x_1 \otimes \cdots \otimes x_p) \otimes \text{Alt}_q(y_1 \otimes \cdots \otimes y_q) \right) = \text{Alt}_{p+q}(x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q)$$

**Solution:** It follows from $\text{Alt}_{p+q} s = \text{sgn}(s) \text{Alt}_{p+q}$ for every $s \in S_{p+q}$ and definition of $\text{Alt}_p, \text{Alt}_q$ that

$$\text{Alt}_{p+q}(\text{Alt}_p \otimes \text{Alt}_q) = \text{Alt}_{p+q}$$

as operators in $V^\otimes(p+q)$.

*10. (a) Show that $\dim \Lambda^{n-1} V = n$

**Solution:** Follows from general formula: $\dim \Lambda^k V = \binom{n}{k}$. 

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*8 Define operators $\varepsilon_i, i_j : \Lambda V \to \Lambda V$ by

$$
\varepsilon_i w = e_i \wedge w \\
i_j(e_j \wedge w) = w \\
i_j(e_{i_1} \wedge \cdots \wedge e_{i_k}) = 0 \quad \text{if none of } i_i = j
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(a) Show that these conditions uniquely define $i_j$. Show that $i_j$ satisfies the skew-symmetric Leibniz identity:

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if $\xi \in \Lambda^d V$. Thus, $i_j$ is the skew-symmetric analogue of the operator $\frac{d}{dx_j}$ on $SV = \mathbb{C}[e_1, \ldots, e_n]$.

**Solution:** It follows from the definition that

$$i_i \varepsilon_i \wedge \cdots \wedge e_{i_k} \wedge \ldots = (-1)^{k-1} i_i(e_{i_k} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \ldots)$$

$$= (-1)^{k-1} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \ldots$$

(we assume all $i_i$ are distinct.) Leibniz identity is straightforward from this.

(b) Prove that the operators $\varepsilon_i, i_j$ satisfy the Clifford algebra relations:

$$\varepsilon_i \varepsilon_j + e_j \varepsilon_i = 0$$

$$i_j i_k + i_k i_j = 0$$

$$i_j \varepsilon_i + \varepsilon_i i_j = \delta_{ij}$$

(compare with: $x_i \frac{d}{dx_j} - \frac{d}{dx_j} x_i = \delta_{ij}$, where $x_i$ is considered as the operator of multiplication by $x_i$ on $\mathbb{C}[x_1, \ldots, x_n]$).

**Solution:** The first two are straightforward. As for the last one, let us apply the right-hand side to a monomial in $\Lambda^d V$. Such a monomial can always be written in the form $x \wedge w$, where $x$ is one of $1, e_i, e_j, e_i \wedge e_j$ and $w$ does not contain $e_i, e_j$. For each of these cases the identity is easily checked by direct calculation.

9. Show that the wedge product is associative:

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i.e.

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**Solution:** It follows from $\text{Alt}_{p+q} s = \text{sgn}(s) \text{Alt}_{p+q}$ for every $s \in S_{p+q}$ and definition of $\text{Alt}_p, \text{Alt}_q$ that

$$\text{Alt}_{p+q}(\text{Alt}_p \otimes \text{Alt}_q) = \text{Alt}_{p+q}$$

as operators in $V^\otimes(p+q)$.

*10. (a) Show that $\dim \Lambda^{n-1} V = n$

**Solution:** Follows from general formula: $\dim \Lambda^k V = \binom{n}{k}$. 

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(b) Construct an isomorphism \( \Lambda^{n-1}V = \Lambda^n V \otimes V^* \) (hint: look at Problem 2)

Solution: We have an obvious map \( \Lambda^{n-1} V \otimes V \to \Lambda^n V : \xi \otimes v \mapsto \xi \wedge v \). As in problem 2, this gives rise to a map \( f: \Lambda^{n-1} V \to \Lambda^n V \otimes V^* \). This map can be described as follows: \( \xi \mapsto \sum (\xi \wedge e_i) \otimes e^i \).
One immediately sees that it is isomorphism.

(c) For a linear operator \( A: V \to V \), consider the corresponding operator \( \Lambda^{n-1} A: \Lambda^{n-1} V \to \Lambda^{n-1} V \).

Write the matrix of \( \Lambda^{n-1} A \) in the basis \( b_1 = e_2 \wedge e_3 \wedge \cdots \wedge e_n, b_2 = e_1 \wedge e_3 \wedge \cdots \wedge e_n, \ldots \).

Solution: There are several ways of doing this. Here is one: assume that \( A \) is invertible. Then it is rather easy to show that if we identify \( \Lambda^{n-1} V = \Lambda^n V \otimes V^* \) as above, then \( \Lambda^{n-1} A = \Lambda^n A \otimes (A^*)^{-1} = \det A (A^*)^{-1} \).

On the other hand, it is known that the matrix \( C = \det A (A^t)^{-1} \) is the matrix of algebraic complements: \( C_{ij} = (-1)^{i+j} M_{ij} \), where \( M_{ij} \) is the determinant of the matrix obtained from \( A \) by removing \( i \)-th row and \( j \)-th column. Thus, the answer (at least for invertible \( A \)) is given by the matrix \( C_{ij} \) defined above; since the entries of this matrix are polynomials in entries of \( A \), the usual arguments show that the condition \( \det A \neq 0 \) can be removed.