The Symplectic Topology of Stein Manifolds

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Preface

All the substantial work in the thesis is my own; there is nothing done in collaboration.

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Summary

Exotic Stein manifolds are Stein manifolds diffeomorphic to $\mathbb{R}^{2k}$ which cannot be embedded symplectically into the Stein manifold $\mathbb{C}^k$. In this thesis we prove two results about exotic Stein manifolds. The first result shows us that in each complex dimension > 2, there exists an exotic Stein manifold which is not symplectomorphic to any finite type Stein manifold. The second result states that in each dimension > 3 there are infinitely many finite type exotic Stein manifolds which are pairwise distinct as symplectic manifolds.

Contents

1. Introduction

This thesis is about the symplectic topology of Stein manifolds. A Stein manifold is a triple (V, J, ϕ) where $\phi : V \to \mathbb{R}$ is an exhausting (i.e. proper and bounded from below) plurisubharmonic function and J is an integrable complex structure. Here plurisubharmonic means that $\omega := -dd^c\phi$ is a symplectic form where d^c is defined by $d^c(a)(X) := da(JX)$. These are examples of symplectic manifolds called exact symplectic manifolds. An exact symplectic manifold (W, θ) is a manifold W with a 1-form θ such that $d\theta$ is a symplectic form. In the case of Stein manifolds, $\theta := -d^c\phi$. There are many choices of ϕ and J, so we wish to study these manifolds up to some sort of deformation, or up to exact symplectomorphism. An exact symplectomorphism between two exact symplectic manifolds (M_1, θ_1), (M_2, θ_2) is a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^*(\theta_2) = \theta_1 + dR$ where $R : M_1 \to \mathbb{R}$ is a smooth function. The kind of deformation we want is called Stein deformation equivalence and is defined later in 2.8. We define an equivalence relation $\sim$ on Stein manifolds by: $A \sim B$ if there exists a sequence of Stein manifolds $F_0, F_1, \cdots, F_n$ such that

(1) $F_0 = A$ and $F_n = B$
(2) F_i is either exact symplectomorphic or Stein deformation equivalent to F_{i+1}.

If we have a complex manifold (V, J) with a holomorphic embedding $i : V \hookrightarrow \mathbb{C}^N$ for some large N such that the image of i is closed, then $\phi := i^*(\sum_{i=1}^N |z_k|^2)$ is an exhausting plurisubharmonic function. This means that (V, J, ϕ) is a Stein manifold. This also means that any smooth affine variety has a Stein structure. In fact all Stein manifolds can be holomorphically embedded into $\mathbb{C}^N$ (See for instance [13]).

If a manifold V carries a Stein structure, then there are restrictions to its topology. We can perturb the associated exhausting plurisubharmonic function ϕ so that it becomes a Morse function. From now on we will always assume these exhausting plurisubharmonic functions are Morse functions. It turns out that the index of the critical points of ϕ is at most n where $2n$ is the real dimension of V. Hence V has the homotopy type of an n dimensional CW complex (see [24]). This is because the stable manifold coming from a critical point of ϕ is actually isotropic (i.e. restricting ω to this manifold is 0) and therefore must have dimension $\leq n$. Conversely we also have the following theorem due to Eliashberg:

Theorem. *([10] [7, Theorem 9.4]) Let W be a manifold of real dimension $2n > 4$ with an almost complex (not necessarily integrable) structure J and an exhausting Morse function $\phi : W \to \mathbb{R}$ such that all the critical points of ϕ have index $\leq n$. Then J is homotopic to an integrable complex structure J_0 such that (W, J_0, ϕ) is a Stein structure.*

In dimension 4 we have a slightly weaker result by Gompf [18]. This says that W is homeomorphic to a Stein manifold if and only if it is the interior of a handlebody with handles of index less than or equal to 2.

Another interesting question is, given a manifold W, what kind of Stein structures can it carry and how many? We can define an invariant $m(W, J, \phi)$ of the $\sim$-equivalence class of Stein structures taking values in $\mathbb{N} \cup \{\infty\}$. For a Stein structure (W, J', ϕ') on W we define $c(W, J', \phi') \in \mathbb{N} \cup \{\infty\}$ as the number of critical points of ϕ'. We define $m(W, J, \phi)$ as the infimum of $c(W, J', \phi')$ over all Stein structures (W, J', ϕ') which are $\sim$-equivalent to (W, J, ϕ). We have a Stein manifold $(\mathbb{R}^{2n}, J_{\text{std}}, \phi_{\text{std}})$ where J_{std} is the standard complex structure on $\mathbb{C}^n$ and $\phi_{\text{std}} := i^*(\sum_{i=1}^N |z_k|^2)$ is an exhausting plurisubharmonic function. Every exhausting plurisubharmonic function must have at least one critical point, hence $m(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}}) = 1$. Conversely if we had a Stein structure (W, J, ϕ) with $m(W, J, \phi) = 1$, then this is in fact Stein deformation equivalent to $(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}})$. Eliashberg asked in [11] whether there are Stein structures $(\mathbb{R}^{2n}, J, \phi)$ such that any Stein structure deformation equivalent to $(\mathbb{R}^{2n}, J, \phi)$ has at least 3 critical points. He also constructed a candidate for a Stein manifold with this property. It is easy to construct Stein manifolds that are not symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\text{std}})$ for reasons to do with volume. For instance the unit disc is not symplectomorphic to $\mathbb{C}$. We can also construct Stein domains of the same volume or of infinite volume but which are not symplectomorphic. For instance, in [12, Prop 3.4.A] we have a Stein manifold constructed as follows: Let $x_1, y_1, \ldots, x_n, y_n$ be real coordinates for $\mathbb{C}^n$. Let B be the unit ball with centre 0 in $\mathbb{C}^n$, and let $H := \{x_1 = 0\} \setminus B$. Then $\mathbb{C}^n \setminus H$ is a Stein manifold which is not symplectomorphic to $(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}})$ by the camel problem. This Stein manifold and $(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}})$ both have infinite volume. But all these manifolds are Stein deformation equivalent to $(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}})$. There are other symplectic structures on $\mathbb{R}^{2n}$ which are very different from $(\mathbb{R}^{2n}, \omega_{\text{std}})$ but the problem is they are not Stein. Generally they don't behave well at infinity. For instance in [19, Corollary 0.4.A_2'], Gromov shows that there exists symplectic forms on $\mathbb{R}^{2n}$ which cannot be symplectically embedded into $(\mathbb{R}^{2n}, \omega_{\text{std}})$. But these examples are not shown to be Stein.

We can also ask the slightly stronger question: is there a Stein structure $(\mathbb{R}^{2n}, J, \phi)$ with $m(\mathbb{R}^{2n}, J, \phi) > 2$ (this question is stronger as we are dealing with $\sim$-equivalence and not just Stein deformation equivalence). An equivalent formulation of this question would be: are there any Stein manifolds diffeormorphic to $\mathbb{R}^{2n}$ but not $\sim$-equivalent to $(\mathbb{R}^{2n}, \omega_{\text{std}})$? Using results by Eliashberg, in real dimension 4 there are no Stein manifolds $(\mathbb{R}^4, J, \phi)$ with $\infty > m(\mathbb{R}^4, J, \phi) > 2$, i.e. there are no finite type Stein surfaces diffeomorphic to $\mathbb{R}^4$ but not $\sim$-equivalent to $(\mathbb{R}^4, \omega_{\text{std}})$ (see the introduction to [34]). It is unknown whether there are Stein manifolds $(\mathbb{R}^4, J, \phi)$ with $m(\mathbb{R}^4, J, \phi) = \infty$ (i.e. non-finite type Stein manifolds diffeomorphic to $\mathbb{R}^4$). The closest we get to such a result is [18]. The case $n = 2k$ $(k > 1)$ is answered in [34] where Seidel and Smith show that there are finite type Stein structures $(\mathbb{R}^{2n}, J, \phi)$ which are not $\sim$-equivalent to $(\mathbb{R}^{2n}, \omega_{\text{std}})$. These satisfy $\infty > m(\mathbb{R}^{2n}, J, \phi) > 2$. They show this by constructing an affine variety which has a Lagrangian torus which cannot be moved off itself by a Hamiltonian isotopy. In fact they show that none of these Stein manifolds can be embedded in a subcritical Stein manifold (i.e. a Stein manifold whose critical points have index at most $n - 1$). In this thesis we will show a similar result for all $n \geq 3$ (Theorem 1.2 covers the case $n > 3$ and section 3.1 covers $n = 3$).

We can also ask if there are Stein structures for which this invariant is infinite (i.e. if there are non-finite type Stein manifolds). We say that a Stein manifold is of finite type if it has a Stein function ϕ with finitely many critical points, each of which is non-degenerate. It is easy to find a Stein manifold (W, J, ϕ) with $m(W, J, \phi)$ equal to infinity (i.e. it is not $\sim$-equivalent to any finite type Stein manifold). For instance we could consider an infinite genus surface. This Stein manifold is not $\sim$-equivalent to any finite type manifold for topological reasons, as it is not homotopic to a finite CW-complex so it cannot even admit a Morse function with finitely many critical points. In complex dimension 2, it is also possible to construct Stein manifolds (W, J, ϕ) homeomorphic to $\mathbb{R}^4$ but with $m(W, J, \phi) = \infty$ (i.e. they are not finite type). In [18], Gompf constructs uncountably many of these Stein manifolds which are homeomorphic to $\mathbb{R}^4$, but they are pairwise non-diffeomorphic. These manifolds are not diffeomorphic to any finite type Stein manifold for the following reason: If they were diffeomorphic to a finite type Stein manifold, they would admit a proper Morse function with finitely many critical points. But in the introduction of [18] near the top of page 622, Gompf says that no such function exists on these manifolds. In particular these manifolds cannot be $\sim$-equivalent to any finite type Stein manifold for reasons to do with the diffeomorphism type of these manifolds. The first main result of this thesis is the following:

Theorem 1.1. *Let $k \geq 3$. There exists a Stein manifold M diffeomorphic to $\mathbb{R}^{2k}$ such that $M \nsim N$ for any finite type Stein manifold N.*

The above theorem shows that there are Stein manifolds not $\sim$-equivalent to any finite type Stein manifold for purely symplectic reasons. The theorem and the author's proof were published in [32] by Seidel.

Another question is: how many $\sim$-equivalence classes of Stein structures are there on a given manifold M? Eliashberg's theorem shows which manifolds have at least 1 $\sim$-equivalence class. In complex dimension $n = 4 + 2k$ where $k \geq 0$, we have already shown there are at least 3 $\sim$-equivalence classes of Stein structures on $\mathbb{R}^{2n}$. We have the standard Stein structure $(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}})$ with $m(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}}) = 1$. We also have a Stein structure $(\mathbb{R}^{2n}, J, \phi)$ with $\infty > m(\mathbb{R}^{2n}, J, \phi) > 2$. Finally, we have a Stein structure $(\mathbb{R}^{2n}, J', \phi')$ with $m(\mathbb{R}^{2n}, J', \phi') = \infty$. In fact we can do better than this: the second result of this thesis is the following theorem:

Theorem 1.2. *Let $k \geq 4$. There exists a family of finite type Stein manifolds X_i diffeomorphic to $\mathbb{R}^{2k}$ indexed by $i \in \mathbb{N}$ such that*

$$i \neq j \Rightarrow X_i \nsim X_j.$$

We also have the following corollary:

Corollary 1.3. *Let M be a compact manifold of dimension 4 or higher. There exists a family of finite type Stein manifolds X_i^M diffeomorphic to T^*M indexed by $i \in \mathbb{N}$ such that*

$$i \neq j \Rightarrow X_i^M \nsim X_j^M.$$

We will prove this corollary at the end subsection 1.2. Theorem 1.1 is stronger than previous results in two ways:

(1) we give examples of finite type exotic Stein manifolds in all complex dimensions ≥ 4 (not just in dimension $4 + 2k$ where $k \geq 0$);
(2) we also show there are countably many pairwise distinct examples in each of these dimensions.

(1) is straightforward but (2) is much harder and involves various new ideas. It is also possible to show that these manifolds cannot be embedded in a subcritical Stein manifold (Corollary 12.5). We hope to address the question of whether Theorem 1.2 holds in dimension 6 in future work.

Any finite type Stein manifold (W, J, ϕ) has a cylindrical end. This means that outside some compact set $K \subset W$, W is exact symplectomorphic to $(A \times [1, \infty), r\alpha)$ where r is the coordinate for $[1, \infty)$ and α is a contact form on A. We call A the contact boundary of W and we will write $\partial W := A$. The boundary of $(\mathbb{C}^n, J_{\text{std}}, \phi_{\text{std}})$ is the standard contact structure α_{std} on the sphere. For $n \geq 3$, the contact boundary of any Stein structure on $\mathbb{R}^{2n}$ is diffeomorphic to a $2n-1$ dimensional sphere. We can ask how many such spheres there are up to contactomorphism. All contact structures that can be filled with a Stein domain diffeomorphic to the unit ball have a hyperplane field which is homotopic to the standard contact hyperplane field (see [33]). The reason why is as follows: Let (R, J_R, ϕ_R) be a Stein manifold diffeomorphic to $\mathbb{R}^{2n}$. Define $\theta := -d^c\phi_R$ and $\omega := d\theta$. We can choose a Morse function $H : R \to \mathbb{R}$ such that it has only one critical point corresponding to its minimum and such that $H = r$ on the cylindrical end $\partial R \times [1, \infty)$. Let X_H be the ω orthogonal vector field to dH. We have that on the cylindrical end $\partial R \times [1, \infty)$, the contact hyperplane field is the ω-orthogonal hyperplane field to the 2-plane field spanned by ∇H and X_H. If c is the minimal value of H, then we can ensure that $(V := H^{-1}(c+\epsilon), \theta|_V)$ is the standard contact sphere for sufficiently small ϵ. Hence we get that the contact hyperplane field on $H^{-1}(1) = \partial R \times \{1\}$ is homotopic to the standard hyperplane field on $H^{-1}(c + \epsilon) = V$. This means that we cannot distinguish such contact structures purely for reasons to do with the homotopy class of the hyperplane field associated to the contact form. If we have a Stein manifold $(\mathbb{R}^{2n}, J, \phi)$ constructed as in [34], Theorem 1.2 or section 3.1, then [32, Corollary 6.5] tells us that its boundary is not contactomorphic to α_{std}. At the moment we can only find two such contact structures on the $2n-1$ dimensional sphere. We will study whether we can find infinitely many contact boundaries on Stein manifolds diffeomorphic to $\mathbb{R}^{2n}$ up to contactomorphism in future work. In Theorem 1.2 we used an invariant called symplectic homology to distinguish these manifolds. But this is not an invariant of the contact boundary. For instance the boundary of $\mathbb{C}$ is contactomorphic to the boundary of a once punctured Riemann surface of genus greater than zero, but the first manifold has symplectic homology 0 while the other has non-trivial symplectic homology. If we had some numerical invariant of the Stein filling of a contact boundary, and we took the supremum of this invariant over all Stein fillings, then this would be an invariant of the contact boundary. The numerical invariant of the Stein filling we hope to use is related to the rank of a localized version of a group called negative equivariant symplectic homology.

We have shown that there is a countably infinite number of finite type Stein manifolds diffeomorphic to $\mathbb{R}^{2n}$ ($n > 3$). There are uncountably many Stein manifolds not $\sim$-equivalent to finite type ones (such as the examples in [18]), so it is natural to ask whether there are uncountably many Stein manifolds diffeomorphic to $\mathbb{R}^{2n}$. Most of these Stein manifolds have to be of non-finite type. Again we will address this question in a future paper.

There are many completely unknown questions related to Stein manifolds. Symplectic homology of convex symplectic manifolds (in particular Stein manifolds) was developed in [15], [8], [9], [39]. It was used to show for instance that the Weinstein conjecture was true for subcritical Stein manifolds (The Weinstein conjecture for Stein manifolds states that the contact boundary of any Stein manifold has at least one Reeb orbit where a Reeb orbit is an embedded circle in a contact manifold (C, α) whose tangent space is in the kernel of $d\alpha$ viewed as a map $TC \to T^*C$). The chain complex for symplectic homology $SH_*(M)$ involves (roughly) $H^{n-*}(M)$ and Reeb orbits of the boundary of M (counted twice). So, if you could show for instance that $SH_*(M) \neq H^{n-*}(M)$, then there must be Reeb orbits on the boundary. We do not know how to calculate symplectic homology for most Stein manifolds. We can calculate this invariant completely for a very small class of these manifolds. We know that symplectic homology $SH_*(W)$ of a subcritical Stein manifold is 0, and that $SH_*(T^*M) = H_*(LM)$ where LM is the loopspace of M. We have a Künneth formula for symplectic homology [26], so we can calculate it for products of these manifolds. We know what symplectic homology is for multiply punctured Riemann surfaces. There are a few more Stein manifolds for which we know a bit about symplectic homology such as examples where we know that it is non-trivial [32, Section 5], or examples (other than the ones mentioned) where it has finitely many idempotents (sections 5.2 and 5.3).

In [32, Section 6] it is shown that it is impossible to calculate the rank of $SH_{2n}(M)$ if we are given a handle decomposition of a general Stein manifold M. In particular it is possible to algorithmically construct a list of simply connected Stein manifolds $M_1, M_2, \ldots$ such that the set $\{i : M_i \sim M_1\}$ is impossible to construct with a computer. This means that in general it might be very difficult to study Stein structures.

1.1. **Sketch proof of the first theorem.** Here we give an outline of the proof of theorem 1.1. For each Stein manifold Y with a trivialisation of the canonical bundle, we have an integer graded commutative $\mathbb{Z}/2\mathbb{Z}$ algebra $SH_{n+*}(Y)$ where $SH_*(Y)$ is called symplectic homology [1]. If Y_1 and Y_2 are Stein manifolds with $Y_1 \sim Y_2$, then $SH_*(Y_1) = SH_*(Y_2)$ (see [32, Section 7]). If we have a compact codimension 0 exact convex submanifold W of Y, then there is a map $SH_*(Y) \to SH_*(W)$ called the transfer map. An exact submanifold is an embedding $i : W \hookrightarrow Y$ such that i is an exact symplectomorphism onto its image. Compact convex symplectic manifolds will be defined later in section 2.1. All we need to know here is that Stein domains are examples of compact convex symplectic manifolds.

First of all we construct a finite type Stein manifold M_k of complex dimension $k \geq 3$ diffeomorphic to Euclidean space such that $SH_*(M_k) \neq 0$. If $k \geq 4$, then these examples will be constructed in the proof of the second theorem which also shows that $SH_*(M_k) \neq 0$. We construct M_3 separately. We will use technology from the proof of the second theorem 1.2 combined with results from [32, Section 5] to show that $SH_*(M_3) \neq 0$.

If we have two Stein manifolds A and B, then it is possible to construct their *end connected sum* $A \#_e B$ (see 2.10). Roughly what we do here is join A and B with a 1-handle, and then extend the Stein structure over this handle. If we have a family of Stein manifolds A_i, $i \in \mathbb{N}$, then we can also form their end connect sum $\#_{e}{}_{i=1}^{\infty} A_i$. The good thing about end connect sums is that symplectic homology of the end connect sum of a family of Stein manifolds is the product of the symplectic homology rings of all the Stein manifolds in that family. We define M_k^∞ to be the infinite end connect sum $\#_{e}{}_{i=1}^{\infty} M_k$.

We now wish to show that M_k^∞ is not $\sim$-equivalent to any finite type Stein manifold. Suppose for a contradiction that it is. Suppose that it is exact symplectomorphic to a finite type Stein manifold, then there exists a compact codimension 0 exact convex submanifold W of M_k^∞ such that the transfer map $SH_*(M_k^\infty) \hookrightarrow SH_*(W)$ is an injection. The manifold M_k^∞ is a union of Stein domains M_k^j, $j \in \mathbb{N}$ where M_k^j is the j-fold end connect sum

[1]With our convention, the pair-of-pants product makes SH_* (and not SH^*) a unital ring.

of M_k. We have $SH_*(M_k^j) = \prod_{i=1}^j SH_*(M_k)$ for $j \in \mathbb{N} \cup \{\infty\}$. For a large enough $j \in \mathbb{N}$, there exists M_k^j such that $W \subset M_k^j$. Hence we get a sequence of maps:

$$SH_*(M_k^\infty) \to SH_*(M_k^{j+1}) \to SH_*(M_k^j) \to SH_*(W).$$

The composition of all these maps is an injection. But the middle map $SH_*(M_k^{j+1}) \to SH_*(M_k^j)$ is a projection $\prod_{i=1}^{j+1} SH_*(M_k) \twoheadrightarrow \prod_{i=1}^j SH_*(M_k)$ with kernel $SH_*(M_k) \neq 0$. Also the first map is surjective because $SH_*(M_k^\infty)$ is an infinite end connect sum. But this means that the transfer map

$$SH_*(M_k^\infty) \hookrightarrow SH_*(W)$$

has non-trivial kernel, contradiction. Hence M_k^∞ is not symplectomorphic to any finite type Stein manifold. A small extension of this argument shows us that M_k^∞ is not $\sim$-equivalent to any finite type Stein manifold.

1.2. **Sketch proof of the second theorem.** We will only consider the theorem in dimension 8, as the higher dimensional case is similar. We will first construct an example of a family of Stein manifolds $(X_n)_{n\in\mathbb{N}}$ as in Theorem 1.2 in dimension 8. Let $V := \{x^7 + y^2 + z^2 + w^2 = 0\} \subset \mathbb{C}^4$ and consider a smooth point, say $p := (0, 0, 1, i) \in V$. Let H be the blowup of $\mathbb{C}^4$ at p. Then $X := H \setminus \widetilde{V}$ is a Stein manifold where $\widetilde{V}$ is the proper transform of V. The variety X is called the Kaliman modification of $(\mathbb{C}^4, V, p)$. We will think of this modification in two stages:

(1) Cut out the hypersurface V in $\mathbb{C}^4$ to get $Z := \mathbb{C}^4 \setminus V$.
(2) Blow up Z at infinity to get X.

In real dimension 4, operation (2) attaches a 2-handle along a knot which is transverse to the contact structure. We let $X_n := \#_{i=1}^n X$ be our family of Stein manifolds.

For each Stein manifold Y, we can define another invariant $i(Y)$ which is the number of idempotents of $SH_*(Y)$ (this invariant might be infinite). Hence all we need to do is show that for $i \neq j$, $i(X_i) \neq i(X_j)$. We have that $SH_*(X_n) = \prod_{i=1}^n SH_*(X)$, and hence $i(X_n) = i(X)^n$. So all we need to do is show that $1 < i(X) < \infty$. If $SH_*(X) \neq 0$, $i(X) > 1$ since we have 0 and 1; but since $SH_*(X)$ can a priori be infinite dimensional in each degree, finiteness of $i(X)$ is much harder. Most of the work in this thesis involves proving $i(X) < \infty$.

For any Stein manifold Y, $SH_*(Y)$ is $\mathbb{Z}$ graded by the Robbin-Salamon index (or the Conley-Zehnder index taken with negative sign). The group $SH_*(Y)$ has a ring structure making $SH_{n+*}(Y)$ into a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. This ring is also graded by $H_1(Y)$. Hence idempotents in $SH_{n+*}(Y)$ have Robbin-Salamon index n and are in the torsion part of $H_1(Y)$. The problem for our example is that $H_1(X) = 0$. In order to find out which elements of $SH_{n+*}(X)$ are idempotents, we will show that $SH_*(X)$ is isomorphic as a ring to $SH_*(Z)$ where $Z = \mathbb{C}^4 \setminus V$ was defined above. Because Z is so much simpler than X and $H_1(Z) \neq 0$, it is possible by a direct calculation to show that $SH_{n+*}(Z)$ has finitely many idempotents.

Proving that $SH_*(X) \cong SH_*(Z)$ relies on the following theorem. This theorem is the heart of the proof. We let $E' \to \mathbb{C}$, $E'' \to \mathbb{C}$ be Lefschetz fibrations, and F' (resp. F'') be smooth fibres of E' (resp. E''). Let F' and F'' be Stein domains with F'' a holomorphic and symplectic submanifold of F'.

Theorem 1.4. *Suppose E' and E'' satisfy the following properties:*

1. *E'' is a subfibration of E'.*
2. *The support of all the monodromy maps of E' are contained in the interior of E''.*
3. *Any holomorphic curve in F' with boundary inside F'' must be contained in F''.*

Then $SH_*(E') \cong SH_*(E'')$.

Remark 1: There exist Lefschetz fibrations E', E'' with the above properties such that $SH_*(E') \cong SH_*(X)$ and $SH_*(E'') \cong SH_*(Z)$. This is because we can choose an algebraic Lefschetz fibration on Z where the closures of all the fibres pass through p. Then blowing up Z at infinity (operation (2) of the Kaliman modification) is the same as blowing up each fibre at infinity and keeping the same monodromy. Hence $SH_*(X) \cong SH_*(Z)$.

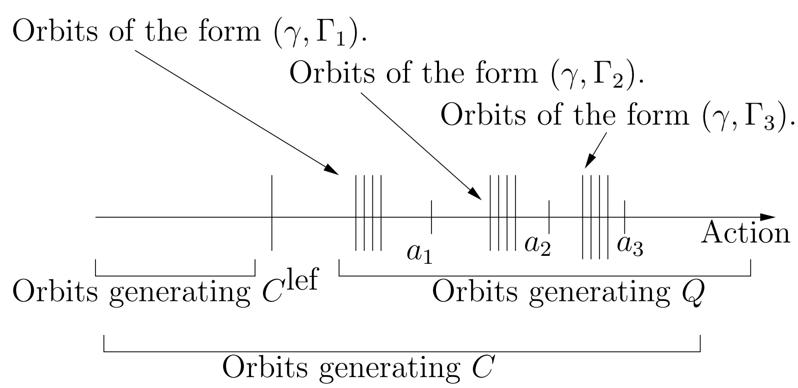
Remark 2: Given varieties X and Z in dimension 4 such that X is obtained from Z by blowing up at infinity, there are Lefschetz fibrations E', E'' satisfying properties (1) and (2) such that $SH_*(E') \cong SH_*(X)$ and $SH_*(E'') \cong SH_*(Z)$. These do not satisfy property (3) because E' is obtained from E'' by filling in a boundary component of the fibres with a disc.

We will prove Theorem 1.4 in two stages. In stage (i), we introduce a new invariant $SH_*^{\text{lef}}(E)$ for a Lefschetz fibration E and show it is equal to symplectic homology. This is covered in sections 6 and 7.1. In stage (ii), we prove that $SH_*^{\mathrm{lef}}(E') \cong SH_*^{\mathrm{lef}}(E'')$. This is covered in section 8. In a little more detail:

(i) Let F be a smooth fibre of E and $\mathbb{D}$ a disc in $\mathbb{C}$. In section 7, we show (roughly) that the chain complex C for $SH_*(E)$ is generated by:

(1) critical points of some Morse function on E;
(2) two copies of fixed points of iterates of the monodromy map around a large circle;
(3) pairs (Γ, γ) where Γ is a Reeb orbit on the boundary of F and γ is either a Reeb orbit of $\partial\mathbb{D}$ or a fixed point in the interior of $\mathbb{D}$.

This is done in almost exactly the same way as the proof of the Künneth formula for symplectic homology [26]. The differential as usual involves counting cylinders connecting the orbits and satisfying the perturbed Cauchy -Riemann equations. The orbits in (1) and (2) actually form a subcomplex C^{lef}, and we define Lefschetz symplectic homology $SH_*^{\mathrm{lef}}(E)$ to be the homology of this subcomplex. Next we need to show that $SH_*(E) \cong SH_*^{\mathrm{lef}}(E)$. We have a short exact sequence $0 \to C^{\mathrm{lef}} \to C \to Q \to 0$ where Q is the quotient complex C/C^{lef}. The chain complex Q is basically generated by the orbits (γ, Γ) in (3). We can choose a filtration of the chain complex Q by action so that the orbits γ are close together compared to Γ (see the diagram below).



This means we can construct a spectral sequence converging to $H_*(Q)$ where each page is equal to $SH_*(\mathbb{D}) = 0$, because the orbits γ generate a chain complex for $SH_*(\mathbb{D})$. Hence $H_*(Q) = 0$ which implies that $SH_*(E) \cong SH_*^{\mathrm{lef}}(E)$.

(ii) Let C' (resp. C'') be the standard chain complex for $SH_*^{\text{lef}}(E')$ (resp. $SH_*^{\text{lef}}(E'')$). The fibration $E' \setminus E''$ is a trivial fibration $\mathbb{D} \times W$. We have a short exact sequence $0 \to B \to C' \to C'' \to 0$ where B is generated by orbits of the form (γ, Γ) in $\mathbb{D} \times W$. The orbit Γ is a critical point of some Morse function on W and γ is either a Reeb orbit of $\partial\mathbb{D}$ or a fixed point in the interior of $\mathbb{D}$. We can use a similar action filtration argument to show that $H_*(B) = 0$. Property (3) in theorem 1.4 is needed here to ensure that the above exact sequence exists. If we didn't have this property, then there would be some spectral sequence from $SH_*^{\text{lef}}(E'')$ (with an extra grading coming from the $H_1(E'')$ classes of these orbits) to $SH_*^{\text{lef}}(E')$.

Lefschetz symplectic homology was partially inspired by Paul Seidel's Hochshild homology conjectures [29], which also relate symplectic homology to Lefschetz fibrations. His conjectures would in particular prove theorem 1.4.

