HW14 – MA322;

4-22; If f & g are both equal to zero except on sets $A_f, A_g \subseteq \mathcal{S}$, $f+g(c)$ is zero except on $A_f \cup A_g$. Similarly, $n \cdot f(c)$ will be zero except on A_f which is finite.

4-25;
$$\int_c \omega = \int_{I^k} c^* \omega$$
$$\int_{c \circ p} \omega = \int_{I^k} (c \circ p)^* \omega = \int_{I^k} p^* \circ c^* \omega \underset{\text{I}}{=} \int_{p([0,1]^k)} c^* \omega = \int_{[0,1]^k} c^* \omega \overset{\text{by def}}{=} \int_c \omega$$

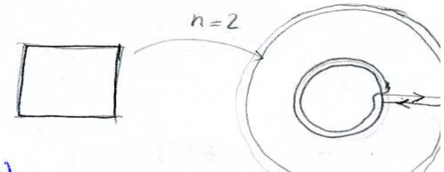
I: In general, by the change of variable formula, for any one-one singular n-cube σ, with $\det \sigma' > 0$, and any k-form, $\varphi = f\, dx^1 \wedge \dots \wedge dx^k$.
$$\int_{\sigma} \varphi = \int_{\sigma(I^k)} f$$

4-26; By Stokes' Thm, $\int_c d\omega = \int_{\partial c} \omega$, for any 1-form ω.

But $d\theta$ is closed, that is $d(d\theta) = 0 \Rightarrow$ for any two-chain we have $\int_c d(d\theta) = 0$, whereas if $\partial c = c_{R,n}$, we have $\int_{\partial c} d\theta = \int_{c_{R,n}} d\theta = 2\pi n$. Contradiction ■

HW14 - MAT322;

4 - 23; Consider the map



$$\begin{cases} \alpha_n : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\} \\ \alpha_n(R,\ t) = \left((R+1)\cos(2\pi n t),\ (R+1)\sin(2\pi n t)\right) \end{cases}$$

$$\partial \alpha_n = (-1)^2 \alpha_{n(R,0)} + (-1)\, \alpha_{n(R,1)} + \underbrace{\alpha_{n(t,0)} - \alpha_{n(t,1)}}_{\emptyset\,!}$$

$$= \alpha_{n(R,0)} - \alpha_{n(R,1)} = c_{1,n} - c_{2,n}$$

- α_n is the desired singular 2-cube.

4-24; Define $g = \int_0^x f - \lambda x$, where $\lambda = \int_0^1 f(x)\,dx$

Then $\Rightarrow$ $dg = f\,dx - \lambda dx$. It is easy to see that

$g(0) = 0 = g(1) = \int_0^1 f - \lambda$

Note; any $\tilde{g} = g + c$, where 'c' is a constant is acceptable.

Uniqueness; let $dg_{\lambda_1} + \lambda_1 dx = dg_{\lambda_2} + \lambda_2 dx \longrightarrow$ integrate:

$$\int_0^1 dg_{\lambda_1} + \lambda_1 dx = 0 + \lambda_1 = 0 + \lambda_2 \Rightarrow \lambda_1 = \lambda_2 .$$

4-31; let $\omega = f\, dx^1 \wedge \dots \wedge dx^n$. as $\omega \not\equiv 0$, $\rightarrow f \not\equiv 0$ $\Rightarrow$ by continuity $\exists$ a rectangle R on which 'f' has a constant sign, say positive. Then,

$$\int_{c} \omega = \int_{R} f\, dx^1 \wedge \dots dx^n > 0$$

where 'c' is a bijection of $[0,1]^n$ to R.

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let $d^2\omega \not\equiv 0 \longrightarrow \int_c d^2\omega = \int_{\partial c} d\omega = \int_{\partial^2 c} \omega = 0$

Stokes' Thm

$\Rightarrow$ But if 'c' is the 'appropriate' chain, $\int_c d^2\omega \neq 0$

Contradiction.

Note: I the proof of the fact that $d^2\omega = 0$, we used the fact that second-order mixed derivatives are equal. Here, we just require $d^2\omega$ be Riemann-integrable; loosely speaking 'continuous'. That is ω to be twice continuously differentiable.

4-32. a) Note that in $\mathbb{R}^n$ any two points can be connected by a segment. Define $c : [0,1]^2 \longrightarrow \mathbb{R}^2$ by:
$c(t, \lambda) = \lambda c_1(t) + (1-\lambda) c_2(t)$. $\partial c = c_1 - c_2 + c_3 - c_4$.,
Where $c_3 = c_1(0)$; $c_4 = c_1(1) = c_2(1)$.

HW14-3.

4-32-a-Cont'd; let $\omega = d\alpha$ (exact)

$$\int_{c_1-c_2+c_3-c_4} \omega = \int_{\partial c} \omega = \int_c d\omega = \int_c d^2\alpha = 0$$

$\Rightarrow$ But $\int_{c_3} \omega = \int_{c_4} \omega = 0$, as c_3 & c_4 are merely points.

$$\Rightarrow \int_{c_1-c_2} \omega = 0 \Rightarrow \int_{c_1} \omega = \int_{c_2} \omega .$$

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let $\begin{cases} c_1 = (\cos(\pi t), \sin(\pi t)) \\ c_2 = (\cos(2\pi t), \sin(2\pi t)) \end{cases}$

$$\int_{c_1} d\theta = 2\pi \qquad \int_{c_2} d\theta = 4\pi .$$

$d\theta$ is closed, but not exact.

b) Fix a point $a \in S$ and define $F(x) = \int_\gamma \alpha$, where 'α' our 1-form, and γ is a curve (1-cube) s.t.: $\begin{cases} \gamma(0) \\ \gamma(1) \end{cases}$
As $\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha$ for any γ_1 & γ_2 with the same start and finish point, $F(x)$ is well-defined. We just have to show that $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = \alpha = \alpha_1 dx + \alpha_2 dy$. Let

$$\gamma_h = (x_0, y_0) + h(1,0) \Rightarrow \lim_{h \to 0} \int_{\gamma_h} \alpha = \frac{\partial F}{\partial x} = \alpha_1$$

and similarly $\frac{\partial F}{\partial y} = \alpha_2$.