!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

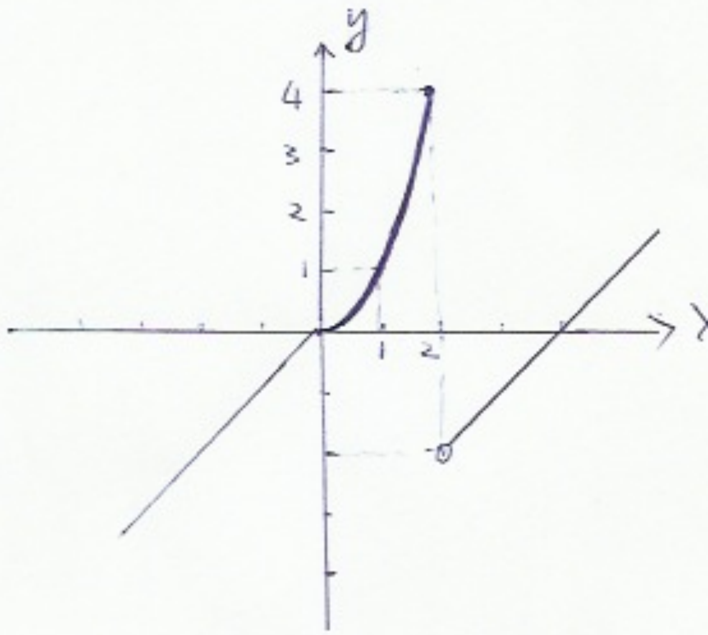
NAME : ID :

LECTURE N.

1. (40pts)

Sketch the graph of the function $f(x)$ and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} -|x| & x < 0 \\ x^2 & 0 \le x \le 2 \\ -4 + x & x > 2. \end{cases}$$



$\lim_{x\to a} f(x)$ exists for all a except for $a = 2$

$\left(\lim_{x\to 2^-} f(x) = 4 \neq \lim_{x\to 2^+} f(x) = -2 \right)$

2. (40pts)

a. State the limit law for the product of two functions $f(x)$ and $g(x)$.

Assume the following two limits exist:

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = M$$

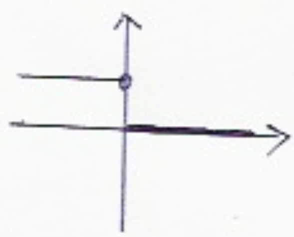
Then

$$\lim_{x \to a} \left(f(x) \cdot g(x)\right) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right) = L \cdot M.$$

b. Show by means of an example that $\lim_{x \to a}[f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$, nor $\lim_{x \to a} g(x)$ exist.

Example: $f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$

$g(x) = \begin{cases} 1 & x < 0 \\ 0, & x \ge 0. \end{cases}$



$f(x) \cdot g(x) \equiv 0.$

3. (45pts)

(a) If $f(x)$ and $g(x)$ are continuous with $f(1) = 2$ and $\lim_{x\to 1}[10f(x) - 5g(x)] = 5$, find $g(1)$.

Because $f(x)$ and $g(x)$ are continuous,

$$5 = \lim_{x\to 1}\left[10 f(x) - 5g(x)\right] = 10\cdot \lim_{x\to 1} f(x) - 5\cdot \lim_{x\to 1} g(x) = 10\cdot f(1) - 5\cdot g(1)$$

so $g(1) = 2f(1) - 1 = 2\times 2 - 1 = 3$.

(b) Is there a number that is exactly 1 more than its thirteenth power?

Yes. Reason: Such a number x satisfies

$$x = x^{13} + 1.$$

or equivalently, $x^{13} - x + 1 = 0$.

Consider the function $f(x) = x^{13} - x + 1$ on the interval $[-2, 1]$.

$f(-2) = (2)^{13} + 2 + 1 = -1024\times 3 + 3 < 0.$

$f(1) = 1^{13} - 1 + 1 = 1 > 0.$

f is a continuous function. By Intermediate Value Theorem, there is $c \in [-2, 1]$ such that

$$0 = f(x) = x^{13} - x + 1.$$

4. (45pts)

Calculate the following three limits (see the next page for the other two):
(a) $\lim_{x\to+\infty}(\sqrt{x^4+2x^2}-\sqrt{x^4+x})$.

$$\lim_{x\to+\infty}\left(\sqrt{x^4+2x^2}-\sqrt{x^4+x}\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{x^4+2x^2}-\sqrt{x^4+x}\right)\cdot\left(\sqrt{x^4+2x^2}+\sqrt{x^4+x}\right)}{\sqrt{x^4+2x^2}+\sqrt{x^4+x}}$$

$$=\lim_{x\to+\infty}\frac{(x^4+2x^2)-(x^4+x)}{\sqrt{x^4+2x^2}+\sqrt{x^4+x}}=\lim_{x\to+\infty}\frac{2x^2-x}{\sqrt{x^4+2x^2}+\sqrt{x^4+x}}$$

$$=\lim_{x\to+\infty}\frac{2-\frac{1}{x}}{\sqrt{1+\frac{2}{x^2}}+\sqrt{1+\frac{1}{x^3}}}=\frac{2-0}{\sqrt{1+0}+\sqrt{1+0}}=1.$$

(b) $\lim_{x\to+\infty} \frac{\cos^2 x}{x}$.

$$0 \le \cos^2 x \le 1$$

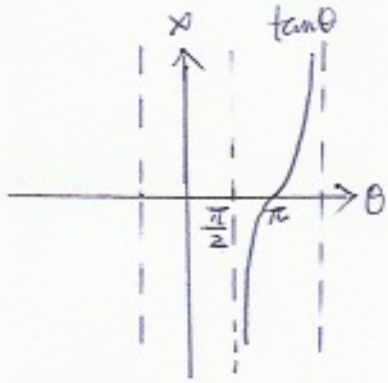
as $x \to +\infty$, $x > 0$. so $0 \le \frac{\cos^2 x}{x} \le \frac{1}{x}$.

