1. Find the interval of convergence for the power series

$$\sum_{n=10}^{\infty} \frac{(3x+2)^n}{n^2}.$$

2. (a) Find the Maclaurin series of the function

$$f(x) = \frac{2}{3x-5}.$$

(b) Find its radius of convergence.

3. Use the Binomial series to find the Maclaurin series for $(1-2x)^{-3}$.

4. Use the Taylor series of the functions you already know to evaluate the sum

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

5. Find the Taylor series for $f(x) = \cos(x)$ centered at $\frac{\pi}{3}$.

6. Let $f(x) = 1 + x + 2x^2 + \frac{2}{3}x^3 + \frac{5}{2}x^4 + \frac{1}{15}x^7$.
(a) Find the 3rd degree Taylor polynomial ,$T_3(x)$, centered at 1 to approximate $f(x)$.
(b) Estimate the error in using this approximation on the interval [.5, 1.5].

7. (a) Approximate $e^{\frac{1}{2}}$ using the 3rd degree Taylor Polynomial for $f(x) = e^x$ centered at 0.

(b) Estimate the error in making this approximation.

8. Find the Maclaurin series for the function

$$f(x) = \int_0^x e^{-t^2}\,dt.$$

9. Find a power series representation for
$$f(x) = \ln\left(\frac{1+x}{1-x}\right).$$

1. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3x+2)^{n+1}}{(n+1)^2} \quad \frac{n^2}{(3x+2)^n}\right|$

$= \frac{n^2}{(n+1)^2} |3x+2|$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} |3x+2|$

$= \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 |3x+2|$

$= |3x+2|$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \iff |3x+2| < 1$

$\iff |x+\frac{2}{3}| < \frac{1}{3}$

$\iff -1 < x < -\frac{1}{3}$

Check endpoints:

$x = -1$: $\sum_{n=10}^{\infty} \frac{(3(-1)+2)^n}{n^2} = \sum_{n=10}^{\infty} \frac{(-1)^n}{n^2}$

converges by alternating series test.

$x = -\frac{1}{3}$: $\sum_{n=10}^{\infty} \frac{(3(-\frac{1}{3})+2)^n}{n^2} = \sum_{n=10}^{\infty} \frac{1}{n^2}$

converges by p-series test.

Interval of convergence $= [-1, -\frac{1}{3}]$.

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(a) $$f(x) = \frac{2}{5\left(\frac{3x}{5}-1\right)} = \frac{-2}{5}\left(\frac{1}{1-\frac{3x}{5}}\right)$$

$$= \frac{-2}{5}\sum_{n=0}^{\infty}\left(\frac{+3x}{5}\right)^n$$

$$= \frac{-2}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n x^n$$

(b) $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{3}{5}\right)^{n+1} x^{n+1} \left(\frac{3}{5}\right)^{-n} x^{-n}\right|$$

$$= \frac{3}{5}|x|$$

$$\lim_{n\to\infty} \frac{3}{5}|x| < 1 \iff |x| < \frac{5}{3}$$

Radius of convergence $= \frac{5}{3}$

$\underline{3}$ $\quad (1-2x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} x^n$

For $n \geq 1$,

$$\binom{-3}{n} = \frac{1}{n!}(-3)(-4)(-5)\cdots(-3-n+1)$$

$$= \frac{1}{n!}(-3)(-4)(-5)\cdots(-n-2)$$

$$= \frac{(-1)^n}{n!}\, 3\cdot 4\cdot 5 \cdots (n+2)$$

$$= (-1)^n \frac{\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdots(n+1)(n+2)}{1\cdot 2\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdots\cancel{n}}$$

$$= (-1)^n \frac{(n+1)(n+2)}{1\cdot 2}$$

Therefore,

$$(1-2x)^{-3} = \binom{-3}{0}x^0 + \sum_{n=1}^{\infty}\binom{-3}{n}x^n$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{(n+1)(n+2)}{2} x^n$$

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$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all } x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Set $x = 1$.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \tan^{-1}(1)$$

$$= \frac{\pi}{4}.$$

b) by definition the Taylor series of $f(x) = \cos x$ Centred at $x = \frac{\pi}{3}$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{3})}{n!} (x - \frac{\pi}{3})^n$$

So we need to find $f^{(n)}(\frac{\pi}{3})$, $\frac{\pi}{3} = \frac{180^\circ}{3} = 60^\circ$

$f(x) = \cos x \implies f(\frac{\pi}{3}) = \frac{1}{2}$

$f'(x) = -\sin x \implies f'(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x \implies f''(\frac{\pi}{3}) = \frac{-1}{2}$

$f^{(3)}(x) = \sin x \implies f'''(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \cos x$ $\quad \frac{1}{2}$

$f^{(5)}(x) = -\sin x$ $\quad -\frac{\sqrt{3}}{2}$

$f^{(6)}(x) = -\cos x$ $\quad -\frac{1}{2}$

$f^{(7)}(x) = \sin x$ $\quad \frac{\sqrt{3}}{2}$

the same patter of four functions repeats.