Proof. of corollary 1.3. There is a standard Stein structure on T^*M such that $SH_0(M)$ is a non-trivial finite dimensional $\mathbb{Z}/2\mathbb{Z}$ vector space. By Lemma 9.6 this means that $i(T^*M) < \infty$. Also $0 \in SH_*(T^*M)$ is an idempotent which means that $0 < i(T^*M)$. We let X_i be defined as in the proof of the main theorem 1.2. We define

$$X_i^M := T^*M \#_e X_i.$$

Then $i(X_i^M) = i(T^*M)i(X_i)$. These numbers are all different as $0 < i(T^*M) < \infty$ and $i(X_i) \neq i(X_j)$ for $i \neq j$. $\square$

1.3. **Notation.** Throughout this thesis we use the following notation:

(1) $M, M', M'', \dots$ are manifolds (with or without boundary).
(2) ∂M is the boundary of M.
(3) $(E, \pi), (E', \pi'), (E'', \pi'')$ are exact Lefschetz fibrations (4.2).
(4) If we have some data X associated to M (resp. E), then $X, X', X'', \dots$ are data associated to $M, M', M'', \dots$ (resp. $E, E', E'', \dots$). For instance $\partial M''$ is the boundary of M''.
(5) ω is a symplectic form on M or E.
(6) θ is a 1-form such that $d\theta = \omega$.
(7) (M, θ) is an exact symplectic manifold.
(8) J is an almost complex structure compatible with ω.
(9) If (M, θ) is a compact convex symplectic manifold (2.1), then $(\widehat{M}, \theta)$ is the completion of (M, θ) (2.5). Similarly by (4.6), (E, π) can

be completed to $(\widehat{E}, \pi)$ (we leave π and θ as they are by abuse of notation).

(10) (M, θ_t) is a convex symplectic or Stein deformation.

(11) F will denote a smooth fibre of (E, π).

(12) If we have some subset A of a topological space, then we will let nhd$A$ be some open neighbourhood of A.

2. General background

This section gives some general background on Stein manifolds and Symplectic homology. The material about Stein manifolds and convex symplectic manifolds in subsection 2.1 is not origional and is mainly taken from [11], [7], [34, Section 2]. The material about symplectic homology in section 2.2 is mainly taken from [14], [39]. A good summary of this is in [25] and in [32]. The symplectic homology calculation in 2.3 has not been published in this form (except the calculation of $SH_*(\mathbb{C})$ which is in [25] for instance). Experts might have been familiar with these calculations due to the fact that they know about symplectic homology of a once punctured surface of genus greater than 0 which involves similar calculations (See [32, Example 3.3] and the comment after the second conjecture in [39, Section 5.2]). The material in section 2.4 has not been published in this form either, but again might be familiar to experts. This way of thinking can be seen for instance in [12, Section 2.4] where an 'alternating trick' is used to show that two symplectic manifolds are symplectomorphic. Similar tricks are used for instance in the proof of Lemma 2.13 to show that two compact convex symplectic manifolds F_t and F_T have the same symplectic homology ring.

2.1. Stein manifolds.

We will define Stein manifolds as in [34]. We let M be a manifold and θ a 1-form where $\omega := d\theta$ is a symplectic form.

Definition 2.1. *(M, θ) is called a* **compact convex symplectic manifold** *if M is a compact manifold with boundary and the ω-dual of θ is transverse to ∂M and pointing outwards. A* **compact convex symplectic deformation** *is a family of compact convex symplectic manifolds (M, θ_t) parameterised by $t \in [0, 1]$.*

We will let λ be the vector field which is ω-dual to θ.

Usually, a compact convex symplectic manifold is called a convex symplectic domain. We have a natural contact form $\theta|_{\partial M}$ on ∂M, and hence we call this the contact boundary of M.

Definition 2.2. *Let M be a manifold without boundary. We say that (M, θ) is a* **convex symplectic manifold** *if there exist constants $c_1 < c_2 < \cdots$ tending to infinity and an exhausting function $\phi : M \to \mathbb{R}$ such that $(\{\phi \leq c_i\}, \theta)$ is a compact convex symplectic manifold for each i. Exhausting here means proper and bounded from below. If the flow of λ exists for all positive time, then (M, θ) is called* **complete**. *If there exists a constant $c > 0$ such that for all $x \geq c$, $(\{\phi \leq c\}, \theta)$ is a compact convex symplectic manifold, then we say that (M, θ) is of* **finite type**.

Definition 2.3. *Let (M, θ_t) be a smooth family of convex symplectic manifolds with exhausting functions ϕ_t. Suppose that for each $t \in [0, 1]$, there are constants $c_1 < c_2 < \cdots$ tending to infinity such that for each s near t and $i \in \mathbb{N}$, $(\{\phi_s \leq c_i\}, \theta)$ is a compact convex symplectic manifold. Then (M, θ_t) is called a* **convex symplectic deformation**.

(We have a notion of a complete and a finite type convex symplectic deformation, which we won't need in this thesis.) The constants $c_1 < c_2 < \cdots$ mentioned in this definition depend on t but not necessarily in a continuous way. The nice feature of convex symplectic manifolds is that we have some control over how they behave near infinity. That is, the level set $\phi^{-1}(c_k)$ is a contact manifold for all k. We say that an exact symplectic manifold has a cylindrical end if outside some relatively compact subset it is exact symplectomorphic to $(N \times [1, \infty), r\alpha)$ where α is a contact form on N and r is the coordinate on $[1, \infty)$. Note that cylindrical ends are not unique. For instance for any function $f : N \to [1, \infty)$ we have a subset of the form $i : N \times [1, \infty) \hookrightarrow N \times [1, \infty)$ where $i(x, r) := (x, rf(x))$. We have $i^*(r\alpha) = r(f\alpha)$ where $f\alpha$ is a new contact form on N. This means that we have a new cylindrical end $(N \times [1, \infty), r(f\alpha))$. Suppose that M is a complete finite type convex symplectic manifold.

Lemma 2.4. *There exists a relatively compact set $K \subset M$ such that $M \setminus K$ is exact symplectomorphic to $(N \times [1, \infty), r\alpha)$ where (N, α) is a contact manifold contactomorphic to $(\phi^{-1}(c), \theta|_{\phi^{-1}(c)})$, $c \gg 0$.*

Proof. Choose c so that it is a regular value of ϕ and such that λ is transverse to $N := \phi^{-1}(c)$. Let $\alpha := \theta|_N$. We can ensure that c is large enough so that

ϕ has no critical values above c. Let $\Phi_t : M \to M$ be the flow of λ (this is well defined as M is complete). We define a map $G : N \times [1, \infty) \to M$ by $G(x, r) := \Phi_{\log r}(x)$ where $x \in N \subset M$ and $r \in [1, \infty)$. We also have that G is a diffeomorphism onto its image. Because there are no critical values of ϕ above c, we have that the complement of the image of G is a relatively compact and equal to $\{\phi < c\}$. Also, $\mathcal{L}_\lambda \theta = \theta$ ($\mathcal{L}$ here means Lie derivative). This means that $G^*(\theta) = r\alpha$. $\square$

Lemma 2.5. *A compact convex symplectic manifold M can be completed to a finite type complete convex symplectic manifold $(\widehat{M}, \theta)$.*

This is explained for instance in [39, section 1.1]. The proof basically involves gluing a cylindrical end onto ∂M. Let (M, θ) be a complete convex symplectic manifold. Let (M', θ') be a compact convex symplectic manifold which is a codimension 0 exact submanifold of (M, θ) (i.e. $\theta|_{M'} = \theta' + dR$ for some smooth function R on M').

Lemma 2.6. *We can extend the embedding $M' \hookrightarrow M$ to an embedding $\widehat{M'} \hookrightarrow M$.*

Proof. There exists a function $R : M' \to \mathbb{R}$ such that $\theta' = \theta + dR$. We can extend R over the whole of M such that $R = 0$ outside some compact subset K containing M'. Let $\theta_1 = \theta + dR$, and λ_1 be the ω-dual of θ_1. Let $F_t : M \to M$ be the flow of λ_1. This exists for all time because M is complete and $\lambda_1 = \lambda$ outside K. We have an embedding $\Phi : (\partial M') \times [1, \infty) \to M$ defined by $\Phi(a, t) = F_{\log t}(a)$. This attaches a cylindrical end to M' inside M, hence we have an exact embedding $\widehat{M'} \to M$ extending the embedding of M'. $\square$

Definition 2.7. *A* **Stein manifold** (M, J, ϕ) *is a complex manifold (M, J) with an exhausting plurisubharmonic function $\phi : M \to \mathbb{R}$ (i.e. ϕ is proper and bounded from below and $-dd^c(\phi) > 0$ where $d^c = J^*d$). A Stein manifold is called* **subcritical** *if ϕ is a Morse function with critical points of index $< \frac{1}{2}\dim_{\mathbb{R}} M$. A manifold with boundary of the form $\phi^{-1}((-\infty, c])$ is called a* **Stein domain**.

We can perturb ϕ so that it becomes a Morse function. From now on, if we are dealing with a Stein manifold, we will always assume that ϕ is a Morse function. The index of a critical point of ϕ is always less than or equal to $\frac{1}{2}\dim_{\mathbb{R}} M$. Note that the definition of a Stein manifold in [11, Section 2] is that it is a closed holomorphic submanifold of $\mathbb{C}^N$ for some N. This has an exhausting plurisubharmonic function $|z|^2$. An important example of a subcritical Stein manifold is $(\mathbb{C}^n, i, |z|^2)$. The Stein manifold (M, J, ϕ) is a convex symplectic manifold $(M, \theta := -d^c\phi)$. Note that $\lambda := \nabla\phi$ where ∇ is taken with respect to the metric $\omega(\cdot, J(\cdot))$. It is easy to see that $\nabla\phi$ is a Liouville vector field transverse to a regular level set of ϕ and pointing outwards. We call a Stein manifold complete or of finite type if the associated convex symplectic structure is complete or of finite type respectively.

Definition 2.8. *If (J_t, ϕ_t) is a smooth family of Stein structures on M, then it is called a* **Stein deformation** *if the function $(t, x) \longrightarrow \phi_t(x)$ is proper and for each $t \in [0, 1]$, there exists $c_1 < c_2 < \dots$ tending to infinity such that for any s near t we have that c_k is a regular value of ϕ_s. This induces a corresponding convex symplectic deformation.*

Example 2.9. *An affine algebraic subvariety M of $\mathbb{C}^N$ admits a Stein structure. This is because it has a natural embedding in $\mathbb{C}^N$, so we can restrict the plurisubharmonic function $\|z\|^2$ to this variety to make it into a Stein manifold. We can also use the following method to find a plurisubharmonic function on M. We first compactify M by finding a projective variety X with complex structure i and an ample divisor D such that $M = X \setminus D$ (for instance we can embed M in $\mathbb{C}^N \subset \mathbb{P}^N$ and then let X be the closure of M in $\mathbb{P}^N$). There exists an ample line bundle $E \longrightarrow X$ associated to the divisor D. Choose a holomorphic section s of E such that $D = s^{-1}(0)$. Then ampleness means that we can choose a metric $\|.\|$ such that its curvature form $\omega := iF_\nabla$ is a positive $(1,1)$-form. Hence we have a Stein structure*

$$(M := X \setminus D, J := i, \phi := -\log\|s\|).$$

Note that by [34, Lemma 8]*, this is of finite type. The Stein structure on this variety has finite volume because we can extend the symplectic structure over to the compactification X of M.*

The following operation constructs a new Stein manifold from two old ones. This is used to construct our infinite family of Stein manifolds. We will let $(M, J, \phi), (M', J', \phi')$ be complete finite type Stein manifolds. Because these manifolds are complete and of finite type, they are the completions of compact convex symplectic manifolds N, N' respectively. In fact $N = \{\phi \leq R\}, N' = \{\phi' \leq R\}$ for some arbitrarily large R. Let p (resp. p') be a point in ∂N (resp. $\partial N'$). The following theorem is proved in greater generality in [10] and [7, Theorem 9.4].

Theorem 2.10. *There exists a connected finite type Stein manifold* (M'', J'', ϕ'') *such that* $N'' := \{\phi'' \leq R\}$ *is biholomorphic to the disjoint union of* N *and* N' *with* $\phi''|_N = \phi$ *on* N *and* $\phi''|_{N'} = \phi'$ *on* N'. *Also, the only critical point of* ϕ'' *outside* N'' *has index* 1.

In this theorem, what we are doing is joining N and N' with a 1-handle and then extending the Stein structure over this handle, and then completing this manifold so that it becomes a Stein manifold. The Stein manifold M'' is called the **end connect sum** of M and M', and we define $M \#_e M'$ as this end connected sum. If M and M' are Stein manifolds diffeomorphic to $\mathbb{C}^n$, then $M \#_e M'$ is also diffeomorphic to $\mathbb{C}^n$. The proof of this theorem also ensures that if we have two Stein *domains* A and B, then we can construct a new Stein domain $A \#_e B$ containing the disjoint union of A and B.

It is also possible to construct infinite end connect sums as follows: Let $A_1, A_2, \cdots$ be a countably infinite family of Stein manifolds. Let ϕ_i be the Stein function associated to A_i. Let $A'_i := \{\phi_i \leq R_i\}$ where R_i is large enough so that A'_i contains all the critical points of ϕ_i in A_i. We will now describe this infinite end connect sum as a union of Stein domains $B := B_1 \cup B_2 \cup B_3 \cdots$ where $B_i \subset B_{i+1}$ for all i. We want the dimensions of the B_i's to be the same, and that B_i is a holomorphic submanifold of B_{i+1}. If ϕ_{B_i} is the Stein function associated to B_i, then we want $\phi_{B_i}|_{B_{i-1}} = \phi_{B_{i-1}}$. This ensures that B is a Stein manifold with Stein function ϕ where $\phi|_{B_i} = \phi_{B_i}$. We define $B_1 := A'_1$. Suppose by induction we have constructed the Stein domain B_{i-1} with the properties described above. Let $B_i := B_{i-1} \#_e A'_i$. Then B_{i-1} is a holomorphic submanifold of B_i and there is a Stein function ϕ_{B_i} on B_i such that $\phi_{B_i}|_{B_{i-1}} = \phi_{B_{i-1}}$ and ∂B_i is a level set of ϕ_{B_i}. Hence we have constructed B_i for all i and the infinite end connect sum is defined to be equal to the union of all these B_i's.

2.2. **Symplectic homology.** In [14] Floer used a new homology theory called symplectic homology for compact symplectic manifolds in order to prove the Arnold conjecture for a large class of compact symplectic manifolds. The Arnold conjecture states that the number of periodic orbits of a generic Hamiltonian must be greater than the sum of the Betti numbers of that manifold. Given a Hamiltonian H, he constructs a chain complex consisting of linear combinations of periodic orbits of H with a differential which is defined by counting holomorphic cylinders connecting these orbits. He then proves that the homology of this chain complex is equal to singular homology. This means that the number of orbits of the Hamiltonian is greater than the sum of the Betti numbers of the manifold and hence the Arnold conjecture is proven for these manifolds. In this section we will discuss symplectic homology as defined by Viterbo in [39] for finite type Stein manifolds. This homology theory is defined in almost exactly the same way as Floer's homology theory above except that our Hamiltonian H behaves in a particular way at infinity. Symplectic homology for many Stein manifolds is not equal to singular homology in contrast to Floer's homology theory for compact symplectic manifolds. For simplicity we will assume that our homology theory has coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Let (M, θ) be a compact convex symplectic manifold. Our manifold $\widehat{M}$ has a cylindrical end symplectomorphic to $(N \times [1, \infty), d(r\alpha))$ where r is a coordinate on $[1, \infty)$ and α is a contact form on N. We choose a smooth function $H : \mathbb{S}^1 \times \widehat{M} \longrightarrow \mathbb{R}$, and an $\mathbb{S}^1$ family of almost complex structures J_t compatible with the symplectic form. We also assume that H is linear at infinity (i.e. $H = ar+b$ for some constants a, b), and J_t is convex with respect to this cylindrical end outside some large compact set (i.e. $\theta \circ J_t = dr$). We call the constant a the **slope at infinity**. We also say that J_t is **admissible**.

We define an $\mathbb{S}^1$ family of vector fields X_{H_t} by $\omega(X_{H_t}, \cdot) = dH_t(\cdot)$. The flow $\text{Flow}^t_{X_{H_t}}$ is called the **Hamiltonian flow**. We choose the cylindrical end and the slope of our Hamiltonians so that the union of the 1-periodic orbits form a compact set. Let $F := \text{Flow}^1_{X_{H_t}}$, then we have a correspondence between 1-periodic orbits o and fixed points p of F. In particular we say that o is non-degenerate if $DF|_p : T_p\widehat{M} \to T_p\widehat{M}$ has no eigenvalue equal to 1. We can also assume that the 1-periodic orbits of our Hamiltonian flow X_{H_t} are non-degenerate. We call a Hamiltonian H satisfying these conditions an **admissible Hamiltonian**.

From now on we will assume that $c_1(M) = c_1(\widehat{M}) = 0$. If we are given a trivialisation of the canonical bundle $\mathcal{K} \cong \mathcal{O}$, then for each orbit o, we can define an index of o called the **Robbin-Salamon** index (This is equal to the Conley-Zehnder index taken with negative sign, but we will not define the Conley-Zehnder index here). The choice of these indices depend on the choice of trivialisation of $\mathcal{K}$ up to homotopy but the indices are canonical if $H_1(M) = 0$. Here is the definition: To any path of symplectic matrices, we can assign an index which is also called the Robbin-Salamon index. It can be calculated as follows (See [21] or [25, Section 3.1]): Let $\Psi : [0,1] \to \mathrm{Sp}(2n)$ be a path of symplectic matrices. A crossing is a number $t \in [0,1]$ such that $\Psi(t)$ has an eigenvalue equal to 1. To each crossing we associate a quadratic form $\Gamma_t : \ker(\mathrm{Id} - \Psi(t)) \to \mathbb{R}$ defined by:

$$\Gamma_t = \omega_{\mathrm{std}}(v, \frac{\partial \Psi(t)}{\partial t} v),$$

where ω_{std} is the standard symplectic form on $\mathbb{R}^{2n}$. We say that a crossing $t_0 \in [0,1]$ is simple if Γ_{t_0} is non-degenerate. We can perturb the path slightly relative to the endpoints so that it only has simple crossings. This ensures that there are only finitely many crossings. We define the Robbin-Salamon index $i_{\mathrm{RS}}(\Psi)$ to be:

$$\frac{1}{2}\mathrm{Sign}(\Gamma_0) + \sum_{t \in (0,1)} \mathrm{Sign}\Gamma_t + \frac{1}{2}\mathrm{Sign}(\Gamma_1).$$

The main properties of the Robbin-Salamon index are:

(1) If we join the end of one path p_1 with the start of another path p_2 to create their concatenation p_3, then $i_{\mathrm{RS}}(p_3) = i_{\mathrm{RS}}(p_1) + i_{\mathrm{RS}}(p_2)$.
(2) If we deform a path relative to its endpoints, then its Robbin-Salamon index doesn't change.
(3) If a_1 is a path in $\mathrm{Sp}(2n_1)$ and a_2 is a path in $\mathrm{Sp}(2n_2)$, then $i_{\mathrm{RS}}(a_1 \bigoplus a_2) = i_{\mathrm{RS}}(a_1) + i_{\mathrm{RS}}(a_2)$.

We let $2n$ be the real dimension of M. Let $x : \mathbb{R}/\mathbb{Z} \to M$ be a non-degenerate periodic orbit of H_t. The differential of the flow $\mathrm{Flow}^t_{X_{H_t}}$ gives us a family of vector space symplectomorphisms $a_t : (T_{x(0)}M, \omega|_{x(0)}) \to (T_{x(t)}M, \omega|_{x(t)})$. The trivialisation of $\mathcal{K}$ induces a symplectic trivialisation of x^*TM up to homotopy because the natural map $\pi_1(Sp(2n)) \to \pi_1(U(1)) \cong \mathbb{Z}$ is an isomorphism. This means that we have a smooth family of vector space symplectomorphims $b_t : (T_{x(t)}M, \omega|_{x(t)}) \to (R^{2n}, \omega_{\mathrm{std}})$, where ω_{std} is the standard symplectic for $\mathbb{R}^{2n}$. Hence the map $l' : t \to b_t \circ a_t \circ b_0^{-1}$ is a path of symplectic matrices.

Definition 2.11. *(See* [32, Section (3a)]*) The* **Robbin-Salamon index** *of the orbit $x : \mathbb{R}/\mathbb{Z} \to M$ is defined to the Robbin-Salamon index of the path $l' : [0,1] \to \mathrm{Sp}(2n)$.*

Given a trivialisation of $\mathcal{K}$ we can see that this index is well defined because it induces a symplectic trivialisation of x^*TM up to homotopy. The grading on symplectic homology is induced by the Robbin-Salamon index (or equivalently, the Conley-Zehnder index *taken with negative sign*). We denote the Robbin-Salamon index by $\mathrm{ind}(x)$. Let

$$CF_k(M,H,J) := \bigoplus_{\mathrm{Flow}^1_{X_{H_t}}(x)=x, ind(x)=k} (\mathbb{Z}/2\mathbb{Z})\langle x\rangle.$$

For a 1-periodic orbit γ we define the **action** $A_H(\gamma)$:

$$A_H(\gamma) := -\int_0^1 H(t,\gamma(t))dt - \int_\gamma \theta.$$

This is the convention of [39] and [26]. This differs in sign from Seidel's convention in [32]. We will now describe the differential

$$\partial : CF_k(M,H,J) \to CF_{k-1}(M,H,J).$$

We consider curves $u : \mathbb{R}\times\mathbb{S}^1 \longrightarrow \widehat{M}$ satisfying the perturbed Cauchy-Riemann equations:

$$\partial_s u + J_t(u(s,t))\partial_t u = \nabla^{g_t} H$$

where ∇^{g_t} is the gradient associated to the metric $g_t := \omega(\cdot, J_t, \cdot)$. For two periodic orbits x_-, x_+ let $\bar{U}(x_-,x_+)$ denote the set of all curves u satisfying the Cauchy-Riemann equations such that $u(s,\cdot)$ converges to $x_\pm$ as $s \to \pm\infty$. This has a natural $\mathbb{R}$ action given by replacing the coordinate s with $s+v$ for $v \in \mathbb{R}$. Let $U(x_-,x_+)$ be equal to $\bar{U}(x_-,x_+)/\mathbb{R}$. If $\mathrm{ind}(x_-) - 1 = \mathrm{ind}(x_+)$, then for a C^∞ generic admissible Hamiltonian and almost complex structure we have that $U(x_-,x_+)$ is a zero dimensional manifold. There is a maximum principle which ensures that all elements of $U(x_-,x_+)$ stay inside a compact set K (see [25, Lemma 1.5]). We have a compactness theorem (see for instance [4]) which ensures that $U(x_-,x_+)$ is compact and hence a finite set. Let $\#U(x_-,x_+)$ denote the number of elements of $U(x_-,x_+)$ mod 2. Then we have a differential:

$$\partial : CF_k(M,H,J) \longrightarrow CF_{k-1}(M,H,J),$$

$$\partial\langle x_-\rangle := \sum_{x_+=\text{index k-1 periodic orbit}} \#U(x_-,x_+)\langle x_+\rangle.$$

By analysing the structure of 1-dimensional moduli spaces, one shows $\partial^2 = 0$ and defines $SH_*(M,H,J)$ as the homology of the above chain complex. As a $\mathbb{Z}/2\mathbb{Z}$ module $CF_k(M,H,J)$ is independent of J, but its boundary operator does depend on J. The homology group $SH_*(M,H,J)$ depends on M,H but is independent of J up to canonical isomorphism. Note that for each $f \in \mathbb{R}$

we have a subcomplex generated by orbits of action $\leq f$. The homology of such a complex is denoted by: $SH_*^{\leq f}(M, H, J)$.

If we have two admissible Hamiltonians $H_1 \leq H_2$ and two admissible almost complex structures J_1, J_2, then there is a natural map:

$$SH_*(M, H_1, J_1) \longrightarrow SH_*(M, H_2, J_2).$$

This map is called the continuation map and is defined as follows: We let G_t, $t \in (-\infty, \infty)$ be a monotone increasing smooth family of Hamiltonians such that $G_t = H_1$ for $t \ll 0$ and $G_t = H_2$ for $t \gg 0$. We define a chain map:

$$\partial : CF_k(M, H_1, J) \longrightarrow CF_k(M, H_2, J),$$

$$\partial \langle x_- \rangle := \sum_{x_+ = \text{index k periodic orbit}} \#U'(x_-, x_+)\langle x_+ \rangle.$$

The symbols $\langle x_- \rangle$ and $\langle x_+ \rangle$ are periodic orbits. The number $\#U'(x_-, x_+) \in \mathbb{Z}/2\mathbb{Z}$ is the number of elements of the set $U'(x_-, x_+)$ where $U'(x_-, x_+)$ is the set of solutions of the parameterised Floer equations:

$$\partial_s u + J_t(u(s,t))\partial_t u = \nabla^{g_t} H_s$$

where $u(s, \cdot)$ converges to $x_\pm$ as $s \to \pm\infty$ and where ∇^{g_t} is the gradient associated to the metric $g_t := \omega(\cdot, J_t, \cdot)$.

If we take the direct limit of all these maps with respect to admissible Hamiltonians ordered by $\leq$, then we get our symplectic homology groups $SH_*(M)$. Supposing we have a family of Hamiltonians $(H_\lambda)_{\lambda \in \Lambda}$ ordered by $\leq$. We say that a family of Hamiltonians $(H_i)_{i \in I \subset \Lambda}$ is *cofinal* if for every $\lambda \in \Lambda$, there exists an $i \in I$ such that $H_\lambda \leq H_i$. Hence we can also define $SH_*(M)$ as the direct limit of all these maps with respect to any cofinal family of Hamiltonians. There is another equivalent way of defining $SH_*(M)$ described in [32, Section 3d]. It is defined as follows: We let H be a Hamiltonian such that $H = h(r)$ on the cylindrical end of M such that $h'(r)$ tends to infinity as r tends to infinity. We define $SH_*(M) := SH_*(M, H, J)$. This definition is equivalent to the previous definition as we can construct a cofinal family of Hamiltonians H_i such that $H_i = H$ on $\{r \leq i\}$ and such that H_i is linear on $\{r > i + \frac{1}{i}\}$. We can also ensure that the only orbits of H_i are in the region $\{H_i = H\}$. We get a sequence of continuation maps $SH_*(M, H_i, J) \to SH_*(M, H, J)$. These induce a map

$$\varinjlim_i SH_*(M, H_i, J) \to SH_*(M, H, J).$$

This map must be an isomorphism as all the orbits of H_i lie in the set $\{H_i = H\}$ so for each orbit of H_i there is a trivial cylinder satisfying the parameterised Floer equations connecting this orbit to the corresponding orbit of H. Any cylinder satisfying the parameterised Floer equations must also decrease action. Hence if we view the above map as a matrix, it will be upper triangular with 1's down the leading diagonal. This kind of matrix is invertible hence must induce an isomorphism.

Suppose that (M', θ') is a compact convex symplectic manifold which is an exact submanifold of M, then there exists a natural map

$$i : SH_*(M) \longrightarrow SH_*(M')$$

called the transfer map. The composition of two of these transfer maps is another transfer map. These maps are introduced in [39, Section 2] and studied in [6, Section 3.3]. In order to construct this map we carefully construct a cofinal family of admissible Hamiltonians H_i, $i \in \mathbb{N}$ on $\widehat{M}$. Here is a sketch of how to do this: We can use Lemma 2.6 to embed $\widehat{M'}$ in $\widehat{M}$. Our cofinal family of admissible Hamiltonians H_i will look cofinal on arbitrarily large compact subsets K_i of $\widehat{M'} \subset \widehat{M}$ (the union of the K_i's is $\widehat{M'}$). We can ensure the orbits of non-negative action lie in $K_i \subset \widehat{M'}$ and that H_i is linear with respect to the cylindrical end of $\widehat{M'}$ near ∂K_i. We also force H_i to be constant on a large region of $\widehat{M'}$ outside K_i to ensure that no Floer cylinders connecting orbits inside K_i escape K_i (see [26, Lemma 1]). Because the Floer differential decreases action, there is a subcomplex C_* consisting of orbits of negative action. The direct limit of the quotient complexes $CH_*(H_i, J)/C_*$ is the chain complex for $SH_*(M')$ because all the orbits and Floer cylinders connecting orbits inside K_i stay inside K_i and because $H_i|_{K_i}$ looks cofinal. So quotienting by C_* induces a map:

$$SH_*(M) \to SH_*(M')$$

which is the transfer map.

