MAT 534 FALL 2000
FINAL

NAME : 

THERE ARE 8 PROBLEMS
SHOW YOUR WORK!!!

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1
1. Let $G$ be a group and $C_G = \{ c \in G \mid \forall g \in G \ cg = gc \}$.

   a) Prove that $C_G$ is a normal subgroup of $G$.

   **Sol's.** Check that if $c \in C_G$ then $c^{-1} \in C_G$. Check that if $c,d \in C_G$, then $cd \in C_G$. Check that $gC_Gg^{-1} \subseteq C_G$.

   b) Prove that if $G$ contains exactly one element $a$ of order two, then $a \in C_G$.

   **Sol's.** $ag = ga$ iff $g^{-1}ag = a$. $(g^{-1}ag)^2 = g^{-1}a^2g = e_G$. It follows that the order of $g^{-1}ag$ is two for every $g \in G$ and by the uniqueness of $a$ we are done.

2. Let $f : G \to H$ be a group homomorphism with kernel $N$. Prove that for every subgroup $K$ of $G$ we have that $f^{-1}(f(K)) = K$ if and only if $N \subseteq K$.

   **Sols.** First prove that $f^{-1}(f(K)) = KN$. Then prove that $K = f^{-1}f(K)$ iff $N \subseteq K$.

3. Let $p$ be the smallest prime number dividing the order of a finite group $G$.

   Show that any subgroup $H$ of $G$ of index $p$ is normal in $G$.

   (Hint. Consider the action of $G$ on $G/H$.)

   **Sols.** Consider $G$ acting on the set of left cosets $G/H$ by left translation: $g \ast (aH) := gaH$. $|G/H| = p$. As discussed in class this defines a group homomorphism $G \to S_p$, where $S_p$ is identified with the permutations of the set $G/H$. The kernel $K$ must be contained in $H$.

   The image is isomorphic to the group $G/K$. The image is a subgroup of $S_p$ so that $|G/K| = [G : K]$ divides $p!$.


   But $K \not\subseteq H \neq G$. So $[G : K] = p$. It follows that $[H : K] = 1$, i.e. $H = K$.

   But $K$ is normal.

4. Let $A = ||a_{ij}|| \in M_{n \times n}(K)$ be a $n \times n$ matrix over a field $K$. Assume that $a_{ii} = a$ for some $a \in K$ and every $1 \leq i \leq n$. Assume also that $a_{ij} = 0$ for every $i < j$.

   a) Prove that $(aI_n - A)^n = 0$. 
**Sol's.** A is lower triangular. The characteristic polynomial is, by expansion, \((x - a)^n\). Apply Cayley-Hamilton.

b) Can one always find an integer \(n'\) such that 0 < \(n'\) < \(n\) and \((A - aI_n)^{n'} = 0\)?

**Sol's.** No:

\[
A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

5. Let

\[
B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}
\]

a) Find the characteristic polynomial and the eigenvalues of \(B\).

**Sol's.** The char. polynomial is \(|xI - B| = x^3 - 12x - 16 = (t + 2)^2(t - 4)|. The eigenvalues are \(l_1 = -2\) and \(l_2 = 4\).

b) Find a maximal set \(S\) of linearly independent eigenvectors of \(B\).

**Sol's.** Solve the linear system \((-2I - B)X^t = 0\). The solutions are multiples of \(v_1 := <1, 1, 0>\). There is a unique eigenvector for \(l_1 = -2\). Similarly, there is a unique eigenvector, \(v_2 := <0, 1, 1>\) for \(l_2 = 4\). \(S = \{v_1, v_2\}\) is a set of the required form.

c) Is \(B\) diagonalizable? If yes, find and invertible matrix \(P\) such that \(PBP^{-1}\) is diagonal.

**Sol's.** No: \(B\) has two eigenvalues. One, \(l_1 = -2\) has algebraic multiplicity two, but the corresponding eigenspace, \(<v_1>\), has dimension one.

It follows that there cannot be a basis of eigenvectors and this implies that \(B\) is not diagonalizable.

6. Let

\[
A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}
\]

a) Find an orthogonal matrix \(P\) such that \(P^tAP\) is diagonal.

