Math 322 Lecture 1 (Tu/Th) MIDTERM I SPRING 2009 Mark de Cataldo

Name and ID#: ____________________, ______________

Problem	**Points Possible**	**Points**
1	10+10+10+30=50	
2	50	
3	50	
4	50	
5	50	
Total	250	

INSTRUCTIONS:

1. Your work will be carefully graded. The correct answers without sufficient work, or not using the method required, will receive minimal or no credit. If you use a theorem from the book, you need to tell us which one you are using (give its name or its statement).

2. The point value of each problem occurs to the left of the problem.

3. Provide clearly written answers in the space provided. You can use the flip sides of the sheets as scrap paper. **Do not tear off any page. You must return all pages.**

4. No books, no cell phones, no PDAs, no calculators.

5. Use the correct notation.

1. (50 total pts)

(a) (10 pts) State the Heine Borel Theorem.
The interval $[a, b]$ *is compact.*

(b) (10pts) Let $f : R^m \to R^n$ be a function and $a \in R^m$, $b \in R^n$.
Write out the definition of

$$\lim_{x \to a} f(x) = b.$$

$\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $|f(x) - f(a)| < \epsilon$ *as soon as* $0 < |x - a| < \delta$.

(c) (10 pts) Write out the definition of "f is differentiable at a."

See textbook.

(d) (20 pts) Give an example where $f : R^2 \to R$ has partial derivatives at $(1, 1)$ but is not differentiable at $(1, 1)$. (Just give an example; do not give a proof).

One example is given by the function with value 1 *on the two lines* $x = 1$ *and* $y = 1$ *and with value* 0 *everywhere else.*

2. (50 pts) Prove directly, that is without using formulas for derivatives, the following theorem.

If $p : R^2 \to R$ is defined by $p(x, y) = x \cdot y$, then

$$Dp(a, b)(x, y) = bx + ay.$$

See textbook.

3. (50 total pts) Let

$$f(x,y) = \frac{x^4 - x^3y + y^4}{x^4 + y^4}.$$

Find its domain of definition and determine whether it can be extended to a continuous function $F : R^2 \to R$.

The domain is $R^2 \setminus (0,0)$.

We have

$$\lim_{(x,0)\to(0,0)} f = 1$$

and

$$\lim_{(x,x)\to(0,0)} f = \frac{1}{2}.$$

This implies there is no limit as $(x,y) \to (0,0)$ *and the function cannot have a continuous extension.*

4. (50 total pts) Let

$$f(x,y) = \frac{x^3 + x^4}{x^2 + y^2}, \qquad if \ \ (x,y) \neq (0,0).$$

Prove that f cannot be extended to a function $F : R^2 \to R$ which is differentiable at $(0,0)$.

We did this example in class, without the extra x^4.

First show that the extension with value 0 at $(0,0)$ is continuous: use the following inequalities:

$$\left|\frac{x^3 + x^4}{x^2 + y^2}\right| \leq \left|\frac{|x|x^2|1+x|}{x^2+y^2}\right| \leq \left|\frac{|x||x^2+y^2||1+x|}{x^2+y^2}\right| \leq |x|(1+|x|) \to 0.$$

Assume differentiability. Then the derivative, being a linear map is of the form $(x,y) \to ax + by$ for two unique real numbers a and b (note we do not use that if the derivative exists, then its obtained using partial derivatives).

Then we should have

$$\lim_{(x,y)\to(0,0)} \left|\frac{\frac{x^3+x^4}{x^2+y^2} - ax - by}{\sqrt{x^2+y^2}}\right| = 0$$

Taking $(x,0)$ we see that $a = 1$. Taking $(y,0)$, we see that $b = 0$. So we should have

$$\lim_{(x,y)\to(0,0)} \left|\frac{\frac{x^3+x^4}{x^2+y^2} - x}{\sqrt{x^2+y^2}}\right| = 0.$$

On the other hand, taking (x,x) we get

$$\lim_{(x,y)\to(0,0)} \left|\frac{\frac{x^3+x^4}{2x^2} - x}{\sqrt{2}|x|}\right| \neq 0,$$

a contradiction.

5. (50 pts) Let $f : R^3 \to R^3$ be the differentiable function

$$f(x, y, z) = (z^2, xy - 3, 2y).$$

We can compose f with itself and obtain $g = f \circ f : R^3 \to R^3$.

Find the 3×3 matrix

$$g'(1, 1, 1)$$

giving the derivative of g at $(1, 1, 1)$.

$$f \circ f(x, y, z) = (4y^2, xyz^2 - 3z^2 - 3, 2xy - 6).$$

The matrix in question is the matrix obtained by computing the derivative of each coordinate function at $(1, 1, 1)$.

Each coordinate function is a polynomial and we know how to compute the derivative of a polynomial at (1,1,1).

We get (sorry, do not have handy the commands for displaying matrices): three rows: $(0, 8, 0)$, $(1, 1, -4)$, $(2, 2, 0)$.