If $SH_*(M) \neq 0$ then it has the structure of a unital ring, where the product is called the pants product. If we shift the grading by n, we have that $SH_{n+*}(M)$ is a $\mathbb{Z}$ graded $\mathbb{Z}/2\mathbb{Z}$ algebra. This is how it is constructed: We start with a disk with two holes in it (i.e. a pair of pants) which we will call Σ. We put a 1-form γ on it such that $d\gamma \leq 0$ and such that near the end of each cylindrical end $I \times \mathbb{S}^1$ we have $\gamma = \delta dt$ where dt is the standard 1-form on $\mathbb{S}^1$ and δ is a constant. Here I is the interval $(-\infty, 1]$ for a negative end or $[1,\infty)$ for a positive end. We give the pair of pants two negative ends with $\gamma = dt$ and a positive end with $\gamma = 2dt$. Let H be an admissible Hamiltonian. Let o_1, o_2 be 1-periodic orbits of H and o_3 a periodic orbit of $2H$. Let $\mathcal{M}(H, J, o_1, o_2, o_3)$ be the set of solutions $u : \Sigma \to M$ satisfying:

$$(du + X_H \otimes \gamma)^{0,1} = 0$$

(note we have a $+$ sign instead of a $-$ sign due to differing sign conventions between this thesis and [32]) where X_H is the Hamiltonian flow of H. We view $X_H \otimes \gamma$ as a map from $T\Sigma$ to TM covering u sending a vector V to $\gamma(V)X_H$. If we have a map $\kappa : T\Sigma \to TM$ covering u, then $(\kappa)^{0,1} := J \circ \kappa - \kappa \circ j$ where J is the almost complex structure on M and j is the complex structure on Σ. We define a chain map:

$$CF_k(H,J) \otimes CF_j(H,J) \to CF_{k+j-n}(2H,J),$$

$$o_1 \otimes o_2 \to \sum_{\text{index k+j-n periodic orbits } o_3 \text{ of } 2H} \#\mathcal{M}(H,J,o_1,o_2,o_3)\langle o_3\rangle.$$

This commutes with continuation maps and hence we can take the direct limit of these maps with respect to the ordering $\leq$ giving us a map:

$$SH_{n+i}(M) \otimes SH_{n+j}(M) \to SH_{n+i+j}(M).$$

The unit is in $SH_n(M)$ and is given by counting holomorphic planes ([32, Section 8]). Note that in [32, Section 8], Paul Seidel is dealing with symplectic cohomology instead of symplectic homology. But, because he uses different sign conventions, both theories are exactly the same up to change of grading. The sign conventions in this thesis are exactly the same as Oancea's ones in [26]. The action functional in [26] is: $A := -\int_\gamma \theta - \int H$ and the one in [32] is: $A := -\int_\gamma \theta + \int H$. Also the other difference in sign convention is that in [26] the Hamiltonian vector field satisfies: $\omega(X, .) = dH$ whereas the one in [32] satisfies: $\omega(., X) = dH$. Both these changes in sign convention mean that the homology theory in [26] is the same as the cohomology theory in [32] up to change of grading (the grading in [26] is $2n$-(the grading in [32])). The periodic orbits in [26] are the same as the ones in [32], but they go in the opposite direction. The Floer trajectories are the same, except that we replace the coordinates (s,t) (for $\mathbb{R} \times \mathbb{S}^1$) with $(-s,-t)$.

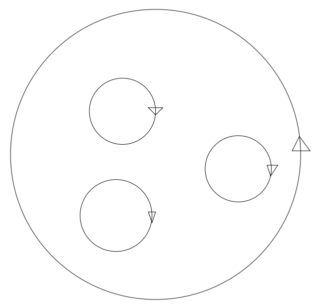
2.3. **A symplectic homology calculation.** In this section we will explicitly calculate $SH_*(C_n)$ as a vector space where C_n is a disc with n holes in it. We will also produce a sketch of how to calculate its ring structure for $n \neq 2$. In this thesis we only need to know $SH_*(C_n)$ for $n = 0$. We also need to know that $SH_*(C_1)$ has finitely many idempotents corresponding to fixed points of the Hamiltonian in the interior of C_1. We will deal with the case $SH_*(C_0)$ first.

2.3.1. *Calculation for the disk.* We let C_0 be the unit disk in $\mathbb{C}$. There is an obvious trivialisation of the canonical bundle $\mathcal{K}$ for C_0. In polar coordinates, the standard symplectic form is $rdr \wedge d\theta$. This has a standard Liouville vector field $\lambda := \frac{r}{2}\frac{\partial}{\partial r}$ where (r, θ) are polar coordinates for $\widehat{C_0} \cong \mathbb{C}$. This is transverse to all the circles with centre the origin, and hence makes C_0 into a compact convex symplectic manifold. Integrating λ gives us a family of diffeomorphisms $F_t : \widehat{C_0} \to \widehat{C_0}$, $F_t(r, \theta) = (re^{t/2}, \theta)$. Let $\mathbb{S}^1$ be the unit circle in $\widehat{C_0}$. We can use F_t to construct a cylindrical end as follows: We define $\Phi : \mathbb{S}^1 \times [1, \infty) \to \widehat{C_0}$ by $\Phi(\theta, b) = F_{\log b}(1, \theta) = (b^{\frac{1}{2}}, \theta)$ where θ is a point in $\mathbb{S}^1$ represented by the angle θ and $b \in [1, \infty)$. Hence a Hamiltonian H is admissible with respect to this cylindrical end if it is of the form $h(r^2)$ near infinity and h' is constant near infinity. When h' is constant we need $h' \neq k\pi$ for some $k \in \mathbb{Z}$ to ensure that there are no orbits near infinity. We choose our cofinal family of Hamiltonians to be of the form $H_k := (k\pi - 1)r^2$. These have only one periodic orbit corresponding to the minimum at the origin. So there are no differentials in the chain complex associated to H_k. We will now calculate the Robbin-Salamon index of this orbit. The flow of this Hamiltonian is: $\phi_t(r, \theta) := (r, \theta - 2(k\pi - 1)t)$. So the linearization of this flow with respect to the trivialisation of $\mathcal{K}$ and coordinates (x, y) is:

$$\begin{pmatrix} \cos 2(k\pi - 1)t & -\sin 2(k\pi - 1)t \\ \sin 2(k\pi - 1)t & \cos 2(k\pi - 1)t \end{pmatrix}.$$

We have that the Robbin-Salamon index is 1 plus twice the number of values of $t \in (0, 1]$ for which this matrix is the identity. Hence, the index is $2k - 1$. Hence $SH_{2k-1}(\widehat{C_0}, H_k, J) = \mathbb{Z}/2\mathbb{Z}$ and $SH_i(\widehat{C_0}, H_k, J) = 0$ for $i \neq 2k - 1$. Taking the direct limit of these groups gives us $SH_*(C_0) = 0$.

2.3.2. *Calculation for multiply punctured disks.* We will now deal with C_n where $n > 0$. The surface C_n is the n-fold end connect sum of C_1 (i.e. the annulus). One would expect that $SH_*(C_n)$ could be calculated in terms of $SH_*(C_1)$ using Theorem 2.17, but the problem is that Theorem 2.17 doesn't work in real dimension 2. The set of trivialisations of $\mathcal{K}$ corresponds to the set of smooth maps $C_n \to U(1) \cong \mathbb{S}^1$ up to smooth homotopy. This in turn corresponds to the set of continuous maps from the wedge sum of n circles to $\mathbb{S}^1$. Hence the set of trivialisations is $\mathbb{Z}^n$. We choose a trivialisation $(l_1, \cdots, l_n) \in \mathbb{Z}^n$. This means that the i'th coordinate tells us that the i'th circle in the wedge sum of n circles wraps round $\mathbb{S}^1$ l_i times. The manifold $\widehat{C_n}$ has $n+1$ cylindrical ends $(S_i^1 \times [1, \infty), r_i d\theta_i)$ where S_i^1 is the i'th boundary circle, r_i is the radial coordinate and $d\theta_i$ is the standard 1-form on S_i^1 giving it a volume of 1. The interior boundaries are oriented in a clockwise direction and the outer boundary circle is oriented in an anticlockwise direction:



Let H be a Hamiltonian such that $H = 0$ away from these cylindrical ends and $H = h(r_i)$ on each cylindrical end where $h : \mathbb{R} \to \mathbb{R}$ and $h' \geq 0$ and $h''(r_i) \to \infty$ as $r_i \to \infty$. We also make sure that $h'' > 0$ when $h' > 0$. To make H smooth we set $h = 0$ near $\{r_i = 1\}$. The problem with this Hamiltonian is that its orbits are degenerate. The constant orbits form a manifold with boundary diffeomorphic to C_n. We will define V to be this manifold with boundary. The other orbits, which correspond to $h'(r_i) = k\pi$ for some k, form manifolds diffeomorphic to $\mathbb{S}^1$. The reason why these orbits are degenerate is because they are not isolated. We need to perturb H by some C^2 small function to make its orbits non-degenerate. Let $L : \widehat{C_n} \to \mathbb{R}$ be a function which is 0 outside V and is Morse inside V. For $\beta > 0$ sufficiently small, we have that $H_1 := H + \beta L$ has non-degenerate orbits inside V. Any Floer trajectory connecting orbits inside V must be entirely contained in V by [25, Lemma 1.5]. If we choose an almost complex structure J which is time dependent outside V but time independent inside V, then [28, Theorem 7.3] says that the only Floer trajectories connecting orbits inside V are Morse flow lines for β sufficiently small. We now need to deal with the orbits outside V. Note that for topological reasons, there are no cylinders satisfying Floer's equations connecting orbits in the cylindrical ends with the constant orbits in the interior. Similarly, there are no Floer cylinders connecting orbits in different cylindrical ends. Hence we may focus on one cylindrical end $(S^1_i \times [1,\infty), r_i d\theta_i)$. On a small neighbourhood of each $\mathbb{S}^1$ family of orbits, we will add a time dependent Hamiltonian K_t to H_1 to make the orbits non-degenerate. Fix some $k \in \mathbb{N}$. Choose $l \in \mathbb{R}$ such that $h'(l) = 2k\pi$. Let N be a small neighbourhood of $\{r_i = l\}$ of the form $\{l - \epsilon \leq r_i \leq l + \epsilon\}$ which doesn't touch any orbits of H_1 and such that H_1 is of the form $h(r_i)$ in N. Let $f : N \to \mathbb{R}$ be a function which is 0 near ∂N and which has two Morse critical points away from $f^{-1}(0)$. We also assume that these critical points lie on $\{r_i = l\}$ and correspond to a maximum of Morse index 1 and a minimum of Morse index 0. Let ϕ_t be the Hamiltonian flow of H_1. Let $f_t := f \circ \phi_{-t}$. We let $H_{2,t} = H_1 + \alpha f_t$ be our new time dependent Hamiltonian where $\alpha > 0$ is very small. The orbits of $H_{2,t}$ in N are non-degenerate for α small enough. We perturb all the orbits in this way until $H_{2,t}$ has only non-degenerate orbits.

We now need to calculate the index of these orbits and the differentials. We will calculate the index first with respect to a standard trivialisation of $\mathcal{K}$ induced by filling all the interior boundaries with disks, and then we will consider the other trivialisations. We can use the index calculations from the previous section as follows: We can fill in the boundary of the cylindrical end $(S^1_i \times [1,\infty), r_i d\theta_i)$ with a disk $\mathbb{D}$ and extend the Hamiltonian $H_{2,t}$ over it such that the only additional critical point is the minimum of Morse index 0 (this is because $H_{2,t}$ is of the form $h(r_i)$ near $\{r_i = 1\}$ with $\{h'(r_i) < 2\pi\}$). This minimum has Robbin-Salamon index 1 as the Robbin-Salamon index is 1 minus the Morse index. We let $K_{k,t}$ be a Hamiltonian which is equal to $H_{2,t}$ in the region $\mathbb{D} \cup \{h' \leq 2k\pi - 1\}$. We also make $K_{k,t}$ linear with respect to r_i in the region $\{r_i \geq 2k\pi - \frac{1}{2}\}$ such that all its orbits are in the region $\{K_{k,t} = H_{2,t}\}$. This Hamiltonian is homotopic through Hamiltonians of the same slope at infinity to a Hamiltonian with 1 non-degenerate critical orbit (corresponding to its minimum) of index $2k - 1$. This is done using calculations from the previous section. The Hamiltonian $K_{2,t}$ has 3 orbits and $SH_*(K_{2,t})$ is $\mathbb{Z}/2\mathbb{Z}$ in degree 3 and 0 elsewhere. Also $K_{2,t}$ has an orbit of degree 1. So the chain complex looks like:

$$\overset{\text{ind}=3}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=2}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{id}} \overset{\text{ind}=1}{\mathbb{Z}/2\mathbb{Z}}.$$

To compute the chain complex for $K_{k,t}$ we proceed by induction. Because $K_{k,t} = K_{k-1,t}$ in some region $r_i \leq C$ and no Floer differential can escape this region by [25, Lemma 1.5], we have that the orbits in this region form a subcomplex (by the induction hypothesis):

$$\overset{\text{ind}=2k-3)}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=2k-4}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{id}} \cdots \overset{\text{ind}=3}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=2}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{id}} \overset{\text{ind}=1}{\mathbb{Z}/2\mathbb{Z}}.$$

The Hamiltonian $K_{k,t}$ has two additional orbits, and we also know that $SH_*(K_{k,t})$ is $\mathbb{Z}/2\mathbb{Z}$ in degree $2k-1$ and 0 elsewhere, hence the extra two orbits have index $2k-1$ and $2k-2$ which ensures that the chain complex is:

$$\overset{\text{ind}=2k-1}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=2k-2}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{id}} \cdots \overset{\text{ind}=3}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=2}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{id}} \overset{\text{ind}=1}{\mathbb{Z}/2\mathbb{Z}}.$$

Let u be a Floer cylinder connecting the orbits of index $2k-1$ and $2k-2$. Suppose that u intersects a point p in the interior of $\mathbb{D}$, then if we let $K_{k,t}$ tend to 0 in $\mathbb{D}$, then u by Gromov compactness converges to a holomorphic curve in $\mathbb{D}$ and hence intersects p a positive number of times (it could theoretically degenerate into a Morse flow line, but the direction of the Morse flow ensures that it cannot intersect the interior of $\mathbb{D}$ either). But this is impossible as any cylinder connecting these orbits has an intersection number of 0 with p. Hence we may assume that any Floer trajectory connecting these orbits cannot leave the region $\{r_i \geq 1\}$. This is true for all k. The other Floer trajectories must meet $\mathbb{D}$ at least once, so when we remove $\mathbb{D}$ and replace it by C_n again, then we get a complex:

$$\overset{\text{ind}=2k-1}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=2k-2}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \cdots \overset{\text{ind}=2}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \overset{\text{ind}=1}{\mathbb{Z}/2\mathbb{Z}}.$$

This is because inserting C_n again doesn't introduce any additional Floer cylinders as they have an intersection number of 0 with each point in the interior of C_n by using this compactness argument explained above. We now need to compute the indices of these orbits with respect to the trivialisation $(l_1, \cdots, l_n) \in \mathbb{Z}^n$. In fact we just add $2l_i$ for each time the orbit wraps around the cylinder. The reason is because for any path Ψ of symplectic matrices, we have

$$i_{\text{RS}}(e^{(2ik\pi t)}\Psi) = i_{\text{RS}}(e^{(2ik\pi t)}) + i_{\text{RS}}(\Psi).$$

The above identity holds because we can use a homotopy fixing the endpoints (similar to the one in [25, Section 3.3]) to deform the path $e^{(2ik\pi t)}$ to the concatenation of the two paths $e^{(2ik\pi t)}$ and Ψ. Also, the Robbin-Salamon index of the orbits corresponding to Morse critical points of $H_{2,t}$ is equal to n minus the Morse index. Putting all of this together we get that $SH_{1+*}(C_n)$

(for $n > 0$) is isomorphic as a $\mathbb{Z}/2\mathbb{Z}$ vector space to the algebra:

$$Z_n := (\mathbb{Z}/2\mathbb{Z})[x_1, y_1, \cdots, x_n, y_n, x_{n+1}, y_{n+1}]/J$$

where J is the ideal

$$(y_1 + \cdots + y_{n+1}, y_i^2, y_i y_j, x_i x_j, 1 \leq i < j \leq n+1)$$

where x_i has degree l_i and y_i has degree -1 for $i \leq n$. Also x_{n+1} has degree $(-\sum_{j=1}^{n} l_i)$ and y_{n+1} has degree -1.

We will show that in fact $SH_{1+*}(C_n)$ (for $n > 2$) is isomorphic as an algebra to Z_n. We will sketch the argument here.

The case $n = 1$ is different. We will deal with the case $n = 1$ before the case $n > 2$. We have that $\widehat{C_1}$ is the same as $T^*(\mathbb{S}^1)$. This means that $SH_*(C_1)$ is the same as the string topology of $\mathbb{S}^1$ using results from [1]. Hence using results from [23] we get that $SH_{1+*}(C_1)$ is isomorphic as a graded algebra to:

$$(\mathbb{Z}/2\mathbb{Z})[x, y, x^{-1}]/(y^2 = 0)$$

where the degree of x is l_1 and the degree of y is -1.

We will now deal with the case $n > 2$. The maximum principle [25, Lemma 1.5] ensures that that any Floer cylinder or pair of pants connecting orbits outside a chosen cylindrical end must stay outside this cylindrical end. So, if we cut off one of the cylindrical ends of C_1, we get a subalgebra:

$$A_l := (\mathbb{Z}/2\mathbb{Z})[x, y]/(y^2 = 0)$$

where $l \in \mathbb{Z}$ and x is of degree l and y is of degree -1. We give the Hamiltonian $H_{2,t}$ constructed above a unique minimum. This corresponds to the unit element in $SH_*(C_n)$ using results from [32, Section 8]. We cannot multiply orbits in different cylindrical ends together for topological reasons, as there are no orbits in the homology class represented by the sum of these orbits in $H_1(C_n)$. This is where the assumption $n > 2$ (rather than $n = 2$) is used. So we only need to focus on a single cylindrical end $(S_i^1 \times [1, \infty), r_i d\theta_i)$ of $\widehat{C_n}$. Making the Hamiltonian in C_n very small we can ensure that any pair of pants trajectory, between orbits in the cylindrical end or in the interior, intersecting this region is a Morse flow line (using the Gromov compactness argument above). Let R be the set of critical points of the Hamiltonian whose stable Morse flowlines intersect this chosen cylindrical end. Then by looking at the model for C_1 (with one cylindrical end missing) we see that multiplying any orbit x_i in $(S^1_i \times [1,\infty), r_i d\theta_i)$ with a critical point of index one in R gives us $x_i y_i$. Any other Floer trajectory stays entirely within this cylindrical end. This gives us enough information to calculate $SH_*(C_n)$.

2.4. **Invariance of symplectic homology.** In this section we define symplectic homology for convex symplectic manifolds (not just compact convex symplectic manifolds). We also show that symplectic homology and transfer maps are invariant under general convex deformations and exact symplectomorphism. Hence they are an invariant of the $\sim$-equivalence class defined in the introduction 1. Finally we state a theorem relating symplectic homology to end connect summation.

Let W be a compact convex symplectic manifold. Let $\partial W \times [1,\infty)$ be the cylindrical end of $\widehat{W}$ and r the coordinate for $[1,\infty)$. The manifold $W^\delta := \{r \leq \delta\}$ is a compact convex symplectic manifold for each $\delta \geq 1$. Let θ_W be the convex symplectic structure on $\widehat{W}$. We will only need the following properties of SH_*:

(1) $SH_*(W^\delta) \cong SH_*(W)$.
(2) The composition of two transfer maps is a transfer map.

Item (1) is true for the following reason: Let H be an admissible Hamiltonian on $\widehat{W}$. Then there exists a diffeomorphism $\Phi : \widehat{W^\delta} \to \widehat{W}$ induced by the backwards Liouville flow pulling back an admissible almost complex structure J on $\widehat{W^1}$ to an admissible almost complex structure J' on $\widehat{W^\delta}$. It is then easy to see that $SH_*(\delta.\Phi^*(H), J') \cong SH_*(H,J)$ as we are solving the same equations (Hamilton's equations and Floer's equations) in each case. Item (2) is true by using continuation maps and looking at the construction of the transfer map in section 12.

Let (M,θ) be a convex symplectic manifold. We have a direct system of codimension 0 compact convex symplectic exact submanifolds of (M,θ) where the morphisms are just inclusion maps. If we have two such manifolds N_1 and N_2 in M such that $N_1 \subset N_2$, then we have a transfer map $SH_*(N_2) \to SH_*(N_1)$. Hence, we have an inverse system $SH_*(N)$. We define $SH_*(M)$ as the inverse limit of the inverse system $SH_*(N)$. First of all we need to show that this definition is consistent with SH_* defined in section 2.2 for compact convex symplectic manifolds.

Theorem 2.12. *Let S be a compact convex symplectic manifold. Then $SH_*(\widehat{S})$ as defined in this section is equal to $SH_*(S)$ as defined in section 2.2.*

Proof. Let r be the radial coordinate on the cylindrical end $\partial S \times [0,\infty)$. Let $N_i := \{r \leq i\}$. The family N_i is a cofinal family of compact convex symplectic manifolds for $\widehat{S}$. Hence $SH_*(\widehat{S})$ is the inverse limit of $SH_*(N_i)$. But the transfer maps $SH_*(N_{i+1}) \to SH_*(N_i)$ are isomorphisms. This means

$$SH_*(\widehat{S}) \cong SH_*(N_1) \cong SH_*(S).$$

□

The ring $SH_*(M)$ is well defined up to exact symplectomorphism as the definition involves the directed system of codimension 0 exact embeddings of compact convex symplectic manifolds. We now need to show that it is invariant under convex symplectic deformation. We first need some preliminary lemmas: Let N be a compact symplectic manifold with convex boundary, and let $p_t : U_t \hookrightarrow N$ be a smooth family of exact symplectic embeddings of some convex manifold with boundary U_t. We also assume $\dim(N) = \dim(U_t)$.

Lemma 2.13. *There exists an isomorphism Φ so that we have a commutative diagram:*

$$\begin{tikzcd} SH_*(N) \arrow[r, "!p_0"] \arrow[dr, "!p_1"'] & SH_*(U_0) \arrow[d, "\cong", "\Phi"'] \\ & SH_*(U_1) \end{tikzcd}$$

$!p_t$ is the transfer morphism induced by p_t.

Proof. let $F_t := p_t(U_t)$.Then $\mathrm{nhd}(\partial F_t)$ is exact symplectomorphic to $\partial F_t \times [1-\epsilon_t, 1+\epsilon_t]$, where ∂F_t is identified with $\partial F_t \times \{1\}$ in $\partial F_t \times [1-\epsilon_t, 1+\epsilon_t]$. Let $F_t^0 = F_t \setminus (\partial F_t \times (1-\epsilon_t, 1])$ and $F_t^1 = F_t \cup (\partial F_t \times (1, 1+\epsilon_t])$. Fix some $T \in [0,1]$. We can assume, locally around T, that ϵ_t varies smoothly with respect to t. Hence, there exists a $\delta > 0$ such that for all $t \in (T-\delta, T+\delta)$ we have that:

$$F_T^0 \subset F_t \subset F_T^1$$

and

$$F_t^0 \subset F_T \subset F_t^1.$$

Hence we have natural morphisms:

$$\begin{tikzcd}
SH_*(F_T^1) \arrow[r, "a"] \arrow[rr, bend right, "e"', "\cong"] & SH_*(F_t) \arrow[r, "b"] & SH_*(F_T^0) & SH_*(F_T) \arrow[l, "c"', "\cong"] \arrow[d, "g"] & SH_*(F_t^1) \arrow[l, "d"'] \arrow[lll, bend right, "f"', "\cong"] \arrow[dl, "h"', "\cong"] \\
 & & & SH_*(F_t^0) &
\end{tikzcd}$$

We know that e, h, c and f are isomorphisms from property (1) mentioned at the start of this subsection. First we show that b is an isomorphism. b is surjective: This is because the image of b contains the image of $b \circ a = e$, and e is surjective. b is injective: d is injective because $h = g \circ d$ is injective. Hence $b = c \circ d \circ f^{-1}$ is injective. Therefore $c^{-1} \circ b$ is an isomorphism between $SH_*(F_t)$ and $SH_*(F_T)$. Also, $c^{-1} \circ b \circ !p_t = !p_T$ where $!p_t$ and $!p_T$ are the natural transfer maps induced by the inclusions of F_t and F_T respectively into N. Therefore because $[0,1]$ is compact, we can choose $F_{T_1} \ldots F_{T_k}$ ($T_0 = 0$, $T_k = 1$) such that we have natural morphisms:

$$\begin{tikzcd}
SH_*(N) \arrow[r, "!p_{T_0}"] \arrow[dr, "!p_{T_1}"] \arrow[ddr, "!p_{T_k}"'] & SH_*(F_{T_1}) \arrow[d, "\cong"] \\
 & SH_*(F_{T_2}) \arrow[d, dashed, "\cong"] \\
 & SH_*(F_{T_k})
\end{tikzcd}$$

$\square$

Let (N, θ_t) be a family of compact convex symplectic manifolds and U_t a compact codimension 0 exact submanifold of (N, θ_t). We also assume $\dim(N) = \dim(U_t)$ and U_t varies smoothly with t.

Lemma 2.14. *We have the following commutative diagram of transfer maps:*

$$\begin{tikzcd}
SH_*(N, \theta_0) \arrow[r] & SH_*(U_0, \theta_0) \\
SH_*(N, \theta_1) \arrow[r] \arrow[u, "\cong"'] & SH_*(U_1, \theta_1) \arrow[u, "\cong"']
\end{tikzcd}$$

Proof. By [34, Lemma 5] we have that $(\widehat{N}, \theta_t)$ is exact symplectomorphic to $(\widehat{N}, \theta_0)$ such that we have a smooth family of exact embeddings of (N, θ_t)

in $(\widehat{N}, \theta_0)$. There exists a large compact convex codimension 0 exact submanifold K of $(\widehat{N}, \theta_0)$ such that (N, θ_t) is contained in K for all $t \in [0,1]$. By lemma 2.13 and the fact that the composition of two transfer maps is a transfer map, we have a commutative diagram:

$$\begin{tikzcd} SH_*(K) \arrow[r] \arrow[dr] & SH_*(N,\theta_0) \arrow[r] \arrow[d, "\cong"] & SH_*(U_0) \arrow[d, "\cong"] \\ & SH_*(N,\theta_1) \arrow[r] & SH_*(U_1) \end{tikzcd}$$

□

Theorem 2.15. *Let (M, θ_t) be a convex symplectic deformation, then*

$$SH_*(M, \theta_0) \cong SH_*(M, \theta_1).$$

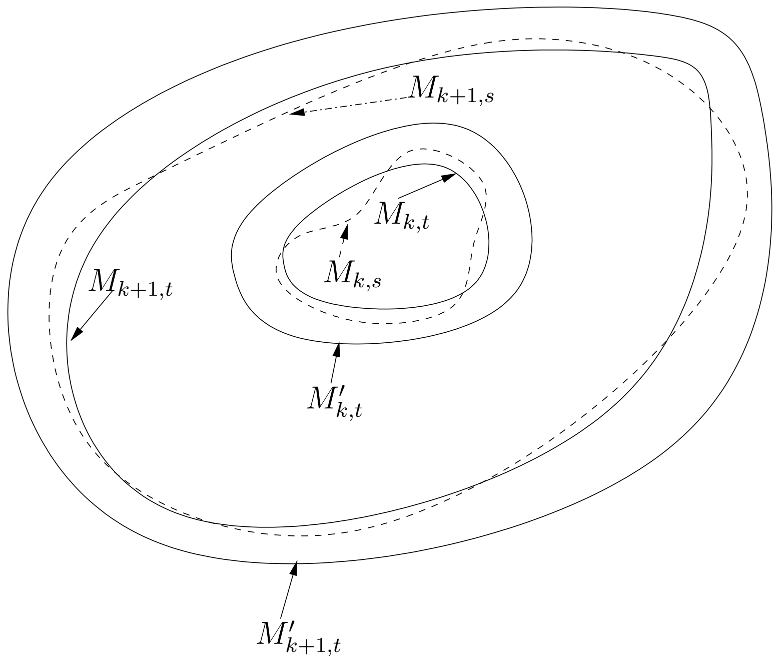
Proof. We may as well assume that the deformation is arbitrarily small because $[0,1]$ is compact. In particular, we may assume that there exist constants $c_1 < c_2 < \dots$ tending to infinity such that the level set $\phi_t^{-1}(c_i)$ is transverse to the Liouville vector field λ_t. We let $M_{i,t} := \phi_t^{-1}(-\infty, c_i]$ which is a compact convex symplectic manifold. Fix some $k > 0$ and $t \in [0,1]$. The convex symplectic manifold (M, θ_t) may not be complete, but we can use the vector field λ_t to flow $M_{k,t}$ to some convex symplectic manifold $M'_{k,t}$ for a very small amount of time. We also construct $M'_{k+1,t}$ in a similar way. We also assume that

$$M_{k,t} \subset M'_{k,t} \subset M_{k+1,t} \subset M'_{k+1,t}.$$

There exists an $\epsilon > 0$ such that for all $s \in (t-\epsilon, t+\epsilon)$, $\partial M_{k,s}$ and $\partial M_{k+1,s}$ are transverse to λ_t. We also assume that ϵ is small enough so that for all such s,

$$M_{k,s} \subset M'_{k,t} \subset M_{k+1,s} \subset M'_{k+1,t}.$$

Here is a picture illustrating our situation:



We have that $(M_{k,s}, \theta_t)$ is a compact convex symplectic manifold as λ_t is transverse to $\partial M_{k,s}$. Because λ_t is transverse to both $\partial M_{k,s}$ and $\partial M'_{k,t}$, we have that the natural transfer map $SH_*(M'_{k,t}, \theta_t) \to SH_*(M_{k,s}, \theta_t)$ is an isomorphism by lemma 2.13. Similarly we have a transfer isomorphism: $SH_*(M'_{k+1,t}) \to SH_*(M_{k+1,s})$. Hence using Lemma 2.14 we get a commutative diagram:

$$\begin{tikzcd}
SH_*(M_{k,t}, \theta_t) & SH_*(M_{k+1,t}, \theta_t) \arrow[l] \\
SH_*(M'_{k,t}, \theta_t) \arrow[u, "\cong"] \arrow[d, "\cong"'] & SH_*(M'_{k+1,t}, \theta_t) \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
SH_*(M_{k,s}, \theta_t) \arrow[d, "\cong"'] & SH_*(M_{k+1,s}, \theta_t) \arrow[l] \arrow[d, "\cong"] \\
SH_*(M_{k,s}, \theta_s) & SH_*(M_{k+1,s}, \theta_s) \arrow[l]
\end{tikzcd}$$

By compactness of the interval $[0,1]$, we get an isomorphism

$$\Phi : SH_*(M_{k,0}, \theta_0) \cong SH_*(M_{k,1}, \theta_1).$$

This map Φ commutes with the transfer maps

$$SH_*(M_{k+1,i}, \theta_i) \to SH_*(M_{k,i}, \theta_i)$$

$(i = 0, 1)$. So we get $SH_*(M, \theta_0) \cong SH_*(M, \theta_1)$. □

We have transfer maps defined for compact convex symplectic manifolds, but we need to extend them to maps whose domain is a general convex symplectic manifold. Let W be a compact codimension 0 exact submanifold of M. We have a direct system Δ of compact convex codimension 0 exact submanifolds N of M containing W. Let N be an element of this direct system, then there is a natural transfer map $SH_*(N) \to SH_*(W)$. Let $SH_*(\Delta)$ denote the respective inverse system whose objects are $SH_*(N)$ and whose maps are transfer maps. Taking the inverse limit K of the inverse system $SH_*(\Delta)$ gives us a natural map $SH_*(M) \cong K \to W$. We call this the transfer map from $SH_*(M)$ to $SH_*(W)$. We can see from the definition that the transfer map only depends on the exact symplectomorphism type of M. This means that if we have an exact symplectomorphism Φ from M to M', then we get a commutative diagram:

$$\begin{tikzcd} SH_*(M) \arrow[r] & SH_*(W) \\ SH_*(M') \arrow[u, "\cong"'] \arrow[r] & SH_*(\Phi(W)) \arrow[u, "\cong"'] \end{tikzcd}$$

Again we need to prove that transfer maps are invariant under deformations. Let (M, θ_t) be a convex deformation such that there is a smooth family of compact codimension 0 exact symplectic submanifolds V_t of (M, θ_t).