**Sol's.** By the spectral theorem (real case) there is a orthonormal basis \(v_1, v_2, v_3\) of eigenvectors of \(A\) such that if \(P\) is the matrix with columns \(v_1, v_2, v_3\) then \(P^{-1}AP = P^tAP\) is diagonal with the corresponding eigenvectors \(l_1, l_2, l_3\) on the diagonal.
We first find the eigenvalues: \( \lambda_1 = 0 \), \( \lambda_2 = -1 \) and \( \lambda_3 = 3 \). We then find eigenvectors by solving \((A - \lambda_i I)X^t = 0\). We find \( v_1' = <1, -1, 1> \) for \( \lambda_1 = 0 \); \( v_2' = <0, 1, 1> \) for \( \lambda_2 = -1 \); \( v_3' = <-2, -1, 1> \) for \( \lambda_3 = 3 \). They are orthogonal. We normalize them and find
\[
v_1 = <\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}>, \quad v_2 = <0, 1/\sqrt{2}, 1/\sqrt{2}>, \quad v_3 = <-2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}>.
\]
It follows that
\[
P = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6}
\end{pmatrix}
\]
is a matrix of the required form.

b) Let \( f(x, y) = \sum_{ij} A_{ij} x_i y_j \) be the corresponding bilinear form on \( \mathbb{R}^3 \). Find the null space and the signature \((p, m)\) of \( f \).

**Sol’s.** By part a), we have one zero, one positive and one negative eigenvalue. The null space is the one generated by the eigenvector corresponding to \( \lambda_1 = 0 \), that is: \( v_1 \). The signature is \((1, 1)\).

7. Let \( M = M_{n \times n}(\mathbb{R}) \), \( S = \{ C \in M \mid C = C^t \} \) and \( A = \{ C \in M \mid C = -C^t \} \).

a) Prove that every \( D \in M \) can be written uniquely as \( D = DS + DA \), where \( DS \in S \) and \( DA \in A \).

**Sol’s.** Define \( DS = \frac{1}{2}(D + D^t) \) and \( DA = \frac{1}{2}(D - D^t) \). Clearly \( DS + DA = D \). We have \( S \cap A = \{ \text{the zero matrix} \} \). Let \( D'_S + D'_A = D \) be another decomposition. Then \( DS - D'_S = D'_A - DA \) and both terms are symmetric and anti-symmetric. It follows they are both zero.

b) Prove that \( \dim_{\mathbb{R}} S = \frac{1}{2}n(n + 1) \).

**Sol’s.** It is the number of pairs \((i, j)\) with \( i \leq j \) which can be counted as \( 1 + 2 + 3 + \ldots + n \).

c) Find the dimension of the trace zero linear transformations on a \( n \)-dimensional real vector space which are symmetric with respect to the dot product.

**Sol’s.** Pick an orthonormal basis for the dot product. Using this basis, the matrices of the linear transformations \( f \) symmetric with respect to the dot product are symmetric and, “viceversa.” Under this identification the required matrices are the trace-zero symmetric matrices. Their dimension is one less than the dimension of the space of symmetric matrices computed above.
8. Let $V$ and $W$ be finite dimensional vector spaces over a field.

a) Prove, using the properties of the tensor product, that $(V \otimes W)^* \simeq V^* \otimes W^*$.

(If you introduce a map, you must check that it is well-defined and that it has the properties that you state and use. The map must be independent of any choice of bases).

**Sol’s.** Define a bilinear map $g : V^* \times W^* \to (V \otimes W)^*$ by setting $g(f, h)(v \otimes w) := f(v)h(w)$.

By the basic property of the tensor product there exists a unique linear map $g' : V^* \otimes W^* \to (V \otimes W)^*$ sending $f \otimes h$ to $g(f, h)$.

Pick bases $v_i$ for $V$, $w_\alpha$ for $W$ and consider the dual bases: $v^*_i$ and $w^*_\alpha$. Let $t := \sum_{i,\alpha} a_{i\alpha} v^*_i \otimes w^*_\alpha$ be an element in $V^* \otimes W^*$. $g'(t)(v_j \otimes w_\beta) = a_{j\beta}$. It follows that $t \in \text{Ker} g'$ iff $t = 0$ and $g'$ is injective.

Since $g'$ is injective between vector spaces of the same finite dimension $\dim V \times \dim W$, $g'$ is also surjective, i.e. it is an isomorphism.

b) Prove that there is a canonical isomorphism

$$\text{Hom}(V, W^*) \simeq (\text{Hom}(V^*, W))^*.$$ 

**Sol’s.** Recall the isomorphisms $\text{Hom}(A, B) \simeq A^* \otimes B$ and $C^{**} \simeq C$. The RHS is isomorphic to $((V^*)^* \otimes W)^* \simeq (V \otimes W)^*$, which, by part a), is isomorphic to $V^* \otimes W^*$ which is isomorphic to the LHS.