So

Taylor series $= f(\frac{\pi}{3}) + \frac{f'(\frac{\pi}{3})}{1!}(x - \frac{\pi}{3}) + \frac{f''(\frac{\pi}{3})}{2!}(x - \frac{\pi}{3})^2 + \frac{f'''(\frac{\pi}{3})}{3!}(x - \frac{\pi}{3})^3 + \cdots$

$$= \frac{1}{2} + \frac{-\frac{\sqrt{3}}{2}}{1!}(x - \frac{\pi}{3}) + \frac{-\frac{1}{2}}{2!}(x - \frac{\pi}{3})^2 + \frac{\frac{\sqrt{3}}{2}}{3!}(x - \frac{\pi}{3})^3 + \cdots$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{4}(x - \frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x - \frac{\pi}{3})^3 + \cdots$$

$\boxed{6}$ $f(x) = 1 + x + 2x^3 + \frac{2}{3}x^3 - \frac{5}{2}x^4 + \frac{1}{15}x^7$

$$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$f(x) = 1 + x + 2x^3 + \frac{2}{3}x^3 - \frac{5}{2}x^4 + \frac{1}{15}x^7$

$f'(x) = 1 + 6x + 2x^2 - 10x^3 + \frac{7}{15}x^6$

$f''(x) = 6 + 4x - 30x^2 + \frac{14}{5}x^5$

$f'''(x) = 4 - 60x + 14x^4$

$f^{(4)}(x) = -60 + 56x^3$

$f(1) = 1 + 1 + 2 + \frac{2}{3} - \frac{5}{2} + \frac{1}{15}$
$= \frac{67}{30}$

$f'(1) = 1 + 6 + 2 - 10 + \frac{7}{15} = -$

$f'''(1) = 4 - 60 + 14 = -4$

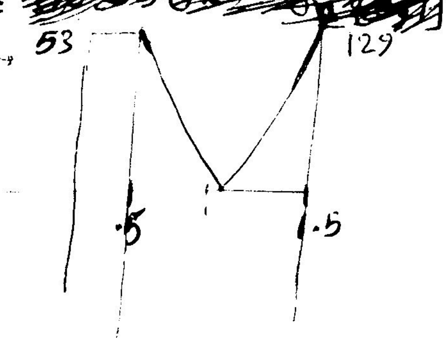
$f''(1) = 6 + 4 - 30 + \frac{14}{5} = \frac{-8}{5}$

$$\Rightarrow T_3(x) = \frac{67}{30} + \frac{-\frac{8}{15}}{1}(x-1) + \frac{-\frac{86}{5}}{2!}(x-1)^2 + \frac{-42}{3!}(x-1)^3$$

$$= \frac{67}{30} - \frac{8}{15}(x-1) - \frac{86}{10}(x-1)^2 - 7(x-1)^3$$

if $|f^{(4)}(x)| \leq M$ on $[0,1]$ then $|\text{error}| \leq \frac{M}{4!}(x-1)^4$

$|f^{(4)}(x)| = |-60 + 56x^3| =$



$\Rightarrow |f^{(4)}(x)| \leq 129$

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then $|\text{error}| \leq \frac{129}{4!} |x-1|^4$

if x is in $[.5, 1.5]$ then $|x-1| < \frac{1}{2}$ so

$$\frac{129}{4!} \cdot |x-1|^4 \leq \frac{129}{4!} \times \left(\frac{1}{2}\right)^4 = \frac{129}{384}$$

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7 $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

a) $\quad T_3\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{48} = \frac{77}{48}$

b) $\quad f(x) = e^x \rightarrow f'(x) = e^x, \ldots, f^{(4)}(x) = e^x$ on $[0, \frac{1}{2}]$

$|f^{(4)}(x)| \leq e^{1/2}$

$$\Rightarrow |\text{error}| = |R_3(x)| \leq \frac{e^{1/2}}{4!} \left|x - \frac{1}{2}\right|^4$$

$$\leq \frac{e^{1/2}}{4!} \left(\frac{1}{2}\right)^4$$

the above inequality we use the fact that for x in $[0, \frac{1}{2}]$

$|x - \frac{1}{2}| \leq \frac{1}{2}$

so if we let $e^{1/2} < e^1 \approx 2.7$

$$\text{error} \leq \frac{2.7}{24} \cdot \frac{1}{16} = \frac{2.7}{384}$$

the Maclaurin series for $f(x) = \int_0^x e^{-t^2}\,dt$

by fundamental theorem of Calculus

$$f'(x) = e^{-x^2}$$

as we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

So $$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

so if $$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

then $$f(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} + C$$

$$f(x) = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!} + C$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + C$$

to find c we plug in $x = 0$ in the above equation.

and get $f(0) = C$ but from definition of $f(x)$

$f(0) = \int_0^0 e^{-t^2}\,dt = 0$ so $C = 0$

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$$f(x) = \ln(1+x) - \ln(1-x)$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$ (see p. 602)

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\ln(1+x) = \ln(1-(-x))$$

$$= -(-x) - \frac{(-x)^2}{2} - \frac{(-x)^3}{3} - \frac{(-x)^4}{4} - \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$f(x) = (x)-(-x) + \left(\frac{-x^2}{2}\right) - \left(-\frac{x^2}{2}\right) + \left(\frac{x^3}{3}\right) - \left(-\frac{x^3}{3}\right) + \cdots$$

$$= 2x$$

$$\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right)$$

$$- \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \right)$$

$$2x \quad + 2\frac{x^3}{3} \quad + 2\frac{x^5}{5} \quad + \cdots$$

$$f(x) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$