**Math 310** **MIDTERM II** **FALL 2016**

LAST NAME, NAME, ID:
RECITATION: (circle one) : 01 (Minoccheri) 02 (Yuan)

| **Problem** | **Points Possible** | **Points** |
|---|---|---|
| 1 | 40 | |
| 2 | 40 | |
| 3 | 30 | |
| 4 | 30 | |
| 5 | 30 | |
| 6 | 30 | |
| Total | 200 | |

**INSTRUCTIONS**:

1. Show your work. Use the correct notation. Use only the methods developed so far in the course (example: do not use methods from later chapters that you may have learned elsewhere). Correct answers without sufficient work, or not using the method required will receive minimal or no credit. If you use a theorem from the book, you need to tell us which one you are using: give its name (example: "dimension theorem"), or its statement (example: "any two bases have the same number of elements").
2. The point-values of each problem are clearly indicated.
3. Provide clearly written answers in the space provided. You can use the flip sides of the pages and page 8 as scrap paper. **Do not tear off any page. You must return all 8 pages.**
4. No books. No notes. Cell phones off and in backpack. No devices. No calculators.

1. (40 pts) Let $A, B$ be an $n \times n$ matrices and let $O$ denote the zero $n \times n$ matrix.

(a) Suppose that $A^2 = O$. Prove that $A$ is not invertible.

(b) Assume that $B \neq O$. Prove or disprove: $AB = O$ implies $A$ not invertible.

a) Assume that $A$ is invertible.
Then $A^{-1}A^2 = A^{-1}O$,

$A^{-1}A \cdot A = O$,

$I \cdot A = O$

$A = O$.

But this is a contradiction, since the zero matrix is not invertible (e.g. $rk(O) = 0 < n$ for any $n$).

b) Assume that $A$ is invertible.
Then $A^{-1}AB = A^{-1}O$

$IB = O$

$B = O$

contradiction, since $B \neq O$ by hypothesis.

Therefore the statement is <u>True</u>.

**2.** (40pts) Let $T : R^2 \to R^2$ be the linear transformation defined by setting:

$$T(a,b) = (a+b, a-b).$$

Let $\beta' = ((1,1),(1,0))$ and $\beta = ((0,1),(1,0))$ be two ordered bases of $R^2$. Determine

(a) $[T]_\beta$

(b) $[T]_{\beta'}$

(c) $Q = [I]_{\beta'}^{\beta}$.

a) $T(0,1) = (1,-1) \qquad T(1,0) = (1,1)$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a = -1,\ b = 1$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a = 1,\ b = 1$

$[T]_\beta = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

b) $T(1,1) = (2,0) \qquad T(1,0) = (1,1)$

$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a = 0,\ b = 2$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a = 1,\ b = 0$

$[T]_{\beta'} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a = 1,\ b = 1$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a = 0,\ b = 1$

$Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$

**3.** (30pts) Determine the ordered basis $\beta^*$ dual to the ordered basis of $R^3$ given by

$$\beta = ((1,0,1),(1,1,1),(0,0,1))$$

as linear combinations of the ordered basis $(x, y, z)$ dual to the standard basis in $R^3$.

$\beta^* = (f_1, f_2, f_3)$.

$f_1 = ax + by + cz$ for some $a, b, c \in R$.

$$\begin{cases} f_1(1,0,1) = a + c = 1 \longrightarrow a = 1 \\ f_1(1,1,1) = a+b+c = 0 \longrightarrow b = -a-c = -1 \\ f_1(0,0,1) = c = 0 \end{cases}$$

$f_2 = ax + by + cz$

$$\begin{cases} f_2(1,0,1) = a + c = 0 \longrightarrow a = 0 \\ a+b+c = 1 \longrightarrow b = 1 \\ c = 0 \end{cases}$$

$f_3 = ax + by + cz$

$$\begin{cases} a + c = 0 \longrightarrow a = -c = -1 \\ a+b+c = 0 \longrightarrow b = -a-c = 0 \\ c = 1 \end{cases}$$

Thus $f_1 = x - y$, $f_2 = y$, $f_3 = -x + z$.

**4.** (30 pts) Express the invertible matrix

$$\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

as a product of elementary matrices.

$$\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow[\underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}}_{= E_1}]{\cdot \text{ left}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow[\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{= E_2}]{\cdot \text{ left}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow[\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{= E_3}]{\cdot \text{ left}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$E_3 E_2 E_1 A = I$, so:

$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

**5.** (30pts) Let $A$ be an $n \times n$ matrix. Prove that $\operatorname{rank}(A) = n$ if and only if $A$ is invertible.

$\operatorname{rk}(A) = n \iff$ (by definition)

$\iff \operatorname{rk}(L_A) = n \iff$ (by definition)

$\iff \dim R(L_A) = n \iff$ (by rank-nullity thm.)

$\iff \dim R(L_A) = n$ and $\dim N(L_A) = 0 \iff$ (by thm: $L_A$ inj. iff $N(L_A) = \{0\}$)

$\iff L_A$ is injective and onto $\iff$

$\iff L_A$ is isomorphism $\iff$ (by thm.)

$\iff A$ is invertible.

**6.** (30pts) Use the "$K = \{s_0\} + K_H$" theorem on solution sets to find all the solutions of the system of equations:

$$x_1 + 2x_2 + x_3 = 4, \quad x_1 - x_2 - x_3 = -1.$$

- Find $K_H$.

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow[R_2 - R_1]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \end{bmatrix}$$

So $rk(A) = 2$, and thus $\dim K_H = 3 - 2 = 1$.

$$\begin{cases} x_1 = x_2 + x_3 \\ 3x_2 + 2x_3 = 0 \end{cases} \qquad \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \text{ is a solution.}$$

So $K_H = \text{span}\left\{\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}\right\}$.

- Find $s_0$.

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_1 - x_2 - x_3 = -1 \end{cases} \longrightarrow \begin{cases} 3x_2 + 2x_3 = 5 \\ x_1 = x_2 + x_3 - 1 \end{cases} \longrightarrow$$

$$\longrightarrow x_2 = 1, \quad x_3 = 1, \quad x_1 = 1.$$

$$s_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

- The general solution is:

$$K = \left\{ t \cdot \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ s.t. } t \in \mathbb{R} \right\}.$$