Theorem 2.16. *We have the following commutative diagram:*

$$\begin{tikzcd} SH_*(M, \theta_0) \arrow[r] & SH_*(V_0) \\ SH_*(M, \theta_1) \arrow[u, "\cong"'] \arrow[r] & SH_*(V_1) \arrow[u, "\cong"'] \end{tikzcd}$$

Proof. We use the same notation as in theorem 2.15. We can find a smooth family of functions R_t on M such that $(V_t, \theta_t + dR_t)$ is an exact symplectic manifold (i.e the associated Liouville vector field $\lambda_t + X_{R_t}$ to $\theta_t + dR_t$ is transverse to ∂V_t and pointing outwards). By Theorem 2.15 we can replace θ_t with $\theta_t + dR_t$. Hence from now on we will assume that λ_t is transverse to

∂V_t and pointing outwards. We can assume that the deformation is small enough so that there exists a $c_1 < c_2 < \cdots$ such that $d\phi_t(\lambda_{M,t}) > 0$ on $\phi_t^{-1}(c_k)$. Fix $t \in [0,1]$ and $k \in \mathbb{N}$. There exists an $\epsilon > 0$ such that for all $s \in (t-\epsilon, t+\epsilon)$, $\partial M_{k,s}$, $\partial M_{k+1,s}$ and ∂V_s are transverse to λ_t. We also assume that ϵ is small enough so that for all such s,

$$V_s \subset M_{k,s} \subset M'_{k,t} \subset M_{k+1,s} \subset M'_{k+1,t}.$$

Hence, we have a diagram:

$$\begin{tikzcd}
SH_*(V_0,\theta_t) \arrow[dd, "\cong"] & SH_*(M_{k,t},\theta_t) \arrow[l] & SH_*(M_{k+1,t},\theta_t) \arrow[l] \\
 & SH_*(M'_{k,t},\theta_t) \arrow[ul] \arrow[dl] \arrow[u, "\cong"'] \arrow[d, "\cong"'] & SH_*(M'_{k+1,t},\theta_t) \arrow[l] \arrow[u, "\cong"] \arrow[d, "\cong"] \\
SH_*(V_1,\theta_t) \arrow[d, "\cong"] & SH_*(M_{k,s},\theta_t) \arrow[l] \arrow[d, "\cong"'] & SH_*(M_{k+1,s},\theta_t) \arrow[l] \arrow[d, "\cong"] \\
SH_*(V_1,\theta_s) & SH_*(M_{k,s},\theta_s) \arrow[l] & SH_*(M_{k+1,s},\theta_s) \arrow[l]
\end{tikzcd}$$

This diagram is commutative, because the right hand side comes from a commutative diagram in theorem 2.15, the left hand triangle is the same as the triangle from lemma 2.13. Using compactness of the interval $[0,1]$ we have a commutative diagram:

$$\begin{tikzcd}
SH_*(V_0,\theta_0) \arrow[d, "\cong"] & SH_*(M_{k,0},\theta_0) \arrow[l] \arrow[d, "\cong"'] & SH_*(M_{k+1,0},\theta_0) \arrow[l] \arrow[d, "\cong"'] \\
SH_*(V_1,\theta_1) & SH_*(M_{k,1},\theta_1) \arrow[l] & SH_*(M_{k+1,1},\theta_1) \arrow[l]
\end{tikzcd}$$

Taking inverse limits proves our theorem.

□

Finally, we need a theorem involving end connect sums:

Theorem 2.17. *Let M, M' be Stein manifolds of real dimension greater than 2, then*

$SH_*(M \#_e M') \cong SH_*(M) \times SH_*(M')$ as rings. Also the transfer map $SH_*(M \#_e M') \to SH_*(M)$ is just the natural projection

$$SH_*(M) \times SH_*(M') \twoheadrightarrow SH_*(M).$$

Cieliebak in [6] showed that the above theorem is true if we view SH_* as a vector space. We can combine Cieliebak's proof with [3, Lemma 7.2] as a substitute for the Annulus Lemma [6, Lemma 3.3] in order to prove a ring isomorphism. We will also prove that we have a ring isomorphism in section 12.3.

3. Proof of the first theorem

3.1. **Constructing our non-finite type examples in dimension 3.**

First we wish to construct a finite type Stein manifold M_3 of complex dimension 3 such that $SH_*(M_3) \neq 0$.

Theorem 3.1. *There exists a contractible Stein manifold W of complex dimension 2 such that $SH_*(W) \neq 0$. This Stein manifold is an affine variety constructed as in example 2.9.*

Before we prove the theorem, we will first construct W. Let $V := \{z_1^2 = z_2^3\} \subset \mathbb{C}^2$. Let $w \in \mathbb{C}^2$ be a point in the smooth part of V. We define W to be $\mathrm{Bl}_w\mathbb{C}^2 \setminus \tilde{V}$ where $\mathrm{Bl}_w\mathbb{C}^2$ is the blowup of $\mathbb{C}^2$ at the point w and $\tilde{V}$ is the proper transform of V in $\mathrm{Bl}_w\mathbb{C}^2$. Another way of saying this is that W is equal to $\mathrm{Kalmod}(\mathbb{C}^2, V, w)$ where Kalmod will be defined later in section 4.3 (this kind of construction is mentioned in [22]). This surface is also contractible (see [22, Theorem 3.5] and Lemma 4.16).

Let $B(\epsilon) \subset \mathbb{C}^2$ be a small ball around the origin which does not contain w ($w \neq 0$ as V has its only singular point at the origin). There exists a Lagrangian torus L in $B(\epsilon) \subset W$ called the linking torus (see [34, section 4]). The inclusion $L \hookrightarrow \mathbb{C}^2 \setminus V$ is π_1 injective. Suppose (for a contradiction) there exists a non-constant holomorphic disk $i : (\mathbb{D}, \partial\mathbb{D}) \to (W, L)$. Then because $L \hookrightarrow C^2 \setminus V$ is π_1 injective, we have that this disk must intersect the exceptional divisor (if it didn't, then the disk bounds a contractible loop on the torus which would mean it had volume zero by Stokes' theorem which is impossible). Let $\mathbb{D}' \subset \mathbb{C}^2$ be the holomorphic disk corresponding to the image of i after applying the blowdown map $\mathrm{Bl} : W \to \mathbb{C}^2$. Then $\mathbb{D}'$ has boundary in $B(\epsilon)$ and it passes through $w \notin B(\epsilon)$. Hence it has an interior maximum outside $B(\epsilon)$ which is impossible. Hence there are no non-constant holomorphic disks in W with boundary in L.

In order to show that $SH_*(W) \neq 0$ we apply the following theorem using L: Let N be a compact convex symplectic manifold, and J an almost complex structure on the completion $\widehat{N}$ which is convex at infinity.

Theorem 3.2. *Suppose that there is a Lagrangian submanifold L_N of N such that there are no non-constant holomorphic disks in $(\widehat{N}, L_N)$. Then $SH_*(N) \neq 0$.*

This theorem is basically proved in [39]. It is also mentioned in the comment after Proposition 5.1 in [32, Section 5].

We now wish to use this theorem to show that $SH_*(W) \neq 0$. We can assume that the Stein manifold W is complete by [34, Lemma 6]. The problem is that the complex structure on W is not necessarily convex at infinity with respect to any cylindrical end. To get around this problem we just apply Theorem 11.1. Hence we can find an almost complex structure J on W (or an some manifold convex deformation equivalent to W - we can ensure that the convex deformation fixes L) such that it is convex at infinity with respect to some cylindrical end and such that there are no non-constant J-holomorphic disks in (W, L). So applying Theorem 3.2 shows us that $SH_*(W) \neq 0$ and hence we have proved Theorem 3.1.

We construct M_3 in the following way: Let q be a point in W. We have a hypersurface $K := W \times \{0\} \subset W \times \mathbb{C}$. Let M' be equal to $W \times \mathbb{C}$ blown up at the point $(q, 0) \in W \times \mathbb{C}$. Let $M_3 := M' \setminus \tilde{K}$ where $\tilde{K}$ is the proper transform of K in M'. Again, another way of saying this is that M_3 is equal to $\text{Kalmod}(W \times \mathbb{C}, K, (q, 0))$. This manifold M_3 is also diffeomorphic to $\mathbb{R}^6$ (see Lemma 4.16).

We need to show that $SH_*(M_3) \neq 0$. By the Künneth formula [26], we have that $SH_*(W \times \mathbb{C}^*) \neq 0$ as $SH_*(W) \neq 0$ and $SH_*(\mathbb{C}^*) \neq 0$ (because $\mathbb{C}^* = T^*\mathbb{S}^1$ and $SH_*(T^*\mathbb{S}^1) = H_{n+*}(\mathbb{S}^1) \neq 0$ by [38]). Later on in the thesis we will show that $SH_*(W \times \mathbb{C}^*) = SH_*(M_3)$ (see Theorem 4.20). So we get that $SH_*(M_3) \neq 0$. Finally we define M_3^∞ as the infinite end connect sum $\#_{e i=1}^{\infty} M_3$.

3.2. **Constructing our non-finite type examples in dimension 4 and higher.** In complex dimension 4 and higher we define $M_k := K_k$ where K_k is constructed in section 5.1. It is shown in sections 5.2 and 5.3 that $SH_*(K_k) \neq 0$. We define M_k^∞ as the infinite end connect sum $\#_{e i=1}^{\infty} M_k$.

3.3. **Main argument.** First of all we need a necessary condition for a convex symplectic manifold M to be of finite type.

Definition 3.3. *We say that M is of algebraic finite type if there exists a codimension 0 compact convex submanifold W of M such that the transfer map $SH_*(M) \to SH_*(W)$ is an injection.*

Note that if we have another codimension 0 compact convex submanifold W' in M containing W, then we have that $SH_*(M) \to SH_*(W')$ is an injection because:

$$SH_*(M) \to SH_*(W') \to SH_*(W)$$

is an injection.

Lemma 3.4. *A finite type Stein manifold N is of algebraic finite type.*

Proof. Let (N, J_N, ϕ_N) be the Stein structure for N. Let $c \gg 0$ be a constant such that ϕ_N has no critical values above c. The manifolds $N_s := \phi_N^{-1}(-\infty, s]$ for $s \geq c$ form a cofinal family of convex symplectic manifolds for N. So $SH_*(N)$ is the inverse limit of $SH_*(N_s)$. We aim to show that the natural map $SH_*(N_s) \to SH_*(N_c)$ is injective. Let λ_N be the associated Liouville flow. Then flowing ∂N_s using $-\lambda$ gives us a concave cylindrical end (note that $-\lambda$ is complete as it flows into the manifold): $(\partial N_s \times (-\infty, 1], r\alpha)$ where $r \in (-\infty, 1]$ and α is a contact form on ∂N_s (this is constructed in a similar way to the convex cylindrical end in lemma 2.4). Also, because there are no critical values of ϕ_N between s and c, we have that ∂N_c is a subset of this cylindrical end. This means that there exists a $K < 1$ such that $\{r \leq K\} \subset N_c \subset N_s = \{r \leq 1\}$. This means we get a sequence of maps:

$$SH_*(\{r \leq 1\}) \to SH_*(N_c) \to SH_*(\{r \leq 1\}).$$

The composition of both these maps is an isomorphism by property (1) of SH_* mentioned at the start of section 2.4. Hence the natural map $SH_*(N_s) \to SH_*(N_c)$ is an injection. $\square$

It is easy to see that being of algebraic finite type is invariant under exact symplectomorphism. We now need to show that this definition is invariant under convex deformation.

Lemma 3.5. *Suppose we have a convex deformation (M, θ_t), then (M, θ_0) is of algebraic finite type if and only if (M, θ_1) is.*

Proof. Because $[0,1]$ is compact, we may as well assume that the deformation is very small. In particular, we will assume that there exist constants $c_1 < c_2 < \ldots$ tending to infinity such that λ_t is transverse to the regular level set $\phi_t^{-1}(c_i)$. Let $V_{t,i} := \phi_t^{-1}(-\infty, c_i]$. If (M, θ_0) is of algebraic finite type, then for a very large i, we have that the transfer map:

$$SH_*(M, \theta_0) \to SH_*(V_{0,i})$$

is an injection. We now have a smooth family of compact codimension 0 convex exact submanifolds $V_{t,i}$ of (M, θ_t). Using theorem 2.16 we get that

$$SH_*(M, \theta_1) \to SH_*(V_{1,i})$$

is also an injection, which implies that (M, θ_1) is also of algebraic finite type. By symmetry, we also have that if (M, θ_1) is of algebraic finite type, then so is (M, θ_0). □

Proof. of theorem 1.1. Suppose for a contradiction that the manifold M_k^∞ (constructed in Sections 3.1 and 3.2) is $\sim$-equivalent to a finite type Stein manifold N. The manifold N is of algebraic finite type. Being of algebraic finite type is invariant under exact symplectomorphism and convex deformation which means that M_k^∞ is also of algebraic finite type. By the comment after Theorem 2.10 we can describe M_k^∞ as a union of Stein domains M_k^j where M_k^j is the j-fold end connect sum of M_k. Because M_k^∞ is a union of compact codimension 0 exact submanifolds M_k^j, we get (by the comment after Definition 3.3) an injective transfer map:

$$SH_*(M_k^\infty) \hookrightarrow SH_*(M_k^j)$$

for some very large j. The above map factors as follows:

$$SH_*(M_k^\infty) \to SH_*(M_k^{j+1}) \to SH_*(M_k^j).$$

By Theorem 2.17, the map $SH_*(M_k^{j+1}) \to SH_*(M_k^j)$ is a projection

$$\prod_{i=1}^{j+1} SH_*(M_k) \twoheadrightarrow \prod_{i=1}^{j} SH_*(M_k)$$

with kernel $SH_*(M_k) \neq 0$. Theorem 2.17 also ensures that the map $SH_*(M_k^\infty) \to SH_*(M_k^{j+1})$ is a surjection. This is because $SH_*(M_k^j)$ is the j-fold product of $SH_*(M_k)$ and $SH_*(M_k^\infty)$ is the inverse limit of these rings where the maps $SH_*(M_k^j) \to SH_*(M_k^{j-1})$ in this inverse limit are the natural projection maps from the j-fold product of $SH_*(M_k)$ to the $j-1$ fold product of $SH_*(M_k)$ (eliminating one of the factors). Hence $SH_*(M_k^\infty)$ is a countably infinite product of $SH_*(M_k)$ and the transfer map $SH_*(M_k^\infty) \to SH_*(M_k^j)$ is a projection. All of this means that the transfer map $M_k^\infty \hookrightarrow SH_*(M_k^j)$ has non-trivial kernel, contradiction. □

4. Background for the second theorem

4.1. **Lefschetz fibrations.** Throughout this section we will let E be a compact manifold with corners whose boundary is the union of two faces $\partial_h E$ and $\partial_v E$ meeting in a codimension 2 corner. We will also assume that Ω is a 2-form on E and Θ a 1-form satisfying $d\Theta = \Omega$. We let S be a surface with boundary. Let $\pi : E \to S$ be a smooth map with only finitely many critical points (i.e. points where $d\pi$ is not surjective). Let $E^{\text{crit}} \subset E$ be the set of critical points of π and $S^{\text{crit}} \subset S$ the set of critical values of π. For $s \in S$, let E_s be the fibre $\pi^{-1}(s)$.

Definition 4.1. *If for every $s \in S$ we have that Ω is a symplectic form on $E_s \setminus E^{\text{crit}}$ then we say that Ω* **is compatible with** *π.*

Note that if Ω is compatible with π then there is a natural connection (defined away from the critical points) for π defined by the horizontal plane distribution which is Ω-orthogonal to each vertical fibre. If parallel transport along some path in the base is well defined then it is an exact symplectomorphism (an exact symplectomorphism is a diffeomorphism Φ between two symplectic manifolds $(M_1, d\theta_1)$ and $(M_2, d\theta_2)$ such that $\Phi^*\theta_2 = \theta_1 + dG$ where G is a smooth function on M_1). From now on we will assume that Ω is compatible with π. We deal with Lefschetz fibrations as defined in [30]. Let J_0 (resp. j_0) be an integrable complex structure defined on some neighbourhood of E^{crit} (resp. S^{crit}). Remember F is some smooth fibre of π.

Definition 4.2. *(E, π) is an* **exact Lefschetz fibration** *if:*

1. *$\pi : E \longrightarrow S$ is a proper map with $\partial_v E = \pi^{-1}(\partial S)$ and such that $\pi|_{\partial_v E} : \partial_v E \to \partial S$ is a smooth fibre bundle. Also there is a neighbourhood N of $\partial_h E$ such that $\pi|_N : N \to S$ is a product fibration $S \times \text{nhd}(\partial F)$ where Ω and Θ are pullbacks from the second factor of this product.*
2. *π is (J_0, j_0) holomorphic near E^{crit} and the Hessian $D^2\pi$ at any critical point is nondegenerate as a complex quadratic form. We also assume that there is at most one critical point in each fibre.*

(3) Ω *is a Kähler form for* J_0 *near* E^{crit}.

Sometimes we will need to define a Lefschetz fibration without boundary. This is defined in the same way as an exact Lefschetz fibration except that E, the fibre F and the base S are open manifolds without boundary. We replace "neighbourhood of $\partial_h E$" in the above definition with an open set whose complement is relatively compact when restricted to each fibre. We also replace "$\partial_v E$" with $\pi^{-1}(S \setminus K)$ where K is a compact set in S. Also π is obviously no longer a proper map, and we assume that the set of critical points is compact. From now on we will let (E, π) be an exact Lefschetz fibration.

Lemma 4.3. [30, Lemma 1.5] *If* β *is a positive two form on* S *then* $\omega := \Omega + N\pi^*\beta$ *is a symplectic form on* E *for* N *sufficiently large.*

We really want our Lefschetz fibrations to be described as finite type convex symplectic manifolds.

Definition 4.4. *A* **compact convex Lefschetz fibration** *is an exact Lefschetz fibration* (E, π) *such that* $(F, \Theta|_F)$ *is a compact convex symplectic manifold. A* **compact convex Lefschetz deformation** *is a smooth family of compact convex Lefschetz fibrations parametrized by* $[0, 1]$.

Note that by the triviality condition at infinity, all smooth fibres of π are compact convex symplectic manifolds as long as the base S is connected. From now on we will assume that (E, π) is a compact convex Lefschetz fibration.

Theorem 4.5. *Let the base* S *be a compact convex symplectic manifold* (S, θ_S). *There exists a constant* $K > 0$ *such that for all* $k \geq K$ *we have:* $\omega := \Omega + k\pi^*(\omega_S)$ *is a symplectic form, and the* ω*-dual* λ *of* $\Theta + k\pi^*\theta_S$ *is transverse to* ∂E *and pointing outwards.*

(The proof is given in section 6.) Note that this theorem also implies that if we have a compact convex Lefschetz deformation, then we have a corresponding compact convex symplectic deformation because we can smooth the codimension 2 corners slightly.

If we have a compact convex Lefschetz fibration, then we wish to extend the Lefschetz fibration structure over the completion $\widehat{E}$ of E. Here is how we naturally complete (E, π): The horizontal boundary is a product $\partial F \times S$. We can add a cylindrical end $G := (\partial F \times [1, \infty)) \times S$ to this in the usual way, extending Θ over this cylindrical end by the 1-form $r(\Theta|_{\partial F})$ where r is the coordinate for $[1,\infty)$. Let E_1 be the resulting manifold. We also extend the map π over E_1 by letting $\pi|_G : G \to S$ be the natural projection. This ensures that π is compatible with the natural symplectic form on E_1 defined as in Lemma 4.3. The fibres of π are finite type complete convex symplectic manifolds. We now need to "complete" the vertical boundary of E_1 so that we have a fibration over the completion $\widehat{S}$ of S. Let $V := \partial_v E_1 := \pi^{-1}(\partial S)$. We then attach $A := V \times [0,\infty)$ to E_1 by identifying $V \subset E_1$ with $V \times \{0\} \subset A$ to create a new manifold $\widehat{E}$. Let $\pi_1 : A \to V$ be the natural projection onto V. We can extend Θ over A by a 1-form $\pi_1^*(\Theta)$ and then perturb this 1-form near $V \times \{0\} \subset A$ so that it is smooth. We can also extend our map π over A by letting $\pi|_A(v,r) = \pi|_V(v)$ where $v \in V \subset E_1$ and $r \in [0,\infty)$.

Definition 4.6. *$(\widehat{E},\pi)$ is called the* **completion** *of (E,π).*

Note that the base of our completed fibration is $(\widehat{S},\theta_S)$.

Definition 4.7. *Any fibration which is the completion of a compact convex Lefschetz fibration is called a* **complete convex Lefschetz fibration**.

Note that if we add a large multiple of $\pi^*\theta_S$ to Θ then $(\widehat{E},\theta)$ is a complete finite type convex symplectic manifold. Lefschetz fibrations have well defined parallel transport maps due to the fact that the fibration is trivial near the horizontal boundary of E. Now we need to deal with almost complex structures on $\widehat{E}$, as this will be useful when we later define SH_*^{lef}. Let J (resp. j) be an almost complex structure on $\widehat{E}$ (resp. $\widehat{S}$). We also assume that π is (J,j)-holomorphic, and that $J = J_0$ near E^{crit} and $j = j_0$ near S^{crit}.

Definition 4.8. *We say that (J,j) are* **compatible** *with $(\widehat{E},\pi)$ if:*

(1) *j is convex at infinity with respect to the convex symplectic structure of $\widehat{S}$ (i.e. $\theta_S \circ j = dr$ for large r where θ_S is the contact form at infinity on $\widehat{S}$ and r is the radial coordinate of the cylindrical end of $\widehat{S}$).*

(2) *J is a product (J_F, j) on the region $C \times S$ where C is the cylindrical end $\partial F \times [1,\infty)$ of S, and J_F is convex at infinity for F. $(\theta|_{\widehat{F}} \circ J_F = dr$ for large r where r is the radial coordinate of the cylindrical end).*

(3) *$\omega(\cdot, J\cdot)$ is symmetric and positive definite.*

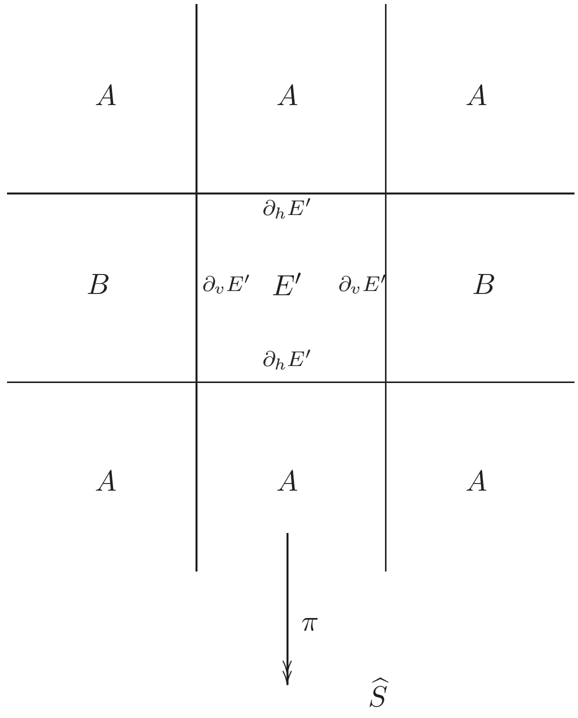
If $(\widehat{E}, \pi)$ is a complete convex Lefschetz fibration then the space of such almost complex structures is nonempty and contractible (see [30, section 2.2]). We wish to have a slightly larger class of almost complex structures.

Definition 4.9. *We define $\mathcal{J}^h(\widehat{E})$ to be the space of almost complex structures on $\widehat{E}$ such that for each J in this space, there exists a (J_1, j_1) compatible with $(\widehat{E}, \pi)$ and a compact set $K \subset \widehat{E}$ with $J = J_1$ outside K and with $\omega(\cdot, J\cdot)$ symmetric and positive definite.*

4.2. **Symplectic homology and Lefschetz fibrations.** We need three theorems which relate symplectic homology to Lefschetz fibrations. These are the key ingredients in proving that our exotic Stein manifolds are pairwise distinct. The proofs of these theorems will be deferred to sections 7 and 8. Theorem 4.11 is very close to Oancea's Künneth formula [26] but theorems 4.13 and 4.14 are new and the main part of the story. Throughout this section we will let $\pi' : E' \to S'$ be a compact convex Lefschetz fibration with fibre F'. From now on we will assume that $c_1(E') = 0$ and to make $SH_*(E')$ graded we will choose a trivialisation of the canonical bundle of E'. Note that when we talk about symplectic homology of a compact convex Lefschetz fibration, we mean the symplectic homology of its completion with respect to the convex symplectic structure. The fibration $\widehat{E'}$ can be partitioned into three sets as follows:

(1) $E' \subset \widehat{E'}$
(2) $A := F'_e \times \widehat{S}'$, where $F'_e := \partial F' \times \mathbb{R}_{\geq 1}$ is the cylindrical end of $\widehat{F'}$.
(3) $B := \widehat{E'} \setminus (A \cup E')$

The set B is of the form $(A_1 \times \mathbb{R}_{\geq 1}) \bigsqcup (A_2 \times \mathbb{R}_{\geq 1}) \cdots \bigsqcup (A_n \times \mathbb{R}_{\geq 1})$, where A_i is a mapping torus of the monodromy symplectomorphism around one of the boundary components of S. Here is a picture of the regions E', A and B.



Let $\pi_1 : A \twoheadrightarrow F'_e$ be the natural projection onto F'_e.

Definition 4.10. *Let $H_{S'}$ be an admissible Hamiltonian for the base $\widehat{S'}$. Let $H_{F'}$ be an admissible Hamiltonian for the fibre $\widehat{F'}$. We assume that $H_{F'} = 0$ on $F' \subset \widehat{F'}$. The map $H : \widehat{E'} \to \mathbb{R}$ is called a* **Lefschetz admissible Hamiltonian** *if $H|_A = \pi^* H_{S'} + \pi_1^* H_{F'}$ and $H|_B = \pi^* H_{S'}$ outside some large compact set. We say that H has* **slope** *(a, b) if H_S has slope a at infinity and H_F has slope b at infinity.*

Let H be a Lefschetz admissible Hamiltonian and let J be an admissible almost complex structure for E'. We will call the pair (H, J) a Lefschetz admissible pair. For generic (H, J) we can define $SH_*(E', H, J)$ (see section 7 for more details). If (H_1, J_1) is another generic Lefschetz admissible pair such that $H \leq H_1$, then there is a continuation map $SH_*(H, J) \to SH_*(H_1, J_1)$ induced by an increasing homotopy from H to H_1 through Lefschetz admissible Hamiltonians. Hence, we have a direct limit $SH^l_*(E') := \varinjlim_{(H,J)} SH_*(H, J)$ with respect to the ordering $\leq$ on Hamiltonians H. This has the natural structure of a ring with respect to the pair of pants product.

