Section 5.4

#2) $|1/x^2 - 1/y^2| = |y-x||x+y|/(x^2y^2) \leq |y-x|\left(\frac{1}{|x|y^2} + \frac{1}{x^2|y|}\right)$. If $x, y \in [1 \to \infty$, then both $\frac{1}{xy^2}$ and $\frac{1}{x^2y}$ are less than or equal to 1. Hence, $|1/x^2 - 1/y^2| \leq 2|x-y|$. So f is Lipschitz on $[1,\infty)$, and hence uniformly continuous.

Suppose f were uniformly continuous on $(0,\infty)$. Then in particular, it would be uniformly continuous on $(0,1)$. By the Continuous Extension Property, it could be extended to a continuous function on $[0,1]$. But continuous functions on closed bounded intervals are bounded, which f is not. Contradiction.

#4) $|f(x) - f(y)| = |x-y|\frac{|x+y|}{(1+x^2)(1+y^2)} \leq |x-y|\left(\frac{|y|}{(1+x^2)(1+y^2)} + \frac{|x|}{(1+x^2)(1+y^2)}\right)$. If $|x| \leq 1$, then $|x|/((1+x^2)(1+y^2)) \leq 1$. If $|x| > 1$, then $|x|/((1+x^2)(1+y^2)) < |x|/(x^2(1+y^2)) < 1/|x| < 1$. Hence, $\frac{|y|}{(1+x^2)(1+y^2)} + \frac{|x|}{(1+x^2)(1+y^2)} \leq 1+1 = 2$. So $|f(x)-f(y)| \leq 2|x-y|$. So f is Lipschitz on $\mathbb{R}$, and hence uniformly continuous.

#7) It is obvious that $f(x) = x$ is uniformly continuous. For all $x, y \in \mathbb{R}$,

$$|\sin(x) - \sin(y)| = 2|\sin(\frac{x-y}{2})||\cos(\frac{x+y}{2})| \leq 2|\frac{1}{2}(x-y)| = |x-y|.$$

So $g(x) = \sin x$ is Lipschitz on $\mathbb{R}$, and hence uniformly continuous. To show that $x \sin x$ is not uniformly continuous, we use the third criterion for nonuniform continuity.

Let (x_n) be the sequence $(2\pi n)$, and let (u_n) be the sequence $(2\pi n + 2\pi/n)$. Then clearly, $\lim(x_n - u_n) = 0$. Also, $\sin(x_n) = 0$ and $\sin(u_n) = \sin(2\pi/n)$. Therefore, $|x_n \sin(x_n) - u_n \sin(u_n)| = (2\pi n + 2\pi/n)\sin(2\pi/n)$. Letting $y_n = 2\pi/n$, we have the sequence $4\pi^2 \frac{\sin y_n}{y_n} + y_n \sin y_n$. As $n \to \infty$, $y_n \to 0$, so $y_n \sin y_n \to 0$. Also, $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, so $\lim(4\pi^2 \frac{\sin y_n}{y_n} + y_n \sin y_n) = 4\pi^2$. In particular, there exists a K such that for $n \geq K$, $|x_n \sin(x_n) - u_n \sin(u_n)| > 1$. So $x \sin x$ is not uniformly continuous.

#8) Let $\epsilon > 0$. By the uniform continuity of f, we can find $\eta > 0$ such that if $|x-u| < \eta$, then $|f(x) - f(u)| < \epsilon$. By the uniform continuity of g, we can find $\delta > 0$ such that if $|x-u| < \delta$, then $|g(x) - g(u)| < \eta$. Therefore, if $|x-u| < \delta$, then $|f(g(x)) - f(g(u))| < \epsilon$, as desired.

#11) Suppose there exists a K such that $\sqrt{x} \leq Kx$ for all $x \in [0,1]$. It follows that $K \geq \frac{1}{\sqrt{x}}$ for all $x \in (0,1]$. But $1/\sqrt{x}$ is unbounded on this interval, contradicting that K is a bound for it. So there does not exist such a K. It follows that $\sqrt{x}$ is not Lipschitz on $[0,1]$ despite being uniformly continuous there.

