Section 5.1

#1) By the definition of continuity, f is continuous at c if and only if $\lim_{x\to c} f = f(c)$. By the Sequential Criterion in Chapter 4, this is equivalent to the statement that if (x_n) is a sequence converging to c such that $x_n \neq c$, then $(f(x_n))$ converges to $f(c)$. Clearly, this condition is satisfied if the Sequential condition in 5.1.3 (which allows for ANY sequence converging to c) is satisfied. Conversely, let (x_n) be any sequence converging to c. Either it is ultimately constant and equal to c, and there is nothing to prove since $f(x_n)$ will be ultimately constantly $f(c)$, or the terms x_{n_k} not equal to c form an infinite subsequece converging to c. Thus, if f is continuous at c, $f(x_{n_k})$ converges to $f(c)$, and hence $f(x_n)$ converges to $f(c)$, since every term not in $f(x_{n_k})$ is equal to $f(c)$.

#3) Let $\epsilon > 0$. If $x_0 \in [a, b)$, then we can find $\delta > 0$ such that $x_0 + \delta < b$, and if $x \in [a, b]$, $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. (Note that if $x \in [a, c]$, $|x - x_0| < \delta$, then automatically, $x \in [a, b)$.) Since $h \equiv f$ on $[a, b)$, then for $x \in [a, c]$, if $|x - x_0| < \delta$, then $|h(x) - h(x_0)| = |f(x) - f(x_0)| < \epsilon$. So h is continuous at every point in $[a, b)$.

Similarly, h is continuous at every point in $(b, c]$.

To show that h is continuous at b, note that we can find $\delta_1, \delta_2 > 0$ such that if $x_1 \in [a, b]$, $x_2 \in [b, c]$, and $|x_i - b| < \delta_i$ for $i = 1, 2$, then $|f(x_1) - f(b)|, |g(x_2) - g(b)| < \epsilon$. Choosing $\delta = \min\{\delta_1, \delta_2\}$, and noting that $|f(x_1) - f(b)| = |h(x_1) - h(b)|$ and $|g(x_2) - g(b)| = |h(x_1) - h(b)|$, we conclude that if $x \in [a, c]$ and $|x - b| < \delta$, then $|h(x) - h(b)| < \epsilon$.

#5) Away from $x = 2$, we see that $\frac{x^2+x-6}{x-2} = x + 3$. So $\lim_{x=2} f = 5$. Therefore, by defining $f(2) = 5$, we extend f to a continuous function on the real line.

#10) Let $\epsilon > 0$. Let $\delta > 0$. Note that $||x| - |c|| \leq |x - c|$, so if $|x - c| < \delta$, then $||x| - |c|| < \epsilon$, So $|x|$ is continuous at every point c.

#15) Since the $\lim_{x\to 0} f$ does not exist, there is some sequence $a_n > 0$ such that a_n converges to 0, but $f(a_n)$ diverges. Since f is a bounded function, $f(a_n)$ is a bounded sequence. By Bolzano-Weierstrass, we can find a convergent subsequence $f(a_{n_k})$, which corresponds to a convergent subsequence a_{n_k}. By 3.4.9, since $f(a_n)$ is a bounded subsequence, if all of its convergent subsequences had the same limit, then $f(a_n)$ would be convergent, contradicting the fact that we chose it to be divergent. Therefore, we can find at least two subsequences $f(a_{n_k})$ and $f(a_{n_j})$ that converge to different limits. Let $x_k = a_{n_k}$ and $y_j = a_{n_j}$.

Section 5.2

#1) All 4 functions are continuous wherever they are defined.

#3) Define $f(x) = \begin{cases} 1 & x > c \\ -1 & x \leq c \end{cases}$ And define $g(x) = -f(x)$. Then f and g are both discontinuous at c, $f + g$ is constant 0 and fg is constant -1, which are both continuous.

#6) Let (x_n) be an arbitrary sequence converging to c that is not ever equal to c. By the Sequential Criterion of Convergence, $f(x_n)$ converges to b. By the Sequential Criterion for Continuity applied to this sequence and g, $g(f(x_n))$ converges to $g(b)$. Therefore, by the sequential criterion for convergence, $g \circ f(x) \to g(b)$ as $x \to c$.

#7) Let $f(x) = 1$ if x is rational, and $f(x) = -1$ if x is irrational. Then f is discontinuous at every point, but $|f|$ is the constant function 1.

#12) Say f is additive and continuous at x_0, and let c be any other point. As $x \to c$, $x - c + x_0 \to x_0$. Since f is continuous at x_0, this implies that $\lim_{x\to c} f(x) = \lim_{x\to c}[f(x - c + x_0) + f(c - x_0)] = f(x_0) + f(c - x_0) = f(c)$, using additivity twice. Thus, f is continuous at c.

#13) First, note that $f(0) = 0$, since $f(0) = f(0+0) = f(0) + f(0) = 2f(0)$. Also note that $0 = f(0) = f(x + -x) = f(x) + f(-x)$ so that $f(-x) = -f(x)$. This last comment shows that it suffices to prove the claim for positive numbers, and it will follow automatically for negative numbers.

Let $c = f(1)$. Let $m \in \mathbb{N}$. Then $c = f(1) = f(1/m + 1/m + \cdots + 1/m) = f(1/m) + f(1/m) + \cdots + f(1/m) = mf(1/m)$, where the each some has m terms. Thus, $f(1/m) = c/m$. Now let $n \in \mathbb{N}$. Then $f(n/m) = f(1/m + \cdots + 1/m) = f(1/m) + \cdots + f(1/m) = nf(1/m) = c(n/m)$, where now each sum has n terms. Therefore, $f(q) = cq$ for all rational numbers q.

Let $x \in \mathbb{R}$ be arbitrary. Let q_n be a sequence of rationals converging to x (this exists by the Density Theorem). Since q_n is rational, $f(q_n) = cq_n$. By The continuity of f, we pass to the limit on both sides to find that $f(x) = cx$ as desired.