EXAMPLE OF USE OF THE DEFINITION OF LIMIT OF A SEQUENCE

1. Show that
\[
\lim \left( \frac{2n}{n + 2} \right) = 2.
\]

Solution.
\[
\left| \frac{2n}{n + 2} - 2 \right| = \left| \frac{4}{n + 2} \right| \leq \frac{4}{n} \quad (\ast).
\]

Fix \( \epsilon > 0 \).
By the Archimedean property, there is \( N_\epsilon \) such that
\[
\frac{4}{\epsilon} < N_\epsilon.
\]

We thus have
\[
\frac{4}{n} < \epsilon \quad \forall n \geq K := N_\epsilon.
\]
By plugging this into (\ast), we see that, for every \( \epsilon > 0 \), there is a natural number \( K \) such that if \( n \geq K \), then
\[
\left| \frac{2n}{n + 2} - 2 \right| < \epsilon
\]
and this means that \( \lim \left( \frac{2n}{n+2} \right) = 2 \).

2. Prove that if \( (a_n) \to A \) and \( (b_n) \to B \), then
\[
(a_n + b_n) \to (A + B).
\]

Solution. By regrouping the terms and by the the Triangle Inequality:
\[
|a_n + b_n - (A + B)| \leq |a_n - A| + |b_n - B| \quad (\ast).
\]

Fix \( \epsilon > 0 \). By assumption there are \( K_1, K_2 \in \mathbb{N} \) such that
\[
|a_n - A| < \frac{\epsilon}{2}, \quad \forall n \geq K_1, \quad |b_n - B| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.
\]
Take \( K := \max \{K_1, K_2\} \). We have
\[
|a_n - A| + |b_n - B| < \epsilon, \quad \forall n \geq K \quad (\ast\ast).
\]
By combining (\ast) with (\ast\ast), we obtain that, for any \( \epsilon > 0 \), there is \( K \in \mathbb{N} \) such that
\[
|(a_n + b_n) - (A + B)| < \epsilon, \quad \forall n \geq K,
\]
that is, we have showed that \( (a_n + b_n) \to (A + B) \).