Theorem 4.11. *There is a ring isomorphsim $SH_*(E') \cong SH^l_*(E')$.*

This will be proved in section 7. Let (H, J) be a Lefschetz admissible pair of slope (a, ϵ) where ϵ is smaller than the length of the shortest Reeb orbit of $\partial F'$. We say that (H, J) is a **half admissible Hamiltonian**. We should think of H as a perturbation of $\pi^* H_{S'}$.

Definition 4.12. *We define*

$$SH_*^{\text{lef}}(E') := \varinjlim_{(H,J)} SH_*(H, J)$$

as the direct limit with respect to the ordering $\leq$ on half admissible Hamiltonians H. This has the structure of a ring as usual.

The difference between $SH_*(E')$ and $SH_*^l(E')$ is that $SH_*(E')$ is defined using Hamiltonians which are linear with respect to some fixed cylindrical end. The ring $SH_*^l(E')$ is defined using Hamiltonians which are linear in the horizontal and vertical directions with respect to some Lefschetz fibration. The difference between $SH_*^{\text{lef}}(E')$ and the other homology theories is that the slopes of a cofinal family of half admissible Hamiltonians do not have to tend to infinity pointwise in the vertical direction. This has to be true for $SH_*(E')$ and $SH_*^l(E')$ where the Hamiltonians have to get steeper and steeper at infinity in all directions. Because a half admissible Hamiltonian is Lefschetz admissible, we have a natural ring homomorphism:

$$\Phi : SH_*^{\text{lef}}(E') \to SH_*^l(E').$$

Theorem 4.13. *If $S' = \mathbb{D}$, the unit disk, then Φ is an isomorphism of rings. Hence by Theorem 4.11,*

$$SH_*(E') \cong SH_*^{\text{lef}}(E')$$

as rings.

This will be proved in section 7.1. Let F' (resp. F'') be a smooth fibre of E' (resp. E''). Let F' and F'' be Stein domains with F'' a holomorphic and symplectic submanifold of F'.

Theorem 4.14. *Suppose E' and E'' satisfy the following properties:*

1. *E'' is a subfibration of E' over the same base.*
2. *The support of all the monodromy maps of E' are contained in the interior of E''.*
3. *Any holomorphic curve in F' with boundary inside F'' must be contained in F''.*

Then $SH_*^{\mathrm{lef}}(E') \cong SH_*^{\mathrm{lef}}(E'')$ *as rings.*

This theorem will be proved in section 8. Combining this theorem with theorem 4.13 proves the key theorem 1.4 in the introduction of this thesis.

4.3. **The Kaliman modification.** In order to produce examples of exotic symplectic manifolds, we first need to construct exotic algebraic varieties. One tool used for constructing these manifolds is called the Kaliman modification. Our treatment follows section 4 of [40].

Consider a triple (M, D, C) where $C \subseteq D \subseteq M$ are complex varieties. Let M and C be smooth, D be an irreducible hypersurface in M, and C be a closed subvariety contained in the smooth part of D such that $\dim(C) < \dim(D)$.

Definition 4.15. *(see* [22]*) The* **Kaliman modification** M' *of* (M, D, C) *is defined by* $M' := \mathrm{Kalmod}\,(M, D, C) = \tilde{M} \setminus \tilde{D}$ *where* $\tilde{M}$ *is the blowup of* M *along* C *and* $\tilde{D}$ *is the proper transform of* D *in* $\tilde{M}$*.*

The Kaliman modification of an affine variety is again an affine variety (see [22]).

Lemma 4.16. [22, Theorem 3.5] *Suppose that (i)* D *is a topological manifold, and (ii)* D *and* C *are acyclic. Then* M' *is contractible iff* M *is.*

Example 4.17. *(***tom Dieck-Petrie surfaces** *see* [36, 35]*) For* $k > l \geq 2$ *with* (k, l) *coprime, the triple* $A_{k,l} := \left(\mathbb{C}^2, \{x^k - y^l = 0\}, \{(1, 1)\}\right)$ *satisfies the conditions of Lemma 4.16. Hence* $X_{k,l} = \mathrm{Kalmod}\,(A)$ *is contractible. Note:* $X_{k,l}$ *is isomorphic to*

$$\left\{ \frac{(xz+1)^k - (yz+1)^l - z}{z} = 0 \right\}.$$

Here x, y, z *are the standard coordinates of* $\mathbb{C}^3$*. Also the numerator of this fraction is divisible by* z*, hence the above fraction is a polynomial.*

Here is another construction:

Example 4.18. *(***Kaliman** [22]*) If we have a contractible affine variety* M *of complex dimension* n*, then we can construct a contractible affine variety*

$$M_k := \mathrm{Kalmod}\,(M \times \mathbb{C}, M \times \{p_1, \ldots, p_k\}, \{(a_1, p_1), \ldots, (a_k, p_k)\})$$

where p_i *are distinct points in* $\mathbb{C}$ *and* a_i *are points in* M*. This variety is contractible by a repeated application of 4.16, because it is a repeated*

Kaliman modification with D isomorphic to M and C a point. There are obvious variants: replace $\mathbb{C}$ and $\{p_1, \ldots, p_k\}$ with some contractible variety and a disjoint union of contractible irreducible hypersurfaces, etc.

At the moment we are only discussing contractibility of varieties. We need to produce varieties diffeomorphic to some $\mathbb{C}^n$. We will use the h-cobordism theorem to achieve this stronger condition.

Corollary 4.19. *(See* [2, Page 174]*,* [27] *and* [40, proposition 3.2]*) Let M be a contractible Stein manifold of finite type. If $n := \dim_{\mathbb{C}} M \geq 3$ then M is diffeomorphic to $\mathbb{C}^n$.*

Proof. Let (J, ϕ) be the Stein structure associated with M. We can also assume that ϕ is a Morse function. For R large enough, the domain $M_R := \{\phi < R\}$ is diffeomorphic to the whole of M as M is of finite type. We want to show that the boundary of $\bar{M}_R := \{\phi \leq R\}$ is simply connected, then the result follows from the h-cobordism theorem.

The function $\psi := R - \phi$ only has critical points of index $\geq n \geq 3$ because the function ϕ only has critical points of index $\leq n$ (see [11, Corollary 2.9]). Viewing ψ as a Morse function, $\bar{M}_R$ is obtained from $\partial \bar{M}_R$ by attaching handles of index ≥ 3. This does not change π_1, hence $\partial \bar{M}_R$ is simply connected because $\bar{M}_R$ is simply connected. $\square$

We now need a theorem which relates the Kaliman modification with symplectic homology. We do this via Lefschetz fibrations. Let X,D,M be as in example 2.9. Let Z be an irreducible divisor in X and $q \in (Z \cap M)$ a point in the smooth part of Z. We assume there is a rational function m on X which is holomorphic on M such that $\overline{m^{-1}(0)}$ is reduced and irreducible and $Z = \overline{m^{-1}(0)}$. Let $M' := \mathrm{Kalmod}(M, (Z \cap M), \{q\})$, and let $M'' := M \setminus Z$. Suppose also that $\dim_{\mathbb{C}} X \geq 3$. We also assume that $c_1(M') = c_1(M'') = 0$.

Theorem 4.20. $SH_*(M'') = SH_*(M')$.

This theorem follows easily from the key theorem 1.4 and the following theorem:

Theorem 4.21. *There exist compact convex Lefschetz fibrations $E'' \subset E'$ respectively satisfying the conditions of theorem 1.4 such that E' (resp. E'') is convex deformation equivalent to M' (resp. M'').*

This will be proved in the appendix (10). The basic idea of the proof is to use Lefschetz fibrations defined in an algebraic way.

5. Proof of the second theorem

5.1. **Construction of our exotic Stein manifolds.** First of all, we will construct a Stein manifold K_4 diffeomorphic to $\mathbb{C}^4$. We will then construct Stein manifolds K_n diffeomorphic to $\mathbb{C}^n$ for all $n > 3$ from K_4. Finally using end connect sums we will construct infinitely many Stein manifolds $(K_n^k)_{k\in\mathbb{N}}$ diffeomorphic to $\mathbb{C}^n$ for all $n > 3$.

We define the polynomial $P(z_0, \ldots, z_3) := z_0^7 + z_1^2 + z_2^2 + z_3^2$ and $V := \{P = 0\} \subset \mathbb{C}^4$. Let $\mathbb{S}^7$ be the unit sphere in $\mathbb{C}^4$.

Theorem 5.1. [5] *$V \cap \mathbb{S}^7$ is homeomorphic to $\mathbb{S}^5$.*

Since V is topologically the cone on the link $V \cap \mathbb{S}^7$,

Corollary 5.2. *V is homeomorphic to $\mathbb{R}^6$.*

Let $p \in V \setminus \{0\}$. We let $K_4 := \mathrm{Kalmod}(\mathbb{C}^4, V, \{p\})$. Now by Corollary 5.2 and Lemma 4.16 we have that K_4 is contractible. Hence by Theorem 4.19 we have that K_4 is diffeomorphic to $\mathbb{C}^4$. We will now construct the varieties K_n by induction. Suppose we have constructed the varieties $K_4, \ldots, K_n$, we wish to construct the variety K_{n+1}. We do this using example 4.18. This means that we will define $K_{n+1} := \mathrm{Kalmod}(K_n \times \mathbb{C}, K_n \times \{0\}, (q, 0))$ where q is a point in K_n. All these are affine varieties and hence have Stein structures by example 2.9. Finally, we define

$$K_n^k := \underset{i=1\ldots k}{\#_e} K_n$$

which is the k fold end connect sum of K_n. The aim of this thesis is to show that if $K_n^k \sim K_n^m$ then $k = m$.

5.2. **Proof of the second theorem in dimension 8.** In this section we wish to show that if $K_4^k \sim K_4^m$ then $k = m$. Let $M' := K_4^1$. By 2.17, $SH_*(K_4^k) = \prod_{i=0}^k SH_*(M')$. Hence if $i(M')$ is finite, $i(K_4^k) = i(M')^k$ where $i(M)$ denotes the number of idempotents of $SH_*(M)$ for any Stein manifold M. So in order to distinguish these manifolds, we need to show that $1 < i(M') < \infty$. Let $M'' := \mathbb{C}^4 \setminus V$ where V is defined in section 5.1. By Theorem 4.20, we have that $SH_*(M'') = SH_*(M')$. We have that $1 < i(M'') < \infty$ by the main results in section 9.3, hence $1 < i(M') < \infty$.

5.3. **Proof of the theorem in dimensions greater than 8.** Let $K_n := K_n^1$. For each $n > 4$ we need to show that $1 < i(K_n) < \infty$ in order to distinguish K_n^k. This is done by induction. Suppose that $1 < i(K_n) < \infty$

for some n, then we wish to show that $1 < i(K_{n+1}) < \infty$. We have by Theorem 4.20, that $SH_*(K_{n+1}) \cong SH_*(K_n \times \mathbb{C}^*)$. Let $B := K_n \times \mathbb{C}^*$. Let $SH_*^{\text{contr}}(\mathbb{C}^*)$ be the subring of $SH_*(\mathbb{C}^*)$ with H_1 grading 0.

One can check that $SH_*^{\text{contr}}(\mathbb{C}^*)$ is a subring isomorphic to $H^{1-*}(\mathbb{C}^*)$. In particular $SH_1^{\text{contr}}(\mathbb{C}^*) \cong \mathbb{Z}/2$. By the Künneth formula (see [26]), we have that

$$SH_{(n+1)+*}(B) \cong SH_{n+*}(K_n) \otimes SH_{1+*}(\mathbb{C}^*).$$

This ring is naturally graded by $H_1(\mathbb{C}^*)$. Hence any idempotent must be an element of

$$SH_{n+*}(K_n) \otimes SH_{1+*}^{\text{contr}}(\mathbb{C}^*) \subset SH_{n+*}(K_n) \otimes SH_{1+*}(\mathbb{C}^*)$$

by Lemma 9.6. The ring $SH_{1+*}^{\text{contr}}(\mathbb{C}^*)$ is naturally graded by the Robbin-Salamon index because $c_1(\mathbb{C}^*) = 0$. This means that any idempotents must live in:

$$SH_{n+*}(K_n) \otimes SH_1^{\text{contr}}(C^*) \cong SH_{n+*}(K_n) \otimes \mathbb{Z}/2 \cong SH_{n+*}(K_n).$$

Hence $i(K_{n+1}) = i(K_n)$. This means that by induction we have $1 < i(K_n) < \infty$ for all $n > 3$ as we proved $1 < i(K_3) < \infty$ in section 5.2. This proves our theorem.

6. Lefschetz fibration proofs

Here is the statement and proof of Theorem 4.5: *Let (E, π) be a compact convex Lefschetz fibration. There exists a constant $K > 0$ such that for all $k \geq K$ we have: $\omega := \Omega + k\pi^*(\omega_S)$ is a symplectic form, and the ω-dual λ of $\theta := \Theta + k\pi^*\theta_S$ is transverse to ∂E and pointing outwards.*

Proof. We let K be a large constant so that $\omega := \Omega + \pi^*(K\omega'_S)$ is a symplectic form (see [30, Lemma 1.5]). Let $\theta'_S := K\theta_S$ and $\omega'_S = d\theta'_S$ and λ'_S be the ω_S-dual of θ_S. Let $U \times V$ be some trivialisation of π around some point $p \in \pi^{-1}(\partial S)$ where $U \subset F$ and $V \subset S$. We let V be some small half disk around $\pi(p)$ and U is some small open ball. Let $\pi^1 : U \times V \twoheadrightarrow U$ be the natural projection. Let λ_F be the $\Omega|_F$-dual of $\Theta|_F$, and λ_Q be the horizontal lift of λ'_S. The ω-dual of Θ is equal to:

$$\lambda_F + W$$

where W is ω-orthogonal to the vertical fibres and is equal to 0 near the horizontal boundary of E. The ω-dual of $K\pi^*\theta'_S$ is equal to:

$$G\lambda_Q$$

where G is some function on $U \times V$. This means that the ω-dual of θ is:

$$\lambda = \lambda_F + W + G\lambda_Q.$$

Because $W = 0$ near the horizontal boundary and because the horizontal subspaces are tangent to the horizontal boundary, we have that λ is transverse to the horizontal boundary. In order to show that λ is transverse to the vertical boundary we need to ensure that we can make G very large compared to $\lambda_F + W$. This can be done by making K sufficiently large.

$\square$

7. A cofinal family compatible with a Lefschetz fibration

In this section we construct a family of Hamiltonians $H_k : \widehat{E} \to \mathbb{R}$ which behave well with respect to the Lefschetz fibration, so that

$$SH^l_*(E) := \varinjlim_k SH_*(E, H_k, J) = SH_*(E).$$

This would be obvious if H_k belonged to the "usual" class (i.e. linear of slope k on the contact cone) but our H_k looks like a product near the codimension 2 corner of E. Throughout this section, (E, π) is a compact convex Lefschetz fibration. We let $\Theta, \Omega, \theta, \omega$ be defined as in section 4.1.

Theorem 7.1. *Let $H : \widehat{E} \to \mathbb{R}$ be Lefschetz admissible for E with non-degenerate orbits. Then the space of regular almost complex structures $\mathcal{J}_{\text{reg}}(\widehat{E}, H)$ is of second category in the space $\mathcal{J}^h(\widehat{E})$ of admissible almost complex structures with respect to the C^∞ topology.*

This theorem comes from using results in [16]. This ensures that the moduli spaces of Floer trajectories are manifolds. For non-generic (H, J), $SH_*(H, J)$ is defined via small perturbations, and is independent of choice of small perturbation via continuation map techniques. We also need a maximum principle to ensure that the Floer moduli spaces have compactifications.

Let W be a connected component of ∂S where S is the base. Now $\widehat{S}$ has a cylindrical end $W \times [0, \infty)$. Let r_S be the coordinate for $[1, \infty)$. Let $u : \mathbb{D} \to \widehat{E}$ satisfy Floer's equations for some $J \in \mathcal{J}^h\widehat{E}$ and some admissible

Hamiltonian H. Here $\mathbb{D}$ is the unit disk parametrized by coordinates (s,t). We can write $H = \pi^* H_S + \pi_1^* H_F$ as in definition 4.10. We assume that $H_F = 0$ on F.

Lemma 7.2. *The function $f := r_S \circ \pi \circ u$ cannot have an interior maximum for r_S large.*

Proof. Let f have an interior maximum at $q \in \mathbb{D}$. Let U be a small neighbourhood of $u(q)$. The symplectic form ω on $\widehat{E}$ splits the tangent space of E into vertical planes and horizontal planes. Let V be the vertical plane field, and let P be the horizontal plane field (the ω-orthogonal of vertical tangent spaces of π). Let ω_S be the symplectic form on the base S, then $\omega_P := \pi^*\omega_S|_P$ is non-degenerate. This means that there exists a function

$$g : \pi^{-1}(W \times [0,\infty)) \to (0,\infty)$$

such that $g\omega_P = \omega|_P$. We may assume that $J(P) \subset P$ because J is compatible with $\widehat{E}$ if r_S is large. Let p be the natural projection $TE \to P$ induced by the splitting $TE = V \bigoplus P$.

Floer's equation for u splits up into a horizontal part associated to P and a vertical part associated to V. The horizontal part can be expressed as:

$$p(\frac{\partial u}{\partial s}) + Jp(\frac{\partial u}{\partial t}) = -J\frac{1}{g}G$$

where G is a vector field on P which is the ω_P-orthogonal to $d\pi^* H_S|_P$ in P. Hence $u' := \pi \circ u$ satisfies the equation:

$$\frac{\partial u'}{\partial s} + j\frac{\partial u'}{\partial t} = -j\frac{1}{u^*(g)}X_{H_S}$$

where j is the complex structure of S, and X_{H_S} is the Hamiltonian vector field of H_S in S. Rearranging the above equation gives:

$$u^*(g)\frac{\partial u'}{\partial s} + ju^*(g)\frac{\partial u'}{\partial t} = -jX_{H_S}.$$

Now locally around the point q, we can choose a reparameterisation of the coordinates (s,t) to new coordinates (s',t') so that u' satisfies:

$$\frac{\partial u'}{\partial s'} + j\frac{\partial u'}{\partial t'} = -jX_{H_S}$$

(i.e. $\frac{\partial s'}{\partial s} = \frac{\partial t'}{\partial t} = \frac{1}{u^*(g)}$ and $\frac{\partial t'}{\partial s} = \frac{\partial s'}{\partial t} = 0$). The above equation is Floer's equation which doesn't have a maximum by [25, Lemma 1.5]. This gives us a contradiction as we assumed f had a maximum at q. $\square$

The same argument above holds if u satisfied the parametrized Floer equations as well. Note we also have a maximum principle in the vertical direction as well. We have that the region A as defined in 4.10 looks like $\partial F \times [1,\infty) \times \widehat{S}$. Let r_F be the coordinate for $[0,\infty)$ in this product. Let $\pi_1 : A \twoheadrightarrow \partial F \times [1,\infty)$ be the natural projection. If a Floer trajectory u has an interior maximum with respect to r_F for r_F large, then $\pi_1 \circ u$ satisfies Floer's equations on F and hence has no maximum by [25, Lemma 1.5]. This gives us a contradiction. Hence $r_F \circ u$ has no maximum for r_F large. The above maximum principles and the regularity result from 7.1 ensures that $SH_*(\widehat{E}, H)$ is well defined.

We also need the 1-forms of our Lefschetz fibration to behave well at infinity. We have that $\pi^{-1}(W)$ is diffeomorphic to the mapping torus T_β of some $\Theta|_{\pi^{-1}(w)}$-exact symplectomorphism β, where $w \in W$. Write $T_\beta = ([0,1] \times F)/\{\sim\}$ where $\sim$ identifies $\{0\} \times F$ and $\{1\} \times F$ via β. Let s be the coordinate for $[0,1]$. Let $\Theta_F := \Theta|_{\pi^{-1}(w)}$. On T_β, let R be a 1-form which is equal to:

$$(1 - g(s))\Theta_F + g(s)\beta^*\Theta_F$$

where $g : [0,1] \to \mathbb{R}$ and near 0, $g = 0$ and near 1, $g = 1$. Also, g has non-negative derivative.

Lemma 7.3. *There is a family of* 1*-forms* Θ_t *in* E *such that* $\Theta_0 = \Theta$*,* $\theta_1|_{T_\beta} = R$*, and* Θ_t *induces a compact convex Lefschetz fibration structure on* (E, π)*.*

Proof. Θ_1 can be constructed as follows: We first extend Θ as in definition 4.6 so that it is defined on $\widehat{E}$. On $\widehat{S}$, we have a cylindrical end C corresponding to W which is symplectomorphic to $[1,\infty) \times W$. Let $B := \pi^{-1}(C)$. Then, $\Theta|_W = q^*\Theta|_{T_\beta}$ where $q : B \to T_\beta$ is the natural projection onto T_β. Let $f : C \to \mathbb{R}$ be a function which is 0 on ∂S and equal to 1 just a bit further out. Then we define $\Theta' := (1-f)\Theta + fR$. Let S' be a surface with boundary in $\widehat{S}$ such that $\partial S'$ is contained in $f^{-1}(1)$. We can choose S' so that there is a diffeomorphism $e : S \to S'$ such that it lifts to a diffeomorphism h from $\pi^{-1}(S)$ to $\pi^{-1}(S')$. Define $\Theta_1 := h^*\Theta'$.

To construct our deformation from Θ to Θ_1 we first deform Θ to $h^*\Theta$ (note, $h^*\Theta$ is well defined because Θ extends to $\widehat{E}$ and hence to $\pi^{-1}(S')$). This deformation comes from a smooth family of embeddings $e_t : S \to \widehat{S}$, where $e_1 = l \circ e$ where $l : S' \to \widehat{S}$ is the embedding of S' into $\widehat{S}$ and e_0 is the embedding of S into $\widehat{S}$.

The deformation from $h^*\Theta$ to Θ_1 is just $\Theta_t := (1-t)h^*\Theta + t\Theta_1$ $t \in [0,1]$. This is sufficient because $h^*\Theta$ and Θ' agree when restricted to each fibre of π.

$\square$

Definition 7.4. *(E,π) is said to be in* **standard form** *when Θ is constructed in the same way as Θ_1 as in Lemma 7.3 for all boundary components of S.*

From now on we assume that (E,π) is of standard form. The completion $\widehat{E}$ has Θ equal to some Θ_1 for every level set of $W \times [1,\infty)$.

Definition 7.5. *Let M be a manifold with contact form α. Let $S : \{\text{Reeb orbits}\} \to \mathbb{R}$, $S(o) := \int_o \alpha$. Then the* **period spectrum** *$\mathcal{S}(M)$ is the set $\operatorname{im}(S) \subset \mathbb{R}$. We say that the period spectrum is discrete and injective if the map S is injective and the period spectrum is discrete in $\mathbb{R}$.*

Definition 7.6. *Let H be a Hamiltonian on a symplectic manifold M. Then the* **action spectrum** *$\mathcal{S}(H)$ of H is defined to be:*

$$\mathcal{S}(H) := \{A_H(o) : o \text{ is a 1-periodic orbit of } X_H\}.$$

A_H is the action defined in section 2.2.

We let F be a smooth fibre of (E,π) and $\Theta_F := \Theta|_F$. Also we let S be the base of this fibration. Let r_S and r_F be the "cylindrical" coordinates on $\hat{S}$ and $\hat{F}$ respectively (i.e. $\omega_S = d(r_S\theta_S)$ on the cylindrical end at infinity and similarly with r_F). Let W be some connected component of the boundary of S. Let $C := \pi^{-1}(W) \times [1,\infty)$. Note: we will sometimes write r_S instead of $\pi^* r_S$ so that calculations are not so cluttered. We hope that this will make things easier to understand for the reader.

Being in standard form means that Θ is of the form:

$$(1-g(s))\Theta_F + g(s)\beta^*\Theta_F$$

in C. Let H be a Lefschetz admissible Hamiltonian of the form $H = \pi^*(H_S) + \pi_1^*(H_F)$ as in definition 4.10. Our symplectic form is $\omega := d\Theta + \pi^* dq$ for some sufficiently positive 2-form dq. On the cylindrical end C we assume that:

$$\pi^* q = r_S ds.$$

Let $v(s) := g'(s)(\beta^*\Theta_F - \Theta_F)$ be a closed 1-form on $\pi^{-1}(s)$. In order to prove theorem 4.11 we need to do some action calculations. Here is a

Lemma containing some of these. The proof of this Lemma also sets up some notation which will be used in the proof of 4.11.

Lemma 7.7. *Let H be a Lefschetz admissible Hamiltonian of the form $H = \pi^*(H_S) + \pi_1^*(H_F)$ as in definition 4.10. Let $\pi^*(H_S)$ be equal to $k(r_S)$ on C. The action of an orbit γ of H in C is given by:*

$$A_H(\gamma) = \int_\gamma \{(-X_{H_F} - k'(r_S)X_{v(s)})\Theta + r_S k'(r_S)\} - \pi_1^* H_F - k(r_S).$$

X_{H_F} and $X_{v(s)}$ are vector fields in W which we will describe in the proof.

Proof. Recall $W = ([0,1] \times F)/\{\sim\}$ where $\sim$ identifies $\{0\} \times F$ and $\{1\} \times F$ via β. Also, X_{H_F} is the Hamiltonian vector field for H_F in the fibre F. Because H_F is invariant under β we have that X_{H_F} is invariant under β. Hence this vector field lifts to the product $([0,1] \times F)$ and then descends to the quotient W. Remember that s is the coordinate for $[0,1]$. We have that $X_{v(a)}$ is the Hamiltonian vector field of $v(a)$ in the fibre $s = a$ in W. Hence the family of vector fields $X_{v(a)}$ parametrized by $a \in [0,1]$ lifts to a universal vector field $X_{v(s)}$ in the product $([0,1] \times F)$ and it descends to the quotient W. On the cylindrical end C we have:

$$\theta = \pi^* q + \Theta,$$

$$\omega = d\Theta + \pi^* dq = d\Theta_F + ds \wedge v + dr_S \wedge ds$$

because $\beta^* d\Theta_F = d\Theta_F$ and

$$dH = d(\pi_1 \circ H_F) + k'(r_S) dr_S.$$

Hence, the Hamiltonian vector field of H is:

$$X_H = X_{H_F} - k'(r_S)\frac{\partial}{\partial s} + k'(r_S)X_{v(s)} + X_{H_F}(v)\frac{\partial}{\partial r_S}.$$

Because u has support in F and $H_F = 0$ in F, we have $X_{H_F}(v) = 0$. Finally the action of an orbit γ of H in C is given by:

$$A_H(\gamma) = \int_\gamma \{(-X_{H_F} - k'(r_S)X_{v(s)})\Theta + r_S k'(r_S)\} - \pi_1^* H_F - k(r_S).$$

□

Theorem 7.8. *Let $H_p : \widehat{E} \to \mathbb{R}$ be Lefschetz admissible for E with slope p on the base and the fibre. We have that by Theorem 4.5 that $(\widehat{E}, \theta)$ is* *a convex symplectic manifold. Then there is a cofinal family of admissible Hamiltonians $K_p : \widehat{E} \to \mathbb{R}$ with respect to the above convex symplectic structure (not the Lefschetz fibration structure) such that:*

(1) *The periodic orbits of K_p of positive action are in 1-1 correspondence with the periodic orbits of H_p. This correspondence preserves index. Also the moduli spaces of Floer trajectories are canonically isomorphic between respective orbits.*
(2) *$K_p < 0$ on $E \subset \widehat{E}$.*
(3) *$K_p|_E$ tends to 0 in the C^2 norm on E as p tends to infinity.*

This theorem means that:

$$\varinjlim_p SH_*^{[0,\infty)}(K_p) = \varinjlim_p SH_*(H_p) \tag{1}$$

$SH_*^{[0,\infty)}(K_p) := SH_*(K_p)/SH_*^{(-\infty,0)}(K_p)$ where $SH_*^{(-\infty,0)}$ is the symplectic homology group generated by orbits of negative action. We also have:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p) \tag{2}$$

This is because there exists a cofinal family of Hamiltonians G_p such that:

(1) $G_p < 0$ on $E \subset \widehat{E}$.
(2) $G_p|_E$ tends to 0 in the C^2 norm on E as p tends to infinity.
(3) All the periodic orbits of G_p have positive action.

Using the fact that both K_p and G_p are cofinal, tending to 0 in the C^2 norm on E and are non-positive on E, there exist sequences p_i and q_i such that:

$$K_{p_i} \leq G_{q_i} \leq K_{p_{i+1}}$$

for all i. Hence:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p).$$

Property (3) implies:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*(G_p).$$

This gives us equation 2. Combining this with equation 1 gives:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*(H_p).$$

This proves Theorem 4.11. Note: the Lefschetz fibration in Theorem 4.11 may not be in standard form, but we can deform it using Lemma 7.3 to a Lefschetz fibration in standard form. This induces an isomorphism between respective symplectic homology groups associated to each Lefschetz fibration.

