The Real numbers: Dedekind cuts

DEFINITION. A *(Dedekind) cut* γ is a set of rational numbers with the following properties:

1. if r is in γ and $r' < r$ is rational, then r' is in γ;
2. there is a rational number NOT in γ;
3. there is no maximum element in γ.

Example. The set of all rational numbers smaller that a FIXED rational number c is a cut; call it $\gamma(c)$. Because of this, rational numbers can be seen as a subset of the set of cuts.

Example. The set of rational numbers which are either negative or whose square is less than 2. This cut is not one of the previous examples ($\sqrt{2}$ is not rational!).

Example. Intuitively: given any real number d, the set of rational numbers smaller than d is a cut; call it $\gamma(d)$.

THE GREAT IDEA IS TO DEFINE THE REAL NUMBERS AS SOMETHING THAT DIVIDES RATIONAL NUMBERS INTO TWO SEPARATE PARTS (LEFT AND RIGHT!).

Theorem. *The set C of cuts admits the operations of*

$$+,\ -,\ \times,\ \textit{division by a nonzero cut.}$$

The relations of

$$<,\ \geq; >,\ \leq$$

make sense.
The set of all cuts C has the following property (called the property of having least upper bounds):
if C' is a set of cuts in C such that there is a rational number r NOT in any γ in c', then there exists a cut $\gamma(C')$ in C such that

$$\gamma \leq \gamma(C'),\ for\ all\ \gamma\ in\ C'$$

and $\gamma(C')$ is the SMALLEST cut in C with this property.

The cut $\gamma(C')$ is unique! It is called the least upper bound of C'.

We call the set of Dedekind cuts: THE REAL NUMBERS!!!

This property allows to define functions like $y = x^a$, or $y = 2^x$ or b^x, $b > 0$ for all real numbers etc: for example x^a is first defined for x rational and a integer; then for a rational; then for a real. Once you have defined x^a for x rational and a real, you can define it for x real!.

Group work: define $+$, $-$, $\times$ and division on the set C of Dedekind's cuts.