#12) Let $\epsilon > 0$. f is uniformly continuous on $[a,\infty)$, so there exists $\delta_1 > 0$ such that if $x, y \in [a,\infty)$ and $|x-y| < \delta_1$, then $|f(x) - f(y)| < \epsilon/2$. f is continuous on the closed bounded interval $[0,a]$, and hence uniformly continuous there. So there exists $\delta_2 > 0$ such that if $x, y \in [0,a]$ and $|x-y| < \delta_2$, then $|f(x)-f(y)| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Suppose $x, y \in [0,\infty)$ such that $|x-y| < \delta$. If $x, y \in [0,a]$, then $|x-y| < \delta_2 \implies |f(x)-f(y)| < \epsilon$. If $x, y \in [a,\infty)$, then $|x-y| < \delta_1 \implies |f(x)-f(y)| < \epsilon$. Finally, if one of x, y (say x) is in $[0,a]$ and the other (say y) is in $[a,\infty)$, then $|x-a| < \delta_1$ and $|y-a| < \delta_2$. It follows that $|f(x)-f(y)| \leq |f(x)-f(a)| + |f(a)-f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$.

#13) Let $\eta > 0$. Choose $\delta > 0$ such that if $|x-y| < \delta$, then $|g_{\epsilon/3}(x) - g_{\epsilon/3}(y)| < \epsilon/3$. (We can do this because $g_{\epsilon/3}$ is uniformly continuous. Therefore by the triangle inequality, if $|x-y| < \delta$, then $|f(x)-f(y)| \leq |f(x) - g_{\epsilon/3}(x)| + |g_{\epsilon/3}(x) - g_{\epsilon/3}(y)| + |g_{\epsilon/3}(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

#14) For any $x \in \mathbb{R}$, there exists an integer n such that $x - np \in [0,p]$. By induction, $f(x-np) = f(x)$. f is continuous, and $[0,p]$ is closed and bounded, so f is a bounded function. Let M be a bound for f on $[0,p]$. Then for $x \in \mathbb{R}$, we choose n as above so that $|f(x)| = |f(x-np)| \leq M$. Therefore, M is a bound for f on $\mathbb{R}$, and so f is bounded.

Now, since f is continuous on $[0,p]$, it is uniformly continuous there. Let $\epsilon > 0$. Then we can find $\delta > 0$ such that if $x, y \in [0,p]$ and $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon/2$. Let $x, y \in \mathbb{R}$ such that $|x-y| < \delta$. We can assume without loss of generality that $x < y$. Let n_x, n_y be the integers such that $x - n_x p, y - n_y p \in [0,p]$. Then it follows from $x < y$ that $n_y \geq n_x$. If $n_x = n_y$, then $|(x - n_x p) - (y - n_y p)| = |x - y| < \delta$. Hence by periodicity and choice of δ, $|f(x) - f(y)| = |f(x - n_x p) - f(y - n_y p)| < \epsilon/2 < \epsilon$. If $n_x < n_y$, then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f((n_x + 1)p)| + \sum_{k=n_x+1}^{n_y-1} |f(kp) - f((k+1)p)| + |f(n_y p) - f(y)| \\ &= |f(x) - f((n_x + 1)p)| + |f(n_y p) - f(y)| \end{aligned}$$

by the periodicity of f, all the middle terms are 0. Also, $x \leq (n_x+1)p \leq n_y p \leq y$, so $|x-(n_x+1)p| < \delta$ and $|y - n_y p| < \delta$. Therefore, $|(x - n_x p) - p| < \delta$, and $x - n_x p, p \in [0,p]$, and so $|f(x) - f((n_x + 1)p)| = |f(x - n_x p) - f(p)| < \epsilon/2$. Likewise, $|f(y) - f(n_y p)| = |f(y - n_y p) - f(0)| < \epsilon/2$. Therefore, $|f(x) - f(y)| < \epsilon$.