Proof. of Theorem 7.8. We will slightly modify the proof of a related result in [26]. Also, we will use the notation set up already in 7.7. We assume that the period spectra of ∂F and ∂S are discrete and injective. The Hamiltonians H_F and H_S have slope $\lambda \notin \mathcal{S}(S) \cup \mathcal{S}(F)$. We assume $H_F = 0$ (resp. $H_S = 0$) in F (resp. S) and $H_F = h_F(r_F)$ (resp. $H_S = h_S(r_S)$) for $r_F \geq 1$ (resp. $r_S \geq 1$). For $r_F \geq 1 + \frac{\delta}{\lambda}$, $h'_F = \lambda$ and for $r_S \geq 1 + \frac{\delta}{\lambda}$ $h'_S = \lambda$. We choose $\delta > 0$ to be a small constant. We can deform H_p into $H := \pi^* H_S + \pi_1^* H_F$. Let NE be a neighbourhood of E which contains all orbits of H. What we want to do is to is choose a Hamiltonian $H_3 : \widehat{E} \to \mathbb{R}$ so that there exist constants c, ϵ such that:

(1) $H_3 = H$ on NE.
(2) Any curve in $\widehat{E}$ with each end converging to an orbit in NE satisfying a Floer type equation (e.g Floer trajectory or pair of pants) is entirely contained in NE.
(3) Outside some large compact set we have that H_3 is linear of slope $\lambda^{\frac{1}{4}}$ with respect to the cylindrical end of the convex symplectic structure associated to $\widehat{E}$.
(4) Any additional orbits of H_3 (i.e. orbits outside NE) have negative action.

We will achieve this in 4 sections $(a) - (d)$. The function H_3 will be constructed in 3 stages in sections (a),(b),(c) respectively (i.e. we first construct H_1 from H in (a) and then H_2 from H_1 in (b) and then H_3 from H_2 in (c)).

In section (a) we will construct a Hamiltonian $H_{F,1} : \widehat{F} \to \mathbb{R}$ so that:

(1) on NF, $H_{F,1}$ is equal to H_F where NF is a small neighbourhood of F which contains all the orbits of H_F.
(2) on $r_F \geq A$, $H_{F,1}$ is constant for some A to be defined later.
(3) $H_{F,1}$ is a function of r_F on the cylindrical end of F.

We also construct a similar Hamiltonian $H_{S,1}$ which is associated with H_S. Finally in this section, we show that the orbits of

$$H_1 := \pi^*(H_{S,1}) + \pi_1^*(H_{F,1})$$

outside NE have negative action. We already know that the orbits inside NE are the same as the orbits of H because $H = H_1$ inside NE.

In section (b) we will construct a Hamiltonian H_2 such that:

(1) $H_2 = H_1$ on $r_S \leq A, r_F \leq A$.
(2) H_2 is constant outside $r_S \leq B, r_F \leq B$ for some constant $B > A$.
(3) Any orbit of H_2 outside $r_S \leq A, r_F \leq A$ has negative action. This ensures that all the orbits of H_2 of positive action are the same as the orbits of H.

In section (c) we will finally construct H_3. We choose some admissible Hamiltonian K with respect to the convex symplectic structure (E, θ) which is equal to 0 on $r_S \leq C, r_F \leq C$ for some chosen $C > B$. Then we let $H_3 := H_2 + K$. We also ensure that K has slope proportional to $\sqrt{\lambda}$ which ensures that the additional orbits created on top of the orbits of H_2 have negative action.

In section (d) we will show that no Floer trajectory of H_3 connecting orbits inside E can intersect $r_F = C$ or $r_S = C$. If we combine this fact with the maximum principle in Lemma 7.2 and also a maximum principle from [25, Lemma 1.5] we find that any Floer trajectory connecting orbits inside E must be contained in E. This ensures that the Floer trajectories connecting orbits inside E are identical to the Floer trajectories of H and hence we get that:

$$SH_*^{[0,\infty)}(H_3) = SH_*(H).$$

And this gives us our result if we set $H_3 = K_p$ where we view λ as a function of p which tends to infinity as p tends to infinity.

Define:

$$\mu_\lambda := \mathrm{dist}(\lambda, \mathcal{S}(S) \cup \mathcal{S}(F)).$$

(a) We first modify a construction due to Herman in [20] which takes some normal admissible Hamiltonian on a finite type convex symplectic manifold and makes it constant near infinity so that the only additional periodic orbits have negative action. We need to modify this argument because we need greater control over the Hamiltonian flow $X_{\pi^* H_S}$. From now on we will assume that $H_S = 0$ on S and is equal to $k(r_S)$ for $r_S \geq 1$. Similarly we assume that $H_F = 0$ on F and is a function of r_F on $r_F \geq 1$. The first thing we need to do is to modify H_S and H_F to $H_{S,1} : S \to \mathbb{R}$ and $H_{F,1} : F \to \mathbb{R}$ so that they are constant at infinity and such that the additional orbits added to $H_1 := H_{F,1} + \pi^* H_{S,1}$ have negative action. We will use all the notation as in the proof of 7.7. Define:

$$R^s := \sup|X_{v(s)}(\Theta_F)|,$$

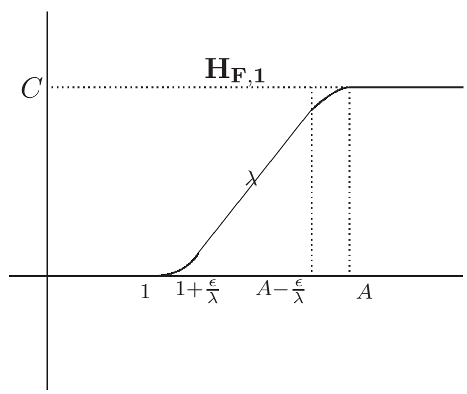
$$R := \sup\{R^s : s \in [0,1]\}.$$

Define:

$$A = A(\lambda) := (6+R)\lambda/\mu_\lambda > 1.$$

We can assume that $A > 1$ because we can choose μ_λ to be arbitrarily small. We define $H_{F,1}$ to be equal to H_F on $r_F < A - \frac{\epsilon}{\lambda}$. Hence on the interior of F, $H_{F,1}$ is 0. Set $H_{F,1} = h_{F,1}(r_F)$ for $r_F \geq 1$ with non negative derivative. We define $h_{F,1}$ as follows: $h'_{F,1}(r_F)$ is equal to λ on $[1+\frac{\epsilon}{\lambda}, A - \frac{\epsilon}{\lambda}]$ For $r_{F,1} \geq A$ set $h_F(r_F)$ to be constant and equal to C where C is arbitrarily close to $\lambda(A-1)$. The Hamiltonian $H_{F,1}$ takes values in $[-\epsilon, \epsilon]$ for $r_F \in [1, 1+\frac{\epsilon}{\lambda}]$ and in $[\lambda(A-1) - 2\epsilon, \lambda(A-1)]$ for $r_F \geq A - \frac{\epsilon}{\lambda}$. Here is a picture:

Figure 7.9.



For notational convenience we will write $H_{F,1}$ instead of $\pi_1^* H_{F,1}$. Assume that $H_{S,1}$ is a Hamiltonian such that on the cylindrical end C we have that $H_{S,1}$ is equal to $k(r_S)$. We define $H_{S,1}$ so that it behaves in a similar way to $H_{F,1}$. (i.e. we have that the graph of $k(r_S)$ is the same as the graph in figure 7.9). We want to show that the additional orbits of $H_1 := H_{F,1} + \pi^* H_{S,1}$ only have negative action. These additional orbits lie in the region $r_S \in (A - \frac{\epsilon}{\lambda}, A)$ and $r_F \in (A - \frac{\epsilon}{\lambda}, A)$. We will first consider the orbits in $r_S \in (A - \frac{\epsilon}{\lambda}, A)$. The orbits of $H_{F,1}$ have action at most λ because $h'_F \leq \lambda$, i.e. $\int_{\text{orbit}} -X_{H_{F,1}}\Theta_F \leq \lambda$. Let p be a point on some orbit o. Remember that the smallest distance between λ and the period spectrum of ∂F is $\geq \mu_\lambda$. Hence near p we have $|k'(r_S)| < \lambda - \mu_\lambda$. Hence $|k'(r_S)X_{v(s)}\Theta| \leq R(\lambda - \mu_\lambda)$ and $r_S k'(r_S) \leq A(\lambda - \mu_\lambda)$. Also, because $H_{F,1}(v) = 0$, we have that the orbit stays in $r_S \in (A - \frac{\delta}{\lambda}, A)$. Hence the action of an orbit of H_1 near $r_S = A$ in the region $r_F \leq 1$ is less than or equal to:

$$\lambda + (R + A)(\lambda - \mu_\lambda) - C,$$

$$\leq (R + 1 + 1 + A - A - (6 + R))\lambda \leq -3\lambda \to -\infty.$$

Now the case for orbits near $r_F = A$ is exactly the same as in Oancea's paper [26]. Near $r_F = A$ we have that $v = 0$, hence the action is at most:

$$\lambda + A(\lambda - \mu_\lambda) - C$$

$$\leq (1 + A - A + 1 - (6 + R))\lambda \leq -3\lambda \to -\infty.$$

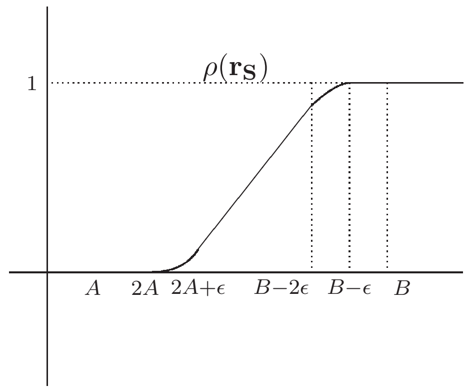
Hence all the additional orbits of H_1 have actions tending to $-\infty$.

(b) Now we modify H_1 so that it is constant and equal to $2C$ outside the compact set $\{r_S \leq B, r_F \leq B\}$ with $B = A\sqrt{\lambda}$. This is true already on $\{r_S \geq A\} \cap \{r_F \geq A\}$, so we only need to consider the case $\{r_S \geq A\} \cap \{r_F \leq A\}$ and $\{r_F \geq A\} \cap \{r_S \leq A\}$. Now the case $\{r_F \geq A\} \cap \{r_S \leq A\}$ is exactly the same as the case Oancea dealt with in [26, section (c)]. (Note: in Oancea's paper, $A = 5\lambda/\mu_\lambda$ instead of $(6 + R)\lambda/\mu_\lambda$ but this makes no difference.) In Oancea's paper he deals with this case by modifying $\pi_1^* H_{F,1}$ to some new Hamiltonian $H_{F,2}$. We will mimic Oancea's paper for the case $\{r_S \geq A, r_F \leq A\}$. This will involve modifying the Hamiltonian $\pi^* H_S$ to some new Hamiltonian $H_{S,2}$. Let:

$$H_{S,2} : W \times [A, \infty) \longrightarrow \mathbb{R},$$

$$H_{S,2} = (1 - \rho(r_S))H_{F,1} + \rho(r_S)C$$

where x is a point in F and s parameterises $[0, 1]$. Also, $\rho : [A, \infty) \to [0, 1]$ with $\rho = 0$ on $[A, 2A]$, $\rho = 1$ for $r_S \geq B - \epsilon$, ρ strictly increasing on $[2A, B - \epsilon]$, and $\rho' = \text{const} \in \left[\frac{1}{B-2A-\epsilon}, \frac{1}{B-2A-3\epsilon}\right]$ on $[2A + \epsilon, B - 2\epsilon]$. The graph of ρ is:



We also have:

$$dH_{S,2} = (1-\rho(r_S))dH_{F,1} + (C - H_{F,1})\rho'(r_S)dr_S,$$

$$X_{H_{S,2}} = (1-\rho(r_S))(X_{H_{F,1}} + X_{H_{F,1}}(v)\frac{\partial}{\partial r_S}) + (C - H_{F,1})\rho'(r_S)(X_{v(s)} - \frac{\partial}{\partial s}).$$

Let $H_2 := H_{S,2} + H_{F,2}$. We have assumed earlier that $X_{H_F}(v) = 0$, and hence $X_{H_{F,1}}(v) = 0$. This means that projecting orbits down to the base S produces orbits of the Hamiltonian H_S. In particular we can assume that the orbits of H_2 on $r_S \geq 1$ stay in each level set $r_S = \text{const}$. For some orbit o of H_2, let:

$$A_1 := -\int_o \left[(1-\rho(r_S))(X_{H_{F,1}}) + (C - H_{F,1})\rho'(r_S)(X_{v(s)})\right](\Theta),$$

$$A_2 := \int_o \left[(C - H_{F,1})\rho'(r_S)r_S\right].$$

The action of this orbit o is equal to:

$$A_1 + A_2 - (C - H_{F,1})\rho(r_S) - C$$

(Remember $H_{F,2} = C$ on $\{r_S \geq A, r_F \leq A\}$). We first consider orbits where $v \neq 0$ on some part of the orbit. Now these orbits are located in the interior of each fibre F. Hence, we can assume that H_F is 0. Also we may assume that $X_{H_{F,1}}(\Theta)$ is bounded above by ϵ. Because $X_{H_{F,1}}(v) = 0$, the r_S coordinate of the orbit is constant, hence we only need to consider 3 cases (i,ii,iii) for these orbits:

(i) $r_S \in [A, 2A] \cup [B - \frac{\epsilon}{\lambda}, \infty)$ Now, $\rho' = 0$ and $X_{H_{F,1}}\Theta$ is 0. Hence the action is bounded above by $-C$.

(ii) $r_S \in [2A, \frac{A+B}{2}]$ $\rho' \leq \frac{1}{B-2A-3\epsilon}$. S is bounded above by $\frac{A+B}{2} + 1$. Also, $|X_{v(s)}(\Theta)|$ is bounded above by the constant R. For large enough λ we also have that $\frac{A+B}{2}\frac{1}{B-2A-3\epsilon}$ is bounded above by $\frac{3}{4}$ because this expression tends to $\frac{1}{2}$ as $\lambda \to \infty$. Also, we can ensure that $\epsilon + C.\frac{1}{B-2A-3\epsilon}.R \leq \frac{1}{8}C$ for large enough λ. Hence our action is bounded above by:

$$\epsilon + C.\frac{1}{B-2A-3\epsilon}.R + C.\frac{A+B}{2}\frac{1}{B-2A-3\epsilon} - C$$
$$\leq -\frac{1}{8}C$$

for large enough λ.

(iii) $r_S \in [\frac{A+B}{2}, B - \frac{\epsilon}{\lambda}]$ In this case we have $\rho \in [\frac{1}{2}, 1]$. Hence for λ big enough we have that the action is bounded above by:

$$C.\frac{1}{B-2A-3\epsilon}.R + C.\frac{1}{B-2A-3\epsilon}.B - C.\frac{1}{2} - C$$
$$\leq -\frac{1}{8}C.$$

Hence all orbits which pass through $v \neq 0$ have negative action in $W \times [A, \infty)$. Now, when $v = 0$ the action of the orbits are the same as in Oancea's paper [26] (although $A = 5\lambda/\mu_\lambda$ instead of $(6+R)\lambda/\mu_\lambda$, but this doesn't matter). Hence, these orbits also have action tending to $-\infty$ as well. Hence we have a Hamiltonian which is equal to H on E and is constant and equal to $2C$ further out, and such that the only additional orbits have negative action.

(c) Finally we need to make this Hamiltonian cofinal by choosing some contact boundary and forcing H to be linear at this contact boundary, and such that the only additional orbits have negative action as well.

Let Z be the Liouville vector field which is ω-dual to $\theta := \Theta + \pi^* q$. Then this vector field is expressed as:

$$Z := Z' + (r_S - Z'(v))(\frac{\partial}{\partial r_S})$$

where Z' is the Liouville vector field in F associated to $\Theta|_{\pi^{-1}(y)}$. We assume that λ is big so that $A\sqrt{\lambda} = B > |Z'(v)|$. Consider the sets:

$$\mathbf{I} = \partial S \times [1, \infty) \times \partial F \times [1, \infty),$$
$$\mathbf{II} = S \times \partial F \times [1, \infty),$$
$$\mathbf{III} = W \times [1, \infty).$$

We define a hypersurface $\Sigma \subset \hat{E}$ such that:

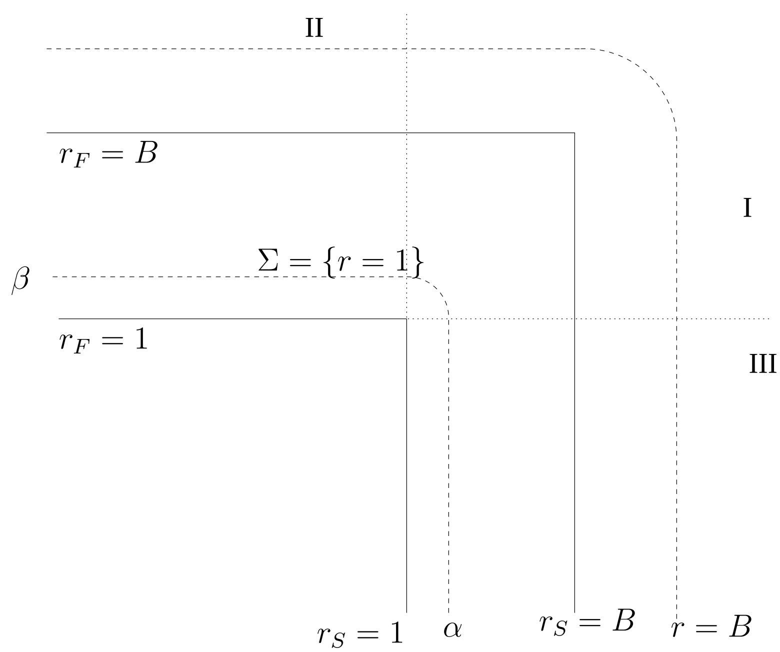
$$r_S|_{\Sigma \cup \mathbf{III}} = \alpha > 1,$$

$$r_S|_{\Sigma\cup\mathbf{I}} \in [1,\alpha],$$
$$r_F|_{\Sigma\cup\mathbf{II}} = \beta > 1,$$
$$r_F|_{\Sigma\cup\mathbf{I}} \in [1,\beta].$$

We can ensure that Z is transverse to this hypersurface, and hence the flow of Z gives us a map:

$$\Psi : \Sigma \times [1,\infty) \to \hat{E}$$

which gives us a cylindrical end for $\hat{E}$. Let r be the radial coordinate for this cylindrical end. Here is a diagram illustrating these regions:



(see Oancea's paper: [26, figure 3]).

Then $\Psi^{-1}(\{r_S \geq B\} \cup \{r_F \geq B\}) \supset \{r \geq B\}$. H_2 is constant and equal to $2C$ on $\{r_S \geq B\} \cup \{r_F \geq B\}$. Let K be a Hamiltonian which is equal to 0 on the region $\{r < B\}$ and is equal to $l(r)$ in $\{r \geq B\}$ where $l'(r) \geq 0$ and for $r \geq B + \epsilon$ we have $l'(r) = \mu \notin \mathcal{S}(\Sigma)$, where μ will be arbitrarily close to $\lambda^{\frac{1}{4}}$. The point is that $K = 0$ on the region

$$\{r_S \leq B\} \cap \{r_F \leq B\}.$$

This means that the orbits lie in the region where H_2 is constant and equal to $2C$. Define:

$$H_3 := H_2 + K.$$

Now the actions of the orbits of K are bounded above by $BV\lambda^{\frac{1}{4}}$ for some constant V. Hence the orbits of H_3 inside $\{r \geq B+\epsilon\}$ have action bounded above by:

$$(B+\epsilon)V\lambda^{\frac{1}{4}} - 2C = V\lambda^{\frac{3}{4}}A + V\epsilon\lambda^{\frac{1}{4}} - \lambda(A-1).$$

For large enough λ we have that this quantity is negative. Hence the actions of the additional orbits are negative.

(d) Finally using [26, Lemma 1], we have that any curve u passing through $\{r_S \in [A, 2A]\}$ must have area greater than cA for some constant c (i.e. $\pi \circ u$ has area less than the area of u, so we can use [26, Lemma 1]). The actions of orbits inside E are bounded above by $P\lambda$ where P is some constant. This means that for small enough μ_λ (i.e. so that $P\lambda < cA$) we have ensured that no Floer trajectory between orbits of positive action can pass through $\{r_S \in [A, 2A]\}$. We have a similar statement for r_F.

Hence by the maximum principle (cf. Lemma 7.2 and [25, Lemma 1.5]) we have that any Floer trajectory connecting orbits of positive action stays within $\{r_S \leq 1, r_F \leq 1\}$ (this uses the fact that on $\{r_F \leq 2A\} \cap \{r_S \leq 2A\}$ we have that our Hamiltonian H_3 is equal to $H_1 = \pi^* H_{S,1} + \pi_1^* H_{F,1}$). Note: Lemma 1 requires that the Hamiltonian be equal to 0 on $\{r_S \in [A, 2A]\}$ which means that it cannot have non-degenerate orbits. This problem can be solved as follows: Let H_k be a sequence of Hamiltonians with non-degenerate orbits and let J_k be a sequence of almost complex structures such that $SH_*(E, H_k, J_k)$ is well defined and (H_k, J_k) C^2 converges to (H, J) as $k \to \infty$. If there is a Floer trajectory passing through $\{r_S \in [A, 2A]\}$ for some sequence of (H_k, J_k)'s converging to (H, J) then by Gromov compactness (see [4]) we have that there is a holomorphic curve passing through $\{r_S \in [A, 2A]\}$ as $H = 0$ in this region (it can't be a Morse flow line because we can ensure that the Morse flow lines of H_k travel in the wrong direction). But this is impossible, hence for some large enough k we have no Floer trajectory passing through $\{r_S \in [A, 2A]\}$. We can use an identical argument with the pair of pants surface satisfying Floer type equations. $\square$

7.1. **A better cofinal family for the Lefschetz fibration.** In this section we will prove Theorem 4.13. We consider a compact convex Lefschetz fibration (E, π) fibred over the disc $\mathbb{D}$. Basically the cofinal family is such that $H_F = 0$. This means that the boundary of F does not contribute to symplectic homology of the Lefschetz fibration. The key idea is that near the boundary of F the Lefschetz fibration looks like a product $\mathbb{D} \times \text{nhd}(\partial F)$ and because symplectic homology of the disc is 0 we should get that the boundary contributes nothing. Statement of Theorem 4.13:

$$SH_*(E) \cong SH_*^{\text{lef}}(E).$$

From now on we will use the same notation as established in the proof of lemma 7.7. Before we prove Theorem 4.13, we will write a short lemma on the $\mathbb{Z}$ grading of $SH_*(E)$.

Lemma 7.10. *Let $\widehat{F} := \pi^{-1}(a) \subset \widehat{E}$ ($a \in \mathbb{D}$). Suppose we have trivialisations of $\mathcal{K}_{\widehat{E}}$ and $\mathcal{K}_{\widehat{S}}$ (these are the canonical bundles for $\widehat{E}$ and $\widehat{S}$ respectively); these naturally induce a trivialisation of $\mathcal{K}_{\widehat{F}}$ away from F. If we smoothly move a, then this smoothly changes the trivialisation.*

Proof. of Lemma 7.10.

We choose a $J \in \mathcal{J}^h(E)$. The bundle E away from E^{crit} has a connection induced by the symplectic structure. Let $A \subset \widehat{E}$ be defined as in 4.10. Let U be a subset of A where

(1) π is J holomorphic.
(2) U is of the form $r \geq K$ where r is the coordinate for $[1, \infty)$ in A (see definition 4.10).

This means that in U, we have that the horizontal plane bundle $\mathbb{H}$ is J holomorphic. Choose a global holomorphic section of $\mathcal{K}_{\widehat{S}}$ and lift this to a section s of $\mathbb{H}$. Choose a global holomorphic section t of $\mathcal{K}_{\widehat{E}}$. The tangent bundle of $\widehat{F}$ is isomorphic to the ω-orthogonal bundle T of $\mathbb{H}$. This is also a holomorphic bundle. There exists a unique holomorphic section w of T such that $s \wedge w = t$. Hence, w is our nontrivial holomorphic section of T in $U \cup \widehat{F}$. This can be extended to $A \cup \widehat{F}$ by property (2). $\square$

In the following proof, whenever we talk about indices of orbits of $\widehat{F}$ outside F, we do this with respect to the trivialisation of Lemma 7.10 above. We do not deal with orbits inside F so this trivialisation is sufficient.

Proof. of Theorem 4.13. We start with a Lefschetz admissible Hamiltonian $H = \pi^* H_S + \pi_1^* H_F$. The idea is to consider the actions of the respective orbits. We assume that the period spectrum of ∂F is discrete and injective. We assume that $H_F = 0$ on F and is equal to $h_F(r_F)$ outside F with $h'_F > 0$. The orbits of H_F consist of constant orbits in F and $\mathbb{S}^1$ families of orbits corresponding to periodic Reeb orbits outside F. We can perturb H_F by a very small amount outside F so that each $\mathbb{S}^1$ family of Reeb orbits becomes a pair of non-degenerate orbits (see [25, Section 3.3]). Hence, we have a Hamiltonian H_F which is equal to 0 inside F and all its orbits outside F are non-degenerate. These perturbations can be made so that the action spectrum of $H_F|_{\widehat{F}\setminus F}$ is discrete and injective. We set the slope of H_F at infinity to be equal to $\lambda \notin \mathbb{S}(\partial F)$. Also we assume $H_S = 0$ on $\mathbb{D}$ and is equal to $h_S(r_S)$ on the cylindrical end $\mathbb{C} \setminus \mathbb{D}$ with $h'_S > 0$. We perturb H_S slightly so that all its orbits are non-degenerate and such that the action spectrum of H_S is discrete and injective. This perturbation is done explicitly in [25, Section 3.3]. The outcome is that we have a periodic orbit corresponding to a fixed point of H_S at the origin of index 1; then we have pairs of orbits of index $2l, 2l+1$ ($l \geq 1$) which came from perturbing the $\mathbb{S}^1$ families of Reeb orbits of $\partial\mathbb{D}$.

Let m be an integer greater than the maximal modulus of the Robbin-Salamon index of orbits of H_F. We think of m as an integer which depends on λ. We also assume that m tends to infinity as λ tends to ∞. Note that all the orbits of H_S are exact (i.e. of the form $\delta(x)$ where x is an SH_*-chain for $\mathbb{D}$) except possibly the orbit o of highest index. We can assume that the slope of H_S is steep enough so o has index $> 2m$. This means that all the orbits in H_S of index $\leq 2m$ are exact. The periodic orbits of H are:

(1) fixed points from the interior E.
(2) fixed points of τ^n for each n where $\tau : F \to F$ is the monodromy symplectomorphism of the loop ∂S. (This isn't quite true, they are actually fixed points in τ^n counted twice modulo a $\mathbb{Z}/n\mathbb{Z}$ action. See lemma 7.7 to see the flow.)
(3) periodic orbits on A.

What we want to do, roughly, is to show that all the orbits in A are exact in the chain complex and hence they do not contribute to $SH_*(E)$. In fact what we do is construct a long exact sequence where one of the terms is chain isomorphic to the Floer chain complex of H, another term is chain isomorphic to the Floer chain complex of $\pi^* H_S + \pi_1^* K_F$ where K_F is some admissible Hamiltonian on $\widehat{F}$ which is equal to 0 on F and has a very small slope at infinity. In fact the slope is smaller than the smallest action value of $\mathbb{S}(\partial F)$ which means there are no periodic orbits of K outside F. This chain complex is used in the definition of Lefschetz symplectic homology. The third term in this long exact sequence is shown to be equal to 0. Hence, taking direct limits (and showing the direct limit structures are compatible) gives us our isomorphism between Lefschetz symplectic homology and normal symplectic homology. Let CF be the closure of $\widehat{F} \setminus F$ in $\widehat{F}$. Then $CF = \partial F \times [1, \infty)$. By abuse of notation we assume that r_F is the coordinate for the $[1, \infty)$ part of CF. Now, $A = CF \times \widehat{S}$ and $H|_A = \pi^* H_S + \pi_1^* H_F$. Hence, orbits of $H|_A$ come in pairs (γ, Γ) where γ is an orbit of H_S and Γ is an orbit of H_F. These orbits are non-degenerate as both γ and Γ are non-degenerate. Let Q be the highest value in the action spectrum of H_S. Also, let L be the highest value in the action spectrum of H. Choose an almost complex structure $J \in \mathfrak{J}^h(E)$ such that $J|_{CF}$ is invariant under translations $r_F \to r_F +$ const. Then by [26, Lemma 1] there exists a $K > 0$ such that any J holomorphic curve which intersects $r_F = 1$ and $r_F = K$ has volume $\geq \max(2Q, L+1)$. Let ϵ be the smallest positive difference in action between two orbits of H_F. Let $F_b \subset \widehat{F}$ be equal to $\{r_F \leq b\}$. The function H_F is a function of (r_F, a) where a is a point in ∂F. Let H_F^b be a Hamiltonian on $\widehat{F}$ which is equal to zero on F_b and which is equal to $H_F(r_F/b, a)$ outside F_b. The Hamiltonian H_F^b has the same orbits as H_F except that the smallest difference in action between two orbits is equal to ϵb. Choose b such that $\epsilon b \geq \max(2Q, L+1)$ and such that $b \geq K$. Define a new Hamiltonian H^b to be the same as H except that we replace H_F with H_F^b (i.e. $H^b = \pi^* H_S + \pi_1^* H_F^b$). This Hamiltonian has the same orbits as H except that the actions of the orbits near $\partial_h E$ have changed. Let

$$B := \{r_F \geq b\} \subset \widehat{E},\ B' := \widehat{E} \setminus B.$$

The Hamiltonian H^b has degenerate periodic orbits. We wish to construct a Hamiltonian H_k^b with some $J_k \in \mathfrak{J}^h(\widehat{E}, H_k^b)$ with the following properties:

(1) H_k^b has non-degenerate periodic orbits
(2) all the orbits have positive action.
(3) $H_k^b|_B = H^b|_B$
(4) The (H_k^b, J_k) Floer trajectories starting at orbits inside B' cannot intersect $r_F = K$.

