3.5 Continued

that is: prove $x_{2j+4} > x_{2j+3}$. To show this, we note that $x_{2j+4} = \frac{1}{2}(x_{2j+3}+x_{2j+2}) = \frac{1}{4}(x_{2j+1}+3x_{2j+2})$ and $x_{2j+3} = \frac{1}{2}(x_{2j+1}+x_{2j+2})$. So $x_{2j+4} - x_{2j+3} = \frac{1}{4}(x_{2j+2} - x_{2j+1}) > 0$ by assumption. Hence, our claim is proven by induction.

Now, for any $n > 1$. $|x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}|$, and so the sequence is contractive. Moreover, we can expand this result by induction to get $|x_{n+1} - x_n| = \frac{1}{2^{n-1}}|x_2 - x_1| = \frac{l}{2^{n-1}}$. Since the sequence is contractive, we know that it is Cauchy, whence convergent. Let its limit be x. We examine the odd-indexed subsequence (x_{2k+1}), and we claim that $x_{2k+1} = x_1 + \sum_{j=0}^{k} \frac{l}{2^{2j-1}}$. For $k = 1$, this is true since $x_3 = \frac{1}{2}(x_2 + x_1) = \frac{1}{2}(l + x_1 + x_1) = x_1 + \frac{l}{2}$ as desired. Assume it is true for some k; we will prove it for $k+1$ and the result follows by induction: $x_{2(k+1)+1} = x_{2k+3} = \frac{1}{2}(x_{2k+2}+x_{2k+1})$. We know that $|x_{2k+2} - x_{2k+1}| = \frac{l}{2^{2k}}$. Moreover, we know that $x_{2k+2} > x_{2k+1}$ and so $x_{2k+2} = x_{2k+1} + \frac{l}{2^{2k}}$. It follows that $x_{2k+3} = \frac{1}{2}(2x_{2k+1} + \frac{l}{2^{2k}}) = x_{2k+1} + \frac{l}{2^{2k+1}} = x_1 + \sum_{j=0}^{k} \frac{l}{2^{2j-1}} + \frac{l}{2^{2k+1}} = x_1 + \sum_{j=0}^{k+1} \frac{l}{2^{2j-1}}$ as desired. Therefore,

$$x_{2k+1} = x_1 + \sum_{j=0}^{k} \frac{l}{2^{2j-1}} = x_1 + \frac{l}{2}\sum_{j=0}^{k-1} \frac{1}{4^j} = x_1 + \frac{l}{2}\left(\frac{1-(\frac{1}{4})^k}{1-\frac{1}{4}}\right) = x_1 + \frac{2l}{3}(1-(1/4)^n)$$

Taking the limit of both sides yields $x = x_1 + \frac{2l}{3} = \frac{1}{3}x_1 + \frac{2}{3}x_2$.

#12) Note that $x_n > 0$ for all n so that $5 + 2x_n > 5$.

$$\begin{aligned}
|x_{n+2} - x_{n+1}| &= \left|\frac{1}{2+\frac{1}{2+x_n}} - \frac{1}{2+x_n}\right| = \left|\frac{2+x_n}{5+2x_n} - \frac{1}{2+x_n}\right| \\
&= \left|\frac{4+4x_n+x_n^2-5-2x_n}{(5+2x_n)(2+x_n)}\right| \le \frac{1}{5}\left|\frac{-1+2x_n+x_n^2}{2+x_n}\right| = \frac{1}{5}\left|\frac{-1}{2+x_n}+x_n\right| \\
&= \frac{1}{5}|x_n - x_{n+1}| = \frac{1}{5}|x_{n+1} - x_n|
\end{aligned}$$

Since $\frac{1}{5} < 1$, this shows the sequence is contractive. Its limit must be positive and satisfy $x = \frac{1}{2+x}$, and so the limit is $x = \sqrt{2} - 1$.

Section 3.5

#1) The sequence $((-1)^n)$ is bounded (by -1 and 1), and not Cauchy (since it is not convergent).

#2a) $\left(\frac{n+1}{n}\right) = (1+1/n)$. Let $\epsilon > 0$. Let $K \in \mathbb{N}$ be large enough so that $1/K < \epsilon/2$. Then for $n, m \geq K$, we have $|(1+1/n) - (1+1/m)| = |1/n - 1/m| \leq 1/n + 1/m \leq 1/K + 1/K < \epsilon/2 + \epsilon/2 = \epsilon$.

#3b) Let $\epsilon = 1$. We will show that for all $K \in \mathbb{N}$, there exist $n, m \geq K$ such that $|x_n - x_m| \geq 1$. Fix arbitrary $K \in \mathbb{N}$. Let m be an odd number greater than K and $n = m+1$. Then $|x_n - x_m| = |(m+1) + \frac{(-1)^{m+1}}{m+1} - m - \frac{(-1)^m}{m}| = |1 + \frac{1}{m+1} + \frac{1}{m}| > 1$ as desired.

#6) Fix $p \in \mathbb{N}$. Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Let $p \in \mathbb{N}$. First, the sequence (s_n) diverges (and so is not Cauchy) by 3.5.6(c) or 3.3.3(b). For all n, $|s_{n+p} - s_n| = \sum_{k=1}^{p} \frac{1}{n+k} \leq \sum_{k=1}^{p} \frac{1}{n} = \frac{p}{n}$, which converges to 0. Hence, $0 < |s_{n+p} - s_n| \leq \frac{p}{n}$, and by the Squeeze Theorem, $\lim |s_{n+p} - s_n| = 0$.

#7) Let $\epsilon = 1$. Since (x_n) is Cauchy, there exists $K \in \mathbb{N}$ such that for all $n, m \geq K$, $|x_n - x_m| < 1$. In particular, $|x_n - x_K| < 1$ for all $n \geq K$. It follows that for all $n \geq K$, $x_n \in (x_K - 1, x_K + 1)$. Since x_K is an integer, it is the only integer in $(x_K - 1, x_K + 1)$. Therefore, since x_n is an integer, we must have $x_n = x_K$ for all $n \geq K$. Therefore, the K-tail of the sequence is constant. Therefore, the sequence is ultimately constant.

#9) Let $m, n \in \mathbb{N}$. Without loss of generality, we may assume $m \geq n$. Let $p = m - n$. Then $|x_m - x_n| = |x_{n+p} - x_n| \leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \cdots + |x_{n+1} - x_n| < r^{n+p-1} + r^{n+p-2} + \cdots + r^n = r^n \sum_{k=0}^{p-1} r^k = r^n \frac{1-r^p}{1-r} < \frac{r^n}{1-r}$. $r^n \to 0$ as $n \to \infty$, so if $\epsilon > 0$, choose K such that $r^K < \epsilon(1-r)$. Then if $m \geq n \geq K$, we get $|x_m - x_n| < \frac{r^K}{1-r} < \epsilon$, and hence the sequence is Cauchy.

#10) Let $l = x_2 - x_1$. First we prove (we will need it later) that $x_{2k+2} > x_{2k+1}$ for all $k \geq 0$. It is true that $x_2 > x_1$, so the base case holds. Assume it is true for some j. We must prove that it is true for $j+1$,