$$\lim_{x\to+\infty} 0 = 0, \quad \lim_{x\to+\infty} \frac{1}{x} = 0.$$

By Squeeze Theorem,

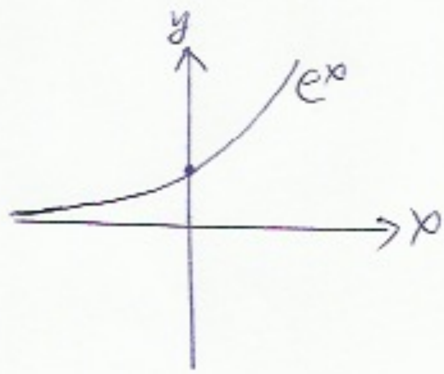
$$\lim_{x\to+\infty} \frac{\cos^2 x}{x} = 0.$$

(c) $\lim_{\theta\to\pi/2^+} e^{\tan\theta}$.



$$\lim_{\theta\to(\frac{\pi}{2})^+} \tan\theta = -\infty.$$

so $\lim_{\theta\to(\frac{\pi}{2})^+} e^{\tan\theta} = e^{-\infty} = 0$



5. (40pts)

(a) Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \sqrt{25 - x^2}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{25-(x+h)^2} - \sqrt{25-x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{25-(x+h)^2} - \sqrt{25-x^2}\right)\cdot\left(\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right)}{h\cdot\left(\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right)}$$

$$= \lim_{h\to 0} \frac{\left(25-(x+h)^2\right) - \left(25-x^2\right)}{h\left(\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right)} = \lim_{h\to 0} \frac{-2xh - h^2}{h\left(\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right)}$$

$$= \lim_{h\to 0} \frac{-2x-h}{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}} = -\frac{2x}{2\sqrt{25-x^2}} = -\frac{x}{\sqrt{25-x^2}}.$$

Domain of $f = [-5, 5]$, Domain of $f' = (-5, 5)$.

(b) Find the equation of the tangent line at $(3, f(3))$. Write your answer in the form $y = mx + b$.

$$(3, f(3)) = (3, \sqrt{25-3^2}) = (3, 4).$$

$$f'(3) = -\frac{3}{\sqrt{25-3^2}} = -\frac{3}{4}.$$

Tangent line: $y - 4 = -\frac{3}{4}(x-3)$.

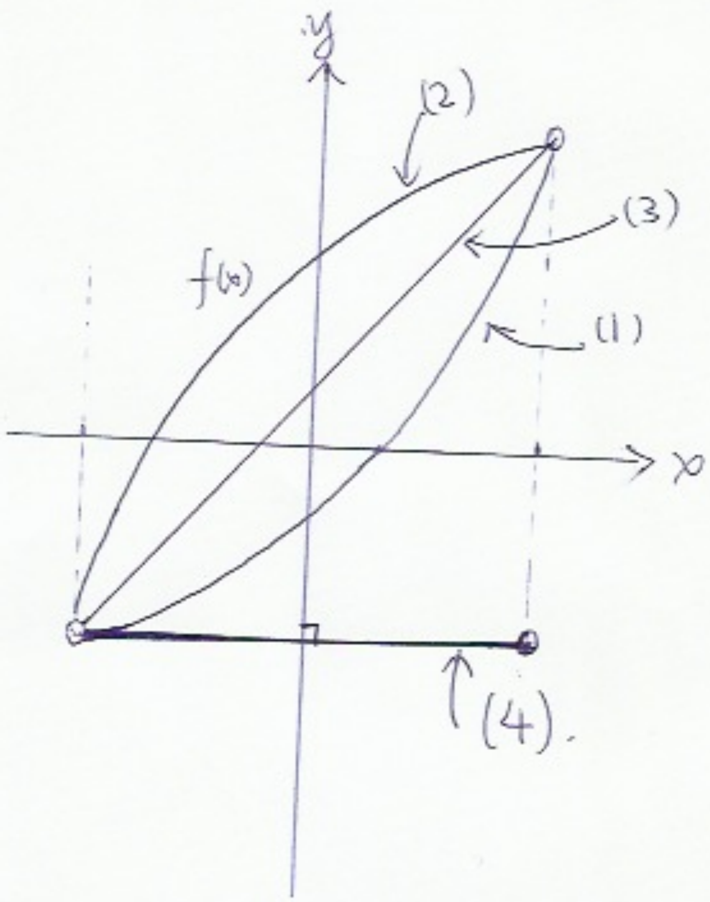
$$\Downarrow$$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

6. (40pts) This problem has two parts: part (a) on this page, part (b) on the next.

(a) Sketch the graph of a function on (-1,1) according to each of the following conditions: (label each of the graphs with the corresponding number (1), ..., (4))

(1) $f' > 0$, $f'' > 0$
(2) $f' > 0$, $f'' < 0$
(3) $f' > 0$, $f'' = 0$
(4) $f' = 0$.



(b) Sketch the graph of a function that satisfies all the following given conditions.

- $\lim_{x\to-\infty} f(x) = 1$, $f(-x) = -f(x)$.
- $f'(x) < 0$ if $|x| < 2$, $f'(x) > 0$ if $|x| > 2$, $f'(-2) = 0$.
- $f''(x) > 0$ if $x < -3$, $f''(x) < 0$ if $-3 < x < 0$,