This is done as follows: Let H_k^b be a sequence of Lefschetz admissible Hamiltonians which C^2 converge to H^b as $k \to \infty$ and such that they have non-degenerate periodic orbits. We can assume that

$$H_k^b|_B = H^b|_B$$

because H^b already has non-degenerate periodic orbits in this region. We can assume that on the region $\{r_F \in [1, b]\}$, H_k^b has no periodic orbits (e.g it could be equal to $\frac{\delta}{k}(r_F - 1)$ for some very small δ; the H_k^b still C^2

converge to H^b as $H^b = 0$ on this region). As a result we can assume that the actions of the orbits of H^b_k in the region B' have action $\leq L$. Let $J_k \in \mathfrak{J}^h(\widehat{E}, H^b_k)$ be such that J_k C^∞ converges to J as $k \to \infty$. Hence all we need to show is that for large enough k, condition (4) is satisfied. Suppose for a contradiction, that there is a sequence $k_i \to \infty$ such that we have a sequence of Floer trajectories u_i in the region B' and passing through $r_F = K$. Then because $H^b_{k_i}$ already satisfies condition (2) and all the orbits inside B' have action $\leq L$ we have that the areas of the u_i are bounded. Then, by Gromov compactness (see [4]) we have that there is a (H^b, J) Floer trajectory u starting inside B' and passing through $r_F = K$. All the orbits of H^b in B' are actually outside $\{r_F \geq 1\}$, hence u intersects $\{r_F = 1\}$ and $\{r_F = K\}$. Also $\pi_1 \circ u$ is holomorphic in $r_F \in [1, K]$ and hence u has volume $\geq \max(2Q, L+1)$. But this is impossible as the maximum action of an orbit in B' must be L as $H^b = H$ in B'. Hence there exists a $k \geq 0$ such that (H^b_k, J_k) satisfies conditions (1) to (4).

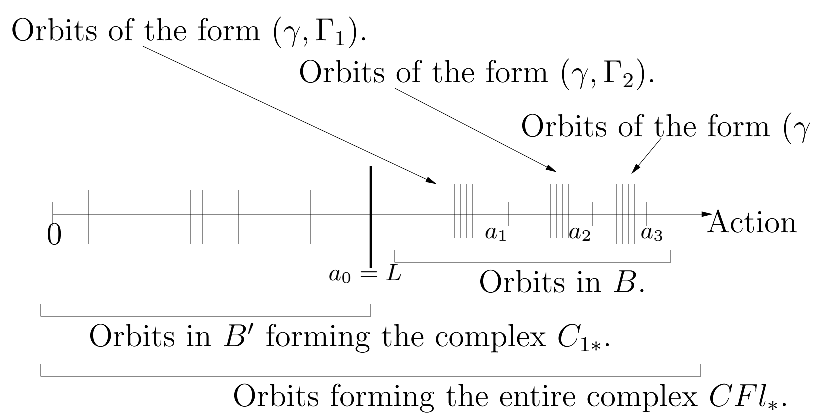
Let O_1 be the set of periodic orbits of H^b_k inside B' and O_2 the set of periodic orbits inside B. Linear combinations of orbits in O_1 form a subcomplex C_{1*} of the Floer complex $CFl := CF_*(H^b_k, J_k)$ by property (4) mentioned earlier. Also, we have a quotient complex $C_{2*} := CFl/C_{1*}$. This gives us a short exact sequence of complexes:

$$0 \to C_{1*} \to CFl_* \to C_{2*} \to 0.$$

We wish to show that all the homology groups of C_{2*} of index $\leq m$ are 0. This will give us our isomorphism between Lefschetz symplectic homology and ordinary symplectic homology as C_{1*} contributes to the chain complex for Lefschetz symplectic homology and CFl_* is the chain complex associated to a Lefschetz admissible Hamiltonian. This is done as follows: Let CFl^a_* be the subcomplex of CFl_* generated by orbits of action $\leq a$. For $a \geq L$ we have that $C_{2*}^{\,a}$ is the subcomplex of C_{2*} equal to CFl^a_*/C_{1*}. The Hamiltonian H^b_k looks like a product Hamiltonian in the region A. This means that periodic orbits come in pairs (γ, Γ) where γ is a periodic orbit of H_S and Γ is a periodic orbit of H^b_F in the region $\{r_F \geq b\} \subset \widehat{F}$. Suppose that the action difference between two orbits (γ, Γ) and (γ', Γ') of H^b_k in B is $\leq Q$. Then, $\Gamma = \Gamma'$. This is because the minimum action difference between any two orbits of H^b_F outside F^b is $2Q$ and the maximum action difference between two orbits of H_S is Q. Choose constants $a_0 < a_1 < \cdots < a_p$ such that:

(1) $a_0 = L$
(2) Any orbit in B has action $\leq a_p$.
(3) For any $0 < i \leq p$, there exists a Γ_i such that the only orbits of action in the region (a_{i-1}, a_i) are of the form (γ, Γ_i) for some γ.

Here is a picture illustrating the situation:



This means we have a filtration of CFl_* as follows:

$$C_{1*} = CFl_*^{a_0} \subset CFl_*^{a_1} \subset CFl_*^{a_2} \subset \cdots \subset CFl_*^{a_p} = CFl_*.$$

Hence we get a filtration of C_{2*}:

$$0 = C_{2*}^{a_0} \subset C_{2*}^{a_1} \subset C_{2*}^{a_2} \subset \cdots \subset C_{2*}^{a_p} = C_{2*}.$$

Let

$$CG_*^i := C_{2*}^{a_{i+1}} / C_{2*}^{a_i} \cong CFl_*^{a_{i+1}} / CFl_*^{a_i}.$$

Let $G_*^i := H_*(CG^i)$. We have a spectral sequence converging to $H_*(C_2)$ with E_1 term equal to $E^1_{i+1,j} = G^i_{i+j}$. In order to show that $H_j(C_2) = 0$ for all $j \leq m$ it is sufficient to show that $G^i_j = 0$. Let Q^i_* be the vector subspace of $CFl_*^{a_{i+1}}$ generated by orbits of the form (γ, Γ_{i+1}). As a vector space, $CFl_*^{a_{i+1}}$ splits as a direct sum $Q^i_* \bigoplus CFl_*^{a_i}$. Because the orbits in Q^i_* have higher action, our differential ∂ is of the form:

$$\begin{pmatrix} \partial_1 & \partial_2 \\ 0 & \partial_3 \end{pmatrix}$$

where $\partial_1 : Q^i_* \to Q^i_*$, $\partial_2 : Q^i_* \to CFl_*^{a_i}$ and $\partial_3 : CFl_*^{a_i} \to CFl_*^{a_i}$. Hence the quotient chain complex CG^i_* is isomorphic to the chain complex Q^i with differential ∂_1. In order to prove our theorem, all we need to show now is that the chain complex Q^i is isomorphic to CS_{*+f} where CS_* is the Floer chain complex of H_S and f is some integer of modulus $\leq m$. This is because $H_{j+f}(CS_*) = 0$ for $|j| \leq m$ because $|j+f| \leq 2m$.

The differential ∂_1 is described by counting Floer trajectories connecting two orbits of Q^i_*. Any Floer trajectory connecting some (γ, Γ_i) with (γ', Γ_i) must have volume $\leq Q$. This means that any Floer trajectory of H^b_k connecting these orbits must be contained in A by using [26, Lemma 1] and Gromov compactness as before (i.e. by considering a sequence of H^b_k's tending to ∞). Now our Hamiltonian is equal to $\pi^* H_S + \pi_1^* H_F$ in A and we also have a product almost complex structure on A. This means that a Floer trajectory connecting (γ, Γ_i) and (γ', Γ_i) must be of the form (u, v) where u is a Floer trajectory of H_S and v is a trivial Floer trajectory of H_F connecting Γ_i with itself. In particular this gives us our chain isomorphism between CS_{*+f} and Q^i.

Hence we have shown that

$$SH_j(\widehat{E}, L) \cong SH_j(\widehat{E}, H^b_k, J_k) \cong SH_j(\widehat{E}, H) \tag{3}$$

for $j \leq m$ and where L is equal to H on B' and $L = \pi^* H_S + K$ where K has slope $\frac{\delta}{k}$. This means that L is Lefschetz admissible. Also H and L have slopes which are functions of λ which tend to infinity as λ tends to ∞ (Remember that m tends to ∞ as λ tends to ∞.). We can also describe the above isomorphism in the following way: There is a monotone increasing family of Lefschetz admissible Hamiltonians joining L to H^b_k by increasing the slope of L near the horizontal boundary. There is also a monotone increasing family of Lefschetz admissible Hamiltonians joining H^b_k to H by letting b tend to 1. These two families can be joined together to create a monotone increasing family of Hamiltonians joining L to H. Using continuation maps, this gives us a natural map $SH_*(L) \to SH_*(H)$. Using energy arguments from [26, Lemma 1] we can show that these continuation maps induce the same isomorphism 3. This means that the above isomorphism is compatible with the pair of pants product. Hence when we take direct limits, we get a ring isomorphism:

$$SH_*(E) \cong SH^{\mathrm{lef}}_*(E).$$

$\square$

8. $SH_*^{\text{lef}}(\widehat{E})$ and the Kaliman modification

In this section we prove theorem 4.14. Throughout this section we assume that E' and E'' are Lefschetz fibrations as described in section 4.2. We recall the situation:

(1) E'' is a subfibration of E' over the same base.
(2) The support of the parallel transport maps of E' are contained in the interior of E''.
(3) There exists a complex structure $J_{F'}$ (coming from a Stein domain) on F' such that any $J_{F'}$-holomorphic curve in F' with boundary in F'' must be contained in F''.

We wish to prove that $SH_*^{\text{lef}}(E') \cong SH_*^{\text{lef}}(E'')$ as rings.

Proof. of Theorem 4.14. Fix $\lambda > 0$. The value λ is going to be the slope of some Hamiltonian, we can always perturb λ slightly so that it isn't in the action spectrum of the boundary. By Theorem 11.1 we can choose an almost complex structure $J_{F',1}$ on $\widehat{F'}$ after a convex deformation away from F' such that it is convex with respect to some cylindrical end at infinity and such that any $J_{F',1}$-holomorphic curve in F' with boundary in F'' must be contained in F''. The reason is because we can ensure that $J_{F',1} = J_{F'}$ in $F' \subset \widehat{F'}$ and that any $J_{F',1}$-holomorphic curve with boundary in $F'' \subset F'$ is contained in F' by Theorem 11.1 hence is contained in F'' by property (3) above. Supposing we have a Hamiltonian $H_{F'}$ which is of the form $h_{F'}(r_{F'})$ on the cylindrical end where $r_{F'}$ is the radial coordinate and $h' \geq 0$ and $H_{F'} = 0$ elsewhere. Then, any curve (Floer cylinder or pair of pants) with boundary in F'' satisfying Floer's equations with respect to $H_{F'}$ and $J_{F',1}$ must be contained in F''. We choose h' small enough so that $H_{F'}$ has no periodic orbits in the region $r_{F'} > 1$. The convex deformation mentioned in Theorem 11.1 fixes $F' \subset \widehat{F'}$ hence it induces a convex deformation on $\widehat{E}$. This is because the region where we deform $\widehat{E}$ looks like a product $\mathbb{C} \times (\widehat{F'} \setminus F')$. From now on we assume that the fibres of $\widehat{E}$ have this almost complex structure $J_{F',1}$ with this cylindrical end.

A neighbourhood of $\partial F''$ in F' is symplectomorphic to $L := (-\epsilon, \epsilon) \times \partial F''$ with the symplectic form $d(r\alpha'')$. Here, r is a coordinate in $(-\epsilon, \epsilon)$ and α'' is the contact form for $\partial F''$. We also choose ϵ small enough so that L is disjoint from the support of the parallel transport maps in F'. Let $\bar{F}'' := F'' \setminus ((-\epsilon/3, 0] \times \partial F'')$. We can choose an almost complex structure $J' \in \mathcal{J}(\widehat{F'})$ with the following properties:

(1) There exists a $\delta > 0$ such that any holomorphic curve meeting both boundaries of $[-\epsilon, -\epsilon/2] \times \partial F'$ has area greater than δ.
(2) $J' = J_{F',1}$ on $\widehat{F'} \setminus \bar{F}''$. This means that any curve (cylinder or pair of pants) satisfying Floer's equations with respect to $H_{F'}$ and J' with boundary in F'' is contained entirely in F''.

Construct an almost complex structure J on $\widehat{E'}$ as follows: The parallel transport maps on $\widehat{F'} \setminus \bar{F}''$ are trivial, hence there is a region W of $\widehat{E'}$ symplectomorphic to $\mathbb{C} \times (\widehat{F'} \setminus \bar{F}'')$. We set $J|_W$ to be the product almost complex structure $J_{\mathbb{C}} \times J'$ where $J_{\mathbb{C}}$ is the standard complex structure on $\mathbb{C}$. We then extend $J|_W$ to some J compatible with the symplectic form ω' such that π' is J-holomorphic outside some large compact set. Let H be a Hamiltonian of the form $\pi^* K + \pi_1^* H_{F'}$ where K is admissible of slope λ on the base $\mathbb{C}$ and $\pi_1 : W \twoheadrightarrow \widehat{F'} \setminus \bar{F}''$ is the natural projection map. J has the following properties:

(1) Any curve u satisfying Floer's equations with respect to H passing through $\partial E''$ must have energy $\geq \delta$. (u can be a cylinder or a pair of pants).
(2) Any such u connecting orbits inside E'' must be entirely contained in E''.

Property (2) is true because: Let u be a curve satisfying Floer's equations connecting orbits in E'', then composing $u|_{u^{-1}(W)}$ with the natural projection $W \twoheadrightarrow \widehat{F'} \setminus \bar{F}''$ gives us a curve w with boundary in F''. This means that w is contained in F'', and hence u is contained in E''. Also if u passes through $\partial E''$, then the projected curve w has energy $\geq \delta$ which means that u has energy $\geq \delta$. Hence Property (1) is true.

We perturb $H_{F'} : \widehat{F'} \to \mathbb{R}$ slightly so that:

(1) It is equal to 0 in F''.
(2) The only periodic orbits of $H_{F'}$ are constant orbits.
(3) The action spectrum of $H|_{\widehat{F'} \setminus F''}$ is discrete and injective.
(4) We leave $H_{F'}$ alone on the cylindrical end.
(5) All the orbits in $\widehat{F'} \setminus F''$ are of negative action and non-degenerate.

Let δ_1 be the smallest positive difference in action between two orbits of $H_{F'}$. Here we fix some integer $m > 0$. We can assume that the critical points of our Lefschetz fibration in $\mathbb{C}$ form a regular polygon with centre the origin. Draw a straight line from the origin to each critical point and let G be the union of these lines. Let $X := \frac{r}{2}\frac{\partial}{\partial r}$ be an outward pointing

Liouville flow. We choose a loop l around G so that the disc V with $\partial V = l$ has volume v where v can be chosen arbitrarily small, and such that X is transverse to this loop. This forms a new cylindrical end for $\mathbb{C}$. Now let H^V_λ be a Hamiltonian on V with slope λ. We assume that H^V_λ has the following properties:

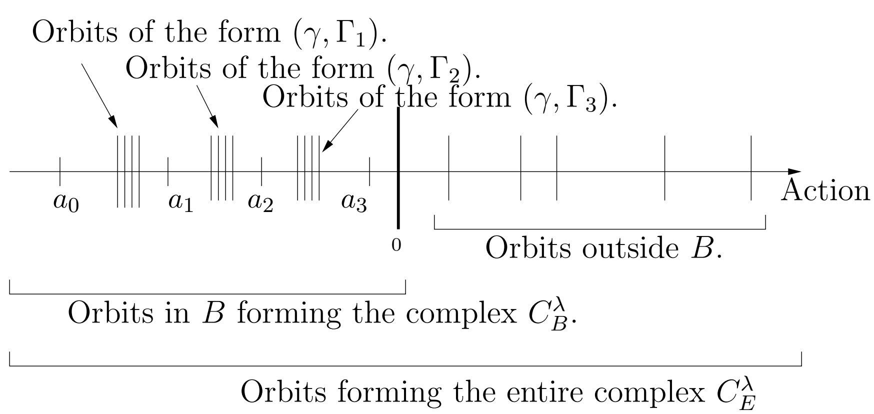
(1) All the orbits are non-degenerate.
(2) The action of any orbit is less than $2v\lambda$.
(3) All orbits of index $\leq m+n$ are exact. ($2n$ is the dimension of our symplectic manifold)
(4) $2v\lambda < \min(\delta, \delta_1)$.

We let $K_\lambda = \pi^*(H^V_\lambda) + \pi_1^* H_{F'}$. Let $B := \mathbb{C} \times (\widehat{F'} \backslash F'') \subset \widehat{E'}$. The Hamiltonian K_λ is of the form $\pi^*(H^V_\lambda) + \pi_1^* H_{F''}$ on B. Hence the orbits on B come in pairs (γ, Γ), where γ corresponds to a periodic orbit of H^V_λ and Γ is a constant periodic orbit of $H_{F''}$. The action of the orbit is $\leq 2v\lambda \leq \delta$ due to property (5) for $H_{F'}$ and properties (2) and (4) for H^V_λ. This implies that any Floer trajectory connecting orbits (γ_1, Γ_1) and (γ_2, Γ_2) inside B must stay inside B, due to property (1) from the properties of J.

Hence we have a subcomplex C^λ_B generated by orbits in B. We also have a quotient complex $C^\lambda_{E''} := C^\lambda_{E'} / C^\lambda_B$ where $C^\lambda_{E'}$ is the the complex generated by all orbits. Let $C^{\lambda,a}_B$ be the subcomplex of C^λ_B generated by orbits of action $\leq a$. There exist constants $a_0 < a_1 < \cdots < a_k$ such that:

(1) a_0 is smaller than the smallest action of an orbit in B.
(2) Any orbit in B has action $\leq a_k$.
(3) For any $0 \leq i < p$, there exists a Γ_i such that the only orbits of action in the region (a_i, a_{i+1}) are of the form (γ, Γ_i) for some γ.

Here is a picture illustrating the situation:



Hence we get a filtration:

$$0 = C_B^{\lambda,a_0} \subset C_B^{\lambda,a_1} \subset \cdots \subset C_B^{\lambda,a_k} = C_B^{\lambda}$$

of C_B^{λ}. We can use a spectral sequence argument as in the proof of Theorem 4.13 to show that:

$$\varinjlim_{\lambda} H_*(C_B^{\lambda}) = 0.$$

In this argument we need to show that $\text{ind}(\gamma, \Gamma_i) = \text{ind}(\gamma) + \text{ind}(\Gamma_i)$ where (γ, Γ_i) is described in property (3) above. But this is obvious because Γ is a constant orbit, so it has a canonical Robbin-Salamon index. This in turn shows us that:

$$\varinjlim_{\lambda} H_*(C_{E''}^{\lambda}) \cong \varinjlim_{\lambda} H_*(C_{E'}^{\lambda}) \cong SH_*^{\text{lef}}(E').$$

Also, because no curve satisfying Floer's equation connecting orbits inside E'' can escape E'', we have that:

$$\varinjlim_{\lambda} H_*(C_{E''}^{\lambda}) \cong SH_*^{\text{lef}}(E'').$$

This gives us our isomorphism:

$$SH_*^{\text{lef}}(E'') \cong SH_*^{\text{lef}}(E').$$

□

9. Brieskorn Spheres

In this section we will mainly be studying the variety V as constructed in section 5.1 and also the variety $M'' := \mathbb{C}^4 \setminus V$.

9.1. **Parallel transport.** There is a natural symplectic form on $\mathbb{C}^4$ (induced from an ample line bundle on its compactification $\mathbb{P}^4$). We have a holomorphic map $P := z_0^7 + z_1^2 + z_2^2 + z_3^2$ with one singular point at 0. We can view P as a fibration which is compatible with this symplectic form. We have $P^{-1}(0) = V$. We prove:

Theorem 9.1. *Parallel transport maps are well defined for P.*

Proof. We first of all compactify $\mathbb{C}^4$ to $\mathbb{P}^4$. We let P' be a holomorphic section of $E := \mathcal{O}_{\mathbb{P}^4}(7)$:

$$P'([z_0 : \cdots : z_4]) := z_0^7 + z_4^5 z_1^2 + z_4^5 z_2^2 + z_4^5 z_3^2.$$

This is equal to P on the trivialisation $z_4 = 1$. We also have another section Q defined by

$$Q([z_0 : \cdots : z_4]) := z_4^7.$$

The map P can be extended to a rational map $P'' : \mathbb{P}^4 \dashrightarrow \mathbb{P}^1$, where $P'' = \frac{P'}{Q}$. Fix an identification $\mathbb{C}^4 = \mathbb{P}^4 \setminus Q^{-1}(0)$. We now have that

$$P = \frac{P'}{Q}.$$

Let $\|.\|_E$ be a positive curvature metric on the ample bundle E. We have a symplectic structure and Kähler form defined in terms of the plurisubharmonic function

$$\phi = -\log \|Q\|_E^2.$$

In order to show that P has well defined parallel transport maps we need to construct bounds on derivatives similar to the main theorem in [17, section 2]. We take the vector field ∂_z on the base $\mathbb{C}$. It has a unique lift with respect to the Kähler metric which is:

$$\xi := \frac{\nabla P}{\|\nabla P\|^2}.$$

Here $\|.\|$ is the Kähler metric and ∇P is the gradient of P with respect to this metric. Take a point p on $D := \{z_4 = 0\}$. We can assume without loss of generality that this lies in the chart $\{z_1 = 1\}$. In this chart we have that the metric $\|.\|_E = e^\sigma |.|$ where σ is a smooth function and $|.|$ is the standard Euclidean metric with respect to this chart. Then:

$$\phi = -\log \|Q\|^2 = -\log |Q|^2 - \sigma.$$

The notation $\lesssim$ means that one term is less than or equal to some constant times the other term. Hence we get:

$$\begin{aligned} B := \left|\xi.\phi\right| \leq \frac{|\langle \nabla P, \nabla \sigma\rangle|}{||\nabla P||^2} + \frac{2|Q| \cdot |\langle \nabla Q, \nabla P\rangle|}{||\nabla P||^2 \cdot |Q|^2} \\ \lesssim \frac{1}{||\nabla P||} + \frac{||\nabla Q||}{||\nabla P|| \cdot |Q|} \end{aligned} \tag{4}$$

We get similar equations to 4 in the other charts $\{z_i = 1\}$. If we can show that for any compact set $T \subset \mathbb{C}$ the function B is bounded above by a constant K in the region $T_1 := P^{-1}(T) \setminus A$ where $A = \{|z_0|, |z_1|, |z_2|, |z_3| \leq 1\}$, then we have well defined parallel transport maps. This is because we get similar bounds if we lift other vectors of unit length (i.e. $c\partial_z$ where $c \in U(1)$). Hence if we have a path, then $\left|\xi.\phi\right|$ is bounded above by a constant on this path. This ensures that the transport maps do not escape to infinity. In the chart $\{z_3 = 1\}$, we have that for $1 \leq i \leq 2$,

$$\partial_i P = 2z_i/z_4^2,$$

$$\partial_0 P = 7z_0^6/z_4^7.$$

We have the following bounds on derivatives:

$$|\nabla Q| \lesssim \frac{|Q|}{|z_4|}.$$

Combining this with equation 4 gives:

$$B \lesssim \frac{1+|z_4|}{|z_4|\left(\sum_{j=0,j\neq 3}^{4} |\partial_i P|\right)}$$

$$\lesssim C := (1+|z_4|)/\left[\frac{7|z_0|^6}{|z_4|^6} + \frac{2|z_1|}{|z_4|} + \frac{2|z_2|}{|z_4|} + |\partial_4 P||z_4|\right]$$

$$\lesssim (1+|z_4|)/\left(|z_0|^6/|z_4|^6 + |z_1|/|z_4| + |z_2|/|z_4|\right).$$

Hence on the chart $\{z_4 = 1\}$,

$$B \lesssim (1+|z_3|^{-1})/\left(|z_0|^6 + |z_1| + |z_2|\right) \tag{5}$$

By symmetry we also have

$$B \lesssim (1+|z_2|^{-1})/\left(|z_0|^6 + |z_1| + |z_3|\right) \tag{6}$$

$$B \lesssim (1+|z_1|^{-1})/\left(|z_0|^6 + |z_2| + |z_3|\right) \tag{7}$$

In the chart $\{z_0 = 1\}$ we have:

$$B \lesssim \frac{1+|z_4|}{|z_4|\left(\sum_{j=1}^{4} |\partial_i P|\right)}$$

$$\lesssim (1+|z_4|)/\left[\frac{2|z_1|}{|z_4|} + \frac{2|z_1|}{|z_4|} + \frac{2|z_2|}{|z_4|} + |\partial_4 P||z_4|\right].$$

So:

$$B \lesssim (1+|z_4|)|z_4|/\left(|z_1| + |z_2| + |z_3|\right)$$

so in the chart $\{z_4 = 1\}$, we get a bound:

$$B \lesssim (1+|z_0|^{-1})/\left(|z_1| + |z_2| + |z_3|\right) \tag{8}$$

Suppose for a contradiction that there is a sequence of vectors $(z_0^i, z_1^i, z_2^i, z_3^i)$ lying in T_1 such that B tends to infinity as i tends to infinity. If (after passing to a subsequence) z_0^i tends to infinity, then equation 8 tells us that

z_1^i, z_2^i, z_3^i are all bounded. But this is impossible as $(z_0^i, z_1^i, z_2^i, z_3^i)$ lies in T_1 which means that z_0^i is bounded. Similarly, using equations 5,6,7 we get that z_j^i is bounded. Hence B is bounded away from the compact set $\{|z_0|, |z_1|, |z_2|, |z_3| \leq 1\}$. This means that B is bounded when restricted to T_1, so we have well defined parallel transport maps. $\square$

Let $(\mathbb{C}^4, \theta)$ be the convex symplectic manifold induced by the compactification $\mathbb{C}^4 \hookrightarrow \mathbb{P}^4$. Because parallel transport maps for P are well defined we can use ideas from [31, section 19b] to deform the 1-form θ on $\mathbb{C}^4$ through a series of 1-forms θ_t such that:

(1) each $\omega_t := d\theta_t$ is compatible with P and θ_t is a convex symplectic deformation on $\mathbb{C}^4$.
(2) (P, ω_1) has trivial parallel transport maps at infinity. This means that near infinity, P looks like the natural projection $C \times \mathbb{C} \twoheadrightarrow \mathbb{C}$ where C is the complement of some compact set in V.
(3) For a smooth fibre F of P, (F, θ_1) is exact symplectomorphic to (F, θ_0).

We have that $M'' = \mathbb{C}^4 \setminus P^{-1}(0)$, so we can restrict P to a fibration $P'' = P|_{M''} : M'' \to \mathbb{C}^*$. Let θ_S be a convex symplectic structure on $\mathbb{C}^*$ with the property that $\theta_{M'',t} := \theta_t|_{M''} + P''^*\theta_S$ is a convex symplectic structure for M''. Let θ'' be a convex symplectic structure on M'' constructed as in example 2.9. It is convex deformation equivalent to $(M'', \theta_{M'',0})$ as follows: Let F be a fibre of P'', then $(F, \theta_{M'',0}|_F)$ is convex deformation equivalent to $(F, \theta''|_F)$ by [32, Lemma 4.4] as both convex structures come from Stein structures constructed algebraically as in Example 2.9. This deformation is $(1-t)\theta_{M'',0}|_F + t\theta''|_F$. The following family of 1-forms $\Theta_t := (1-t)\theta_{M'',0} + t\theta''$ induces a convex symplectic deformation (we might have to add $\pi^*\theta'_S$ to $\theta_{M'',0}$ and θ'' where θ_S is a convex symplectic structure on $\mathbb{C}^*$ and $d\theta_S$ is sufficiently large). The reason why it is a convex symplectic deformation is as follows: We can ensure that θ_S comes from a Stein function ϕ_S on $\mathbb{C}^*$. Also, θ_0 comes from some Stein function $\phi : \mathbb{C}^4 \to \mathbb{R}$, hence $\theta_{M'',0}$ comes from a Stein function $\phi_0 := \phi|_{M''} + P''^*\phi_S$. The 1-form θ'' comes from a Stein function ϕ_1. Hence Θ_t comes from a Stein function of the form $\phi_t := (1-t)\phi_0 + t\phi_1$. The set of singular points of $\phi_t|_F$ for all t lie inside a compact set K_F (independent of t) for each fibre F. Let K be the union of all the compact sets K_F for each fibre F in M''. We can choose θ_S large enough so that outside some annulus A in $\mathbb{C}^*$, ϕ_t has no singularities in

$K \cap P''^{-1}(\mathbb{C}^* \setminus A)$. Also, there are no singularities of ϕ_t outside K. Hence, all the singularities of ϕ_t stay inside some compact set independent of t. This means that ϕ_t is a Stein deformation. This means that $(M'', \theta_{M'',1})$ is convex deformation equivalent to $(M'', \theta_{M'',0})$ which is convex deformation equivalent to (M'', θ'').

Hence on (M'', θ''), we have that the parallel transport maps of P'' are trivial at infinity after a convex symplectic deformation to $(M'', \theta_{M'',1})$.

