1) Cauchy Convergence Criterion: A sequence \((x_n)\) is Cauchy if and only if it is convergent.

Proof. Suppose \((x_n)\) is a convergent sequence, and \(\lim(x_n) = x\). Let \(\epsilon > 0\). We can find \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(|x_n - x| < \epsilon/2\). Therefore, by the triangle inequality, for all \(m, n \geq N\), \(|x_m - x_n| + |x - x_n| < \epsilon/2 + \epsilon/2 = \epsilon\). So \((x_n)\) is Cauchy.

Conversely, suppose \((x_n)\) is Cauchy. Let \(\epsilon > 0\). By a result proved in class, \((x_n)\) is bounded. By Bolzano-Weierstrass, it has a convergent subsequence \((x_{n_k})\) with \(\lim(x_{n_k}) = x\) for some \(x\). We can find \(K \in \mathbb{N}\) such that for all \(k \geq K\), \(|x_{n_k} - x| < \epsilon/2\). We can also find \(M\) such that for all \(m, n \geq M\), \(|x_m - x_n| < \epsilon/2\). Let \(N = \sup\{K, M\}\). Then since \(n_k \geq k\) for all \(k\), if \(k \geq N\), we have that \(k, n_k \geq M\) and \(n_k \geq K\). Therefore, for all \(k \geq N\), \(|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon\) by the Triangle Inequality. Therefore, \((x_n)\) is Cauchy.

2) (a) There is no example. Suppose \((x_n)\) is monotone and bounded. Then by the Monotone Convergence Theorem, it converges. Therefore, \((x_n)\) is not divergent. Therefore, \((x_n)\) is not properly divergent.

(b) Let \((x_n)\) be the sequence given by \(x_n = \begin{cases} 1 + 1/n & \text{if } n \text{ is even} \\ 1/n & \text{if } n \text{ is odd} \end{cases}\)

The even-indexed subsequence converges to 1 and the odd subsequence converges to 0.

(c) \((-1)^n\) is a bounded sequence. It is a subsequence of itself, and it diverges.

(d) Let \((y_n)\) be the sequence given by \(y_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}\)

It is unbounded because the even-indexed subsequence is properly divergent. The odd-indexed sequence is constant, and so bounded.

3) Let \(\epsilon_0 = 1\), and let \(H \in \mathbb{N}\) be arbitrary. Let \(n = H\) and \(m = H + 1\) so that \(n, m \geq H\). We will show that \(|x_m - x_n| > \epsilon_0\) (where \(x_n = \pi^n\)) so that the sequence is not Cauchy. Indeed, since \(\pi > 3\), we see that \(|x_m - x_n| = |\pi^n - 1| > 3^n \cdot 2 > 1\).

4) (a) divergent
(b) convergent
(c) convergent
(d) convergent

5) \(a\)

\[
\lim \left(\frac{2 + 1}{2n+2}\right)^n = \lim \frac{2n(1 + \frac{1}{2n})^n}{2n+2} = \lim \left(\frac{1 + \frac{1}{2n}}{4}\right)^n = \frac{1}{4} \lim (1 + \frac{1}{2n})^n
\]
Note that \((1 + \frac{1}{2^n})^{2n}\) is a subsequence of \((1 + \frac{1}{2^n})^n\) (take \(n_k = 2k\)). Since
the latter converges to \(e\), so does the former. Therefore, \(\lim(1 + \frac{1}{2^n})^n = \lim \sqrt[n]{(1 + \frac{1}{2^n})^{2n}} = \sqrt[n]{\lim(1 + \frac{1}{2^n})^{2n}} = \sqrt[e]{4}\). Threfore, the desired limit is
\[
\frac{\sqrt e}{4}
\]

(b) We first prove the following: if \(A \subseteq \mathbb{R}, \ c \in \mathbb{R}\) is a cluster point of \(A\), \(f : A \to \mathbb{R}\) and \(f(x) \geq 0\) for all \(x \in A\). Furthermore, we assume that \(\lim_{x \to c} f\) exists. Then \(\lim_{x \to c} \sqrt f\) exists, and is equal to \(\sqrt{\lim_{x \to c} f}\).

Proof. \(\lim_{x \to c} f \geq 0\) by a result in class, so \(\sqrt{\lim_{x \to c} f}\) exists. Let \((x_n)\) be an arbitrary sequence in \(A\) that converges to \(c\) such that \(c \notin \{x_n | n \in \mathbb{N}\}\). Then \((f(x_n))\) is a nonnegative sequence that converges to \(\lim_{x \to c} f\). Therefore, by a result in class about sequences, \((\sqrt f(x_n))\) that converges to \(\sqrt{\lim_{x \to c} f}\). Thus, by the sequential criterion for limits, \(\lim_{x \to c} \sqrt f\) exists, and is equal to \(\sqrt{\lim_{x \to c} f}\) as claimed. \[\square\]

For the problem, Since we are interested in \(x\) near 0, we may restrict our attention to the interval \(A := (-1/2, 1/2)\). Thus, \(1 + x\) and \(1 - x\) are positive on this interval, and so we calculate (using the above claim in the last step):

\[
\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x - x^2} = \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x - x^2} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \lim_{x \to 0} \frac{2x}{(x - x^2)(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{2}{(1-x)(\sqrt{1+x} + \sqrt{1-x})} = 2/(1+1) = 1
\]

6) Let \(\epsilon > 0\). Set \(M = |c| + 1 > 0\). Set \(\delta = \inf \{1, \frac{\epsilon}{M + M|c| + |c|^2}\} > 0\). We see that \(|x^3 - c^3| = |x - c||x^2 + xc + c^2| \leq |x - c|(|x|^2 + |x||c| + |c|^2)\). If \(|x - c| < \delta\), then \(|x - c| < 1\), and so \(|x - c| \leq |x - c| < 1\); implying that \(|x| < |c| + 1 = M\). Therefore, \(|x^3 - c^3| < |x - c|(M^2 + M|c| + |c|^2) < \frac{\epsilon}{M + M|c| + |c|^2}(M^2 + M|c| + |c|^2) = \epsilon\). Therefore, \(\lim_{x \to c} x^3 = c^3\).