9.2. **Indices.** Let $P'' : M'' \to \mathbb{C}^*$, $P''(z) = P(z)$. Let F be a smooth fibre of P''. This fibre has a natural exhausting plurisubharmonic function ϕ as in example 2.9. We can modify ϕ to an exhausting plurisubharmonic function ϕ' which is complete by [34, Lemma 6]. We denote this new Stein manifold by $\widehat{F}$. The following theorem is about indices of a cofinal family of Hamiltonians on $\widehat{F}$.

Theorem 9.2. *There is a cofinal family of Hamiltonians H_λ on $\widehat{F}$ with the following properties:*

1. *There exists some convex symplectic submanifold T of F such that $\widehat{T}$ (the symplectic completion of T) is symplectomorphic to $\widehat{F}$.*
2. *$H_\lambda = 0$ on T.*
3. *if y is a periodic orbit of H_λ not in T then it has Robbin-Salamon index ≥ 2.*
4. *For each $k \in \mathbb{Z}$ there exists an $N > 0$ (independent of λ) such that the number of periodic orbits of H_λ of index k is bounded above by N.*
5. *If we don't count critical points from the interior, then there is exactly one orbit of index 2 and one orbit of index 3 such that the action difference between these two orbits tends to 0 as λ tends to infinity. Also the number of Floer cylinders connecting these orbits is even.*

This theorem is proved by analysing the Conley-Zehnder indices of a Reeb foliation on the Brieskorn sphere $V \cap S$, where S is the unit sphere in $\mathbb{C}^4$. This result needs the following two lemmas:

Lemma 9.3. *$\widehat{F}$ is the completion of some convex symplectic submanifold T with boundary the Brieskorn sphere $V \cap S$.*

Proof. of Lemma 9.3. By 9.1, we have that $V \setminus 0$ is symplectomorphic to $F \setminus K$ where K is a compact set. Hence there exists a cylindrical end of $\widehat{F}$

which is symplectomorphic to the cylindrical end of V induced by flowing $V \cap S$ by parallel transport. $\square$

Lemma 9.4. *There is a contact form on the Brieskorn sphere $V \cap S$ such that all the Reeb orbits are non-degenerate and they have Conley-Zehnder indices ≥ 2. Also, there is exactly one orbit of index 2 and no orbits of index 3.*

Proof. In [37] Ustilovsky constructs a contact form such that all the Reeb orbits are non-degenerate and such that their reduced Conley-Zehnder index is $\geq 2(n-2)$ where $n = 3$ in our case. Ustilovsky defines the reduced Conley-Zehnder index to be equal to the Conley-Zehnder index $+(n-3)$. This means that the Reeb orbits have Conley-Zehnder index $\geq n-1 = 2$. He also shows for each $k \in \mathbb{Z}$, there are finitely many orbits of Conley-Zehnder index k. He shows that there are no orbits of odd index and the orbit of lowest index has index 2. $\square$

Proof. of Theorem 9.2.

By Lemma 9.3 $\widehat{F}$ has a convex cylindrical end which is symplectomorphic to $[1,\infty)\times\Sigma$ where Σ is the Brieskorn sphere $V\cap S$. We choose a Hamiltonian which is constant on the interior of F and equal to $h(r)$ on the cylindrical end, where r parameterises $[1,\infty)$. We also assume that $h'(r)$ is constant and not in the period spectrum of B at infinity. Also, near each orbit in the cylindrical end, we assume that $h'' > 0$. The flow of the Hamiltonian at the level $r = k$ is the same as the flow of $X_H := -h'(k)R$, where R is the Reeb flow. The Conley-Zehnder indices from Lemma 9.4 are computed by trivialising the contact plane bundle. We can trivialise the symplectic bundle by first trivialising the contact plane bundle and then trivialising its orthogonal bundle. We trivialise the orthogonal bundle by giving it a basis $(\frac{\partial}{\partial r}, R)$. The symplectic form restricted to this basis is the standard form:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hamiltonian flow in this trivialisation is the matrix:

$$\begin{pmatrix} 1 & 0 \\ h''t & 1 \end{pmatrix}$$

along the orthogonal bundle. This is because R is invariant under this flow and the Lie bracket of $X_H = -h'R$ with $\frac{\partial}{\partial r}$ is $h''R$. The Robbin-Salamon index of this family of matrices is $\frac{1}{2}$. We calculate this index by perturbing this family of matrices by a function $\xi : [0,1] \to \mathbb{R}$ where $\xi(0) = \xi(1) = 0$ as follows:

$$\left(\begin{array}{cc} 1 & \xi(t) \\ h''t & 1 \end{array} \right).$$

Choosing ξ so that its derivative is non-zero whenever $\xi = 0$ ensures that the path is generic enough to enable us to compute its Robbin-Salamon index.

Remember that the Robbin-Salamon index of an orbit is equal to the Conley-Zehnder index taken with negative sign. Lemma 9.4 tells us the Conley-Zehnder indices of all the Reeb orbits. The flow $X_H = -h'R$ of the Hamiltonian has orbits in the opposite direction to Reeb orbits. Hence the Robbin-Salamon index (restricted to the contact plane field) of an orbit of $X_H = -h'R$ is the same as the Conley-Zehnder index of the corresponding Reeb orbit. Hence the Robbin-Salamon index of some orbit of the Hamiltonian on the level set $r = k$ is equal to $C + \frac{1}{2}$ where C is the Conley-Zehnder index of the associated Reeb orbit as calculated in Lemma 9.4. Hence the indices of these orbits are $\geq 2 + \frac{1}{2}$.

The problem is that these orbits are degenerate. This is why their index is not an integer. As in [25, section 3] we can perturb each circle of orbits to a pair of non-degenerate orbits. Let C' be a circle of orbits. We choose a Morse function f on C'. If we flow f along X_H (the Hamiltonian flow of H) we get a time dependent Morse function $f_t = f \circ \phi_{-t}$ (ϕ_t is the Hamiltonian flow). Extend f_t so that it is defined as a function on a neighbourhood of C'. Let $H + f_t$ be our new Hamiltonian. The orbits near C' now correspond to critical points p of f. The Robbin-Salamon index of such an orbit is:

$$i(C') + \frac{1}{2}\mathrm{sign}(\nabla^2_p f)$$

where $i(C')$ is the Robbin-Salamon index of the manifold of orbits. The symbol 'sign' means the number of positive eigenvalues minus the number of negative eigenvalues. In our case we can choose f so that it has 2 critical points p_1, p_2 such that

$$\mathrm{sign}(\nabla^2_{p_1} f) = 1,\ \mathrm{sign}(\nabla^2_{p_2} f) = -1.$$

Hence, if the Conley-Zehnder index of a Reeb orbit C is k, then we can perturb H so that the associated Hamiltonian orbits have Robbin-Salamon index (or equivalently Conley-Zehnder index taken with negative sign) $k+0$ and $k+1$. This means all the non-constant orbits of H have Robbin-Salamon index ≥ 2.

We now need to show that there are a finite number of orbits in each degree. This follows directly from Lemma 9.4 which says that there are finitely many Reeb orbits in each degree. Finally this same lemma says that there is only one Reeb orbit with Conley-Zehnder index 2 and no Reeb orbits with Conley-Zehnder index 3. So the Hamiltonian H has one orbit of Robbin-Salamon index 2 and one orbit of index 3. We can also ensure that the actions of these orbits are arbitrarily close by letting the associated Morse function f be C^2 small. There are an even number of Floer cylinders connecting the orbit of index 3 with the orbit of index 2 by [9, Proposition 2.2]. $\square$

Lemma 9.5. *We have $H^i(M'') = 0$ for $i \geq 2$.*

Proof. $M'' = \mathbb{C}^4 \setminus V$. Theorem 5.1 tells us that V is homeomorphic to $\mathbb{R}^6$. This means that there is a neighbourhood B of V which retracts onto V whose boundary ∂B satisfies $H^i(\partial B) = 0$ for $i \geq 2$. The Mayor-Vietoris sequence involving B, M'' and $B \cup M'' = \mathbb{C}^4$ ensures that $H^i(M'') = 0$ for $i \geq 2$. $\square$

9.3. **Symplectic homology of these varieties.** We wish to show that the symplectic homology of the variety $M'' := \mathbb{C}^4 \setminus V$ has only finitely many idempotents using the results of the previous two sections. We will then show that it has at least two idempotents: 0 and 1. First of all we need the following lemma:

Lemma 9.6. *Let $R = \bigoplus_{g \in G} R_g$ be an algebra over $\mathbb{Z}/2$ which is graded by some finitely generated abelian group G. If a is an idempotent in R then $a \in \bigoplus_{g \in G_n} R_g$ where G_n is the subgroup of torsion elements of G.*

Proof. We have $a = a_{g_1} + \cdots + a_{g_n}$ where $g_i \in G$ and $a_{g_i} \in R_{g_i}$. Suppose for a contradiction we have that $a = a^2$ and g_1 is not torsion. Then $a^2 = a_{g_1}^2 + \cdots + a_{g_n}^2$. The group G/G_n is a free $\mathbb{Z}$ algebra, hence there is a group homomorphism $p : G \to G/G_n \to \mathbb{Z}$ such that $p(g_1) \neq 0$. The map p gives R a $\mathbb{Z}$ grading. Let b be an element of R. It can be written uniquely as $b = b_1 + \cdots + b_k$ where b_i are non-zero elements of R with grading $q_i \in \mathbb{Z}$. We can define $f(b)$ as $\min_{j \in \{i | q_i \neq 0\}} |q_j|$. Note that $f(b)$ is well defined only if at least one of the q_i's are non-zero. Because $p(g_1) \neq 0$, we have that $f(a)$ is well defined and positive. We also have that $f(a^2) \geq 2f(a)$ which means that $a^2 \neq a$. This contradicts the fact that a is an idempotent. $\square$

The vector space $SH_{4+*}(M'')$ is a ring bi-graded by the Robbin-Salamon index and the first homology group. We write $4+*$ here because the unit has Robbin-Salamon index 4. The previous lemma shows us that any idempotent must have grading 4 in $SH_*(M'')$ and be in a torsion homology class.

We have a map $P'' : M'' \to \mathbb{C}^*$. At the end of section 9.1 we had a convex symplectic structure $(M'', \theta_{M'',1})$. Let A be a large annulus in the base $\mathbb{C}^*$ which is a compact convex symplectic manifold. Let $(F'', \theta_{M'',1})$ be a fibre of P''. Choose a compact convex symplectic manifold (with corners) $\bar{M}''$ such that $(\bar{M}'', \bar{P}'' := P''|_{\bar{M}''}, \theta_{M'',1})$ is a compact convex Lefschetz fibration with fibres $\bar{F}''$ and base $A \subset \mathbb{C}^*$. We can also ensure that $\partial \bar{F}''$ is transverse to λ_1 (the associated Liouville vector field of $\mathbb{F}''$) and there are no singularities of λ_1 outside $\bar{F}''$ in F''. Hence the completion of $\bar{M}''$ is $\widehat{M}''$.

Let $(\widehat{E}'', \pi'')$ be the completion of $(\bar{M}'', \bar{P}'', \theta_{M'',1})$ (so that $\widehat{\bar{M}''} = \widehat{E''}$). We wish to use the results of section 7 to show that $SH_*(E'')$ has finitely many idempotents, and hence $SH_*(M'')$ has finitely many idempotents. Let H be a Lefschetz admissible Hamiltonian for $\widehat{E''}$. Let C be the cylindrical end of $\widehat{F''}$. We may assume that this cylindrical end is of the form $(S_V \times [1,\infty), r_F \alpha_F)$ where (S_V, α_F) is the Brieskorn sphere described in 9.2 and r_F is the coordinate for $[1,\infty)$. The Hamiltonian H is of the form $\pi''^* H_{S''} + \pi_1''^* H_{F''}$ as in definition 4.10. By Lemma 9.6, we have that any idempotent must come from a linear combination of orbits of H in torsion homology classes as long as H is large enough (i.e. it is large enough in some cofinal sequence of Lefschetz admissible Hamiltonians). Away from $C \times \mathbb{C}^* \subset \widehat{E''}$ we have that the Hamiltonian flow of H is the same as the flow of $L := \pi''^* H_{S''}$. Let X be the Hamiltonian vector field associated to L, and let $X_{S''}$ be the Hamiltonian vector field in $\mathbb{C}^*$ associated to $H_{S''}$. Then the value of X at a point p is some positive multiple of the horizontal lift of $X_{S''}$ to the point p. We can assume that $H_{S''}$ has exactly two contractible periodic orbits of index 0 and 1 corresponding to Morse critical points of $H_{S''}$ (as any Reeb orbit of $\mathbb{C}^*$ is not contractible). We can also make $H_{S''}$ C^2 small away from the cylindrical ends of $\mathbb{C}^*$ so that the only Floer cylinders connecting contractible orbits correspond to Morse flow lines. Hence, any contractible orbit of X must project down to a constant orbit of $X_{S''}$. We let $H_{F''}$ be a Hamiltonian as in Lemma 9.2 above in Section 9.2. We let our almost complex structure J when restricted to $C \times \mathbb{C}^* \subset \widehat{E''}$ be equal to the product almost complex structure $J_F \times J_{\mathbb{C}^*}$ where J_F is an admissible almost complex structure on $\widehat{F}$ and $J_{\mathbb{C}^*}$ is the standard complex structure on $\mathbb{C}^*$. The contractible orbits in this cylindrical end come in pairs (Γ, γ) where Γ is an orbit in $\widehat{F}$ and γ is a contractible orbit in $\mathbb{C}^*$. Because there are only 2 contractible orbits in $\mathbb{C}^*$ and there are finitely many orbits in each degree in $\widehat{F}$, we have finitely many contractible orbits of index 4 for H. Hence:

Theorem 9.7. *The ring $SH_{4+*}(M'')$ has only finitely many idempotents.*

We now wish to show that $SH_*(M'')$ has at least 2 idempotents. To do this we show that $SH_*(M'') \neq 0$, and hence has a unit by [32, Section 8]. This means that $SH_*(M'')$ has 0 and 1 as idempotents. The Hamiltonian H has non-degenerate orbits in $C \times \mathbb{C}^* \subset \widehat{E''}$, so we perturb H away from this set to make all its orbits non-degenerate. In $E'' \subset \widehat{E''}$ we can ensure that H is C^2 small and J is independent of t, hence the only orbits in this region are critical points of H and the only Floer cylinders correspond to Morse flow lines. The orbits corresponding to critical points of H have Robbin-Salamon index ≥ 3 because $H^i(M'') = 0$ for $i > 1$ by Lemma 9.5. Hence all orbits have index ≥ 2. There is only one orbit of index 2. This orbit is in the cylindrical end $C \times \mathbb{C}^* \subset \widehat{E''}$. Hence the orbit is of the form (Γ_m, γ_m) where γ_m has index 0 and Γ_m has index 2. This orbit is closed because there are no orbits of lower index. Suppose for a contradiction this orbit is exact, then there exists a Floer cylinder connecting an orbit β of index 3 with (Γ_m, γ_m). This orbit β must be contractible, so it is either a critical point, or it is of the form (Γ_1, γ_1) in $C \times \mathbb{C}^* \subset \widehat{E''}$. The action of (Γ_m, γ_m) is larger than the action of a critical point and hence β cannot be a critical point. Hence β is of the form (Γ_1, γ_1). Suppose that the index of Γ_1 is 3. We have $\gamma_m = \gamma_1$ and by Theorem 9.2 we can ensure that the action difference between Γ_1 is arbitrarily close to Γ_m. Similarly if γ_1 has index 1 then we can ensure that $\Gamma_m = \Gamma_1$ and the action difference between γ_m and γ_1 is arbitrarily small. This means that the action difference between (Γ_m, γ_m) and (Γ_1, γ_1) is arbitrarily small. This means that if we have a Floer cylinder connecting (Γ_m, γ_m) and (Γ_1, γ_1) then Gromov compactness ensures that it must stay in the region $C \times \mathbb{C}^* \subset \widehat{E''}$ (Because the action of (Γ_m, γ_m) tends to the action of (Γ_1, γ_1), we get a sequence of Floer cylinders converging to a Floer cylinder of energy 0 which cannot exit $C \times \mathbb{C}^* \subset \widehat{E''}$). Because all the Floer cylinders stay inside $C \times \mathbb{C}^* \subset \widehat{E''}$, the number of Floer cylinders connecting (Γ_m, γ_m) and (Γ_1, γ_1) is equal to the number of Floer cylinders connecting Γ_m and Γ_1 multiplied by the number of Floer cylinders connecting γ_m and

γ_1. We need to show that the number of Floer cylinders connecting (Γ_1, γ_1) and (Γ_m, γ_m) is even and by the previous comment, this means we only need to show that the number of Floer cylinders connecting Γ_1 and Γ_m is even or the number of Floer cylinders connecting γ_1 and γ_m is even. But the number of Floer cylinders connecting Γ_1 and Γ_m is even if Γ_1 has index 3 (by part (5) of 9.2) and similarly γ_1 is closed if it has index 1 (so there are an even number of Floer cylinders connecting γ_1 and γ_m). Hence the number of Floer cylinders connecting these two orbits is even and so (Γ_m, γ_m) is not exact. Hence $SH_*(M'') \neq 0$.

This completes the proof of the second theorem 1.2 subject to checking ring addition under end connect sums.

10. Appendix A: Lefschetz fibrations and the Kaliman modification

Let X,D,M be as in example 2.9. Let Z be an irreducible divisor in X and $q \in (Z \cap M)$ a point in the smooth part of Z. We assume there is a rational function m on X which is holomorphic on M such that $\overline{m^{-1}(0)}$ is reduced and irreducible and $Z = \overline{m^{-1}(0)}$. Let $M' := \text{Kalmod}(M, (Z \cap M), \{q\})$, and let $M'' := M \setminus Z$. Suppose also that $\dim_{\mathbb{C}} X \geq 3$. *Here is the statement of theorem 4.21: There exist Lefschetz fibrations $E'' \subset E'$ respectively satisfying the conditions of Theorem 1.4 such that E' (resp. E'') is convex deformation equivalent to M' (resp. M'').*

The rest of this section is used to prove this theorem. We will start with several preliminary lemmas.

Lemma 10.1. *There are Stein functions ϕ' (resp. ϕ'') on M' (resp. M'') such that M'' becomes a symplectic submanifold of M'.*

Proof. Let m' be the pullback of m to $\text{Bl}_q X$. Let Z' be the divisor defined by the zero set of m'. Let Z'' be the divisor defined by the zero set of $\frac{1}{m'}$, so that Z' is linearly equivalent to Z''. By abuse of notation, we write D as the total transform of D in $\text{Bl}_q X$.

Let $\tilde{Z}$ be the proper transform of Z. We can choose an effective ample divisor D' with support equal to $\tilde{Z} \cup D$ (as a set) so that $D' - Z''$ is effective. We have that $Y_1 := D'$ and $Y_2 := Y_1 - Z'' + Z'$ are linearly equivalent effective ample divisors. Let E be a line bundle associated to Y_1 and let s_1, s_2 be sections so that $s_i^{-1}(0) = Y_i$. There is a metric $\|.\|$ of positive curvature on E. We define $\phi' := -dd^c \log(s_1)$ and $\phi'' := -dd^c \log(s_2)$. $\square$

Moving the point q within the smooth part of $Z \cap M$ induces a Stein deformation of M' and M'' by a slight modification of the above lemma.

We now need a technical lemma involving convex symplectic manifolds of finite type. Let (M, θ_1), (M, θ_2) be convex symplectic manifolds. Suppose that $\theta_1 = \theta_2$ inside some codimension 0 submanifold C such that (C, θ_1) is a compact convex symplectic manifold.

Lemma 10.2. *If all the singular points of θ_1 and θ_2 are contained in C, then (M, θ_1) is convex deformation equivalent to (M, θ_2).*

Proof. The interior C^o of C has the structure of a finite type non-complete convex symplectic manifold constructed as follows: The boundary of C has a collar neighbourhood in C of the form $N := (-\epsilon, 1] \times \partial C$, with $\theta_1 = r\alpha$. Here r is the coordinate on $(-\epsilon, 1]$, and α is a contact form on ∂C. We let $\psi : C^o \to \mathbb{R}$ be an exhausting function, which is of the form $h(r)$ on N and such that $h(r) \to \infty$ as $r \to 1$. For some $N \gg 0$, we have that $\psi^{-1}(l)$ is transverse to the associated Liouville field λ_1 for all $l \geq N$. Let ϕ_1 be the function associated to the convex symplectic structure (M, θ_1). We may assume that $\phi_1^{-1}(l)$ is transverse to λ_1 for all $l \geq N$ as well. We can smoothly deform the function ϕ_1 into the function ψ through a series of exhausting functions ϕ_t (the domain of ϕ_t smoothly changes within M as t varies) such that $\phi_t^{-1}(N+k)$ is transverse to λ_1 for each $k \in \mathbb{N}$. This induces a convex symplectic deformation from (M, θ_1) to $(C^o, \theta_1|_{C^o})$. Similarly we have a convex symplectic deformation from (M, θ_2) to $(C^o, \theta_1|_{C^o})$. Hence, (M, θ_1) is convex deformation equivalent to (M, θ_2). □

We need another similar lemma about deformation equivalence.

Lemma 10.3. *Suppose (M, θ_1) and (M, θ_2) are convex symplectic manifolds such that $\theta_1 = \theta_2 + dR$ for some function R, then (M, θ_1) is convex deformation equivalent to (M, θ_2).*

Proof. Let ϕ_1 (resp. ϕ_2) be the function associated with the convex symplectic structure (M, θ_1) (resp. (M, θ_2)). Choose constants $c_1 < c_2 < \cdots$ and $d_1 < d_2 < \cdots$ tending to infinity such that $M_i^1 := \phi_1^{-1}(-\infty, c_i]$ (resp. $M_i^2 := \phi_2^{-1}(-\infty, c_i]$) are compact convex symplectic manifolds. Also we assume that:

$$M_i^1 \subset M_i^2 \subset M_{i+1}^1 \subset M_{i+1}^2$$

for all i. Let $R' : M \to \mathbb{R}$ be a function such that $R' = 0$ on a neighbourhood of ∂M_i^1 and $R' = R$ on a neighbourhood of ∂M_i^2 for all i. Let $\theta_3 :=$

θ_1+dR'. We will show that both (M,θ_1) and (M,θ_2) are convex deformation equivalent to (M,θ_3). Let $R_t : M \to \mathbb{R}$ be a family of functions such that $R_t = 0$ on a neighbourhood of ∂M_i^1 for all i and such that $R_0 = 0$ and $R_1 = R'$. Then $(M, \theta_1 + dR_t)$ is a convex deformation from (M,θ_1) to (M,θ_3) because $(M_i^1, \theta_1 + dR_t)$ is a compact convex symplectic manifold for all i. Also let $R'_t : M \to \mathbb{R}$ be a family of functions such that $R'_t = R$ on a neighbourhood of ∂M_i^2 and such that $R'_0 = R$ and $R'_1 = R'$. Then $(M, \theta_1+dR'_t)$ is a convex deformation from (M,θ_2) to (M,θ_3). Hence (M,θ_1) is convex deformation equivalent to (M,θ_2). $\square$

We let E be an ample line bundle on X, and s,t sections of E. We assume that s is non-zero on M. Let t be a holomorphic section of E, and let $p := t/s$ be a map from M to $\mathbb{C}$.

Definition 10.4. *We call* (M,p) *an* **algebraic Lefschetz fibration** *if:*

(1) $\overline{t^{-1}(0)}$ *is smooth, reduced and intersects each stratum of* D *transversally.*
(2) p *has only nondegenerate critical points and there is at most one of these points on each fibre.*

An algebraic Lefschetz fibration (M,p) has a symplectic form ω constructed as in example 2.9. This means that ω is compatible with p. These are not exact Lefschetz fibrations since the horizontal boundary is not trivial, but they are very useful since our examples arise in this way.

Theorem 10.5. *Parallel transport maps for an algebraic Lefschetz fibration are well defined.*

This is basically proved in [17, section 2], but, there is a subtle distinction between the above theorem and theirs. In [17, section 2], the Stein structure and the Lefschetz fibration are constructed from the same compactification (X,D) of M. In our case they come from different compactifications. The proof can be easily adjusted to this case. This is due to the fact that if we have two different metrics on M induced from compactifications (X_1,D_1) and (X_2,D_2), then the C^2 distance between them is bounded.

We need the following technical theorem so that we can relate algebraic Lefschetz fibrations with ordinary Lefschetz fibrations. We let (E',π'), (E'',π'') be algebraic Lefschetz fibrations such that $\pi''|_{E''} = \pi'$. Let θ' (resp. θ'') be a convex symplectic structure on E' (resp. E'') constructed as in example 2.9 such that $d\theta'' = d\theta'|_{E''}$.

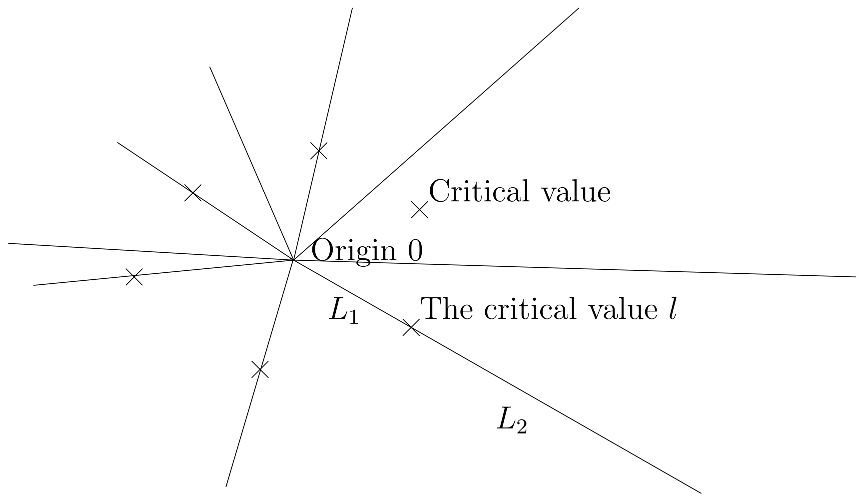
Lemma 10.6. *Suppose that all the singular points of π' are contained in E''. Then: there exists a convex symplectic structure θ_1' (resp. θ_1'') on E' (resp. E'') such that:*

(1) *(E', π', θ_1') (resp. (E'', π'', θ_1'')) are Lefschetz fibrations without boundary.*
(2) *$d\theta_1'|_{E''} = d\theta_1''$.*
(3) *All the parallel transport maps are trivial on a neighbourhood N of $E' \setminus E''$. We can ensure that $E' \setminus N$ is relatively compact when restricted to each fibre.*
(4) *For each smooth fibre F' of π', $\theta'|_{F'} = \theta_1'|_{F'} + dR$ for some compactly supported function R. We have a similar statement for (E'', π'').*
(5) *(E', θ_1') (resp. (E'', θ_1'')) is convex symplectic deformation equivalent to (E', θ') (resp. (E'', θ'')).*

Proof. of Theorem 10.6 We divide this proof into 3 sections. In the first section we construct the Lefschetz fibration without boundary (E', π', θ_1'). In the second section we construct (E'', π'', θ_1''). In section 3 we show that (E', θ_1') (resp. (E'', θ_1'')) is convex deformation equivalent to (E', θ') (resp. (E'', θ'')).

Step 1 We will use ideas from [31, section 19b]. The map π' has well defined parallel transport maps by Theorem 10.5. We have the same for (E'', π''). Suppose without loss of generality that $0 \in \mathbb{C}$ is a regular point of these fibrations. Let $Q' := {\pi'}^{-1}(0)$, $Q'' := Q \cap E''$. Consider the family of radial lines in $\mathbb{C}$ coming out of 0. Let L be one of these radial lines which passes through a critical value l of π'. We can write $L = L_1 \cup L_2$ where L_1 is the line joining 0 and l, and $L_1 \cap L_2 = \{l\}$.

Radial lines coming from the origin



We now have vanishing thimbles V_1 and V_2 of l covering L_1 and L_2. Let V be the union of all such thimbles for all radial lines passing through critical values of π'. We can use this to construct a map $\rho : E' \backslash V \to \mathbb{C} \times Q'$. The map is constructed as follows: Let x be a point in $E' \setminus V$. Then we can parallel transport x along a radial line to a point a in Q'. Then $\rho(x) := (\pi'(x), a)$. Let $X = V \cap Q'$ and $W := Q' \setminus X$. Then $\rho|_{E' \setminus V} : E' \setminus V \to \mathbb{C} \times W$ is a diffeomorphism. Let $\varpi := (\rho|_{E' \setminus V})^{-1}$. Let $\theta_{Q'} := \theta'|_{Q'}$. From now on, if we have a differential form q on $\mathbb{C} \times W$, then we will just write q instead of $\rho^* q$ to clean up notation.

Because parallel transport maps are exact, we have: $\theta'|_{E' \setminus V} = \theta_{Q'} + \kappa' + dR'$ where κ' is a 1-form satisfying $i^* \kappa' = 0$ for all maps i where i is the inclusion map of any fibre of π' into E', and R is some function on $\mathbb{C} \times W$. Let $\bar{f} : W \to \mathbb{R}$ be a function which is equal to 1 near X and is 0 outside some relatively compact neighbourhood of X. We extend $\bar{f}$ by parallel transport along these radial lines to a map $g : E' \setminus V \to \mathbb{R}$. Then we extend g to a map $f : E' \to \mathbb{R}$ as g is constant near V. We will also assume that g is only non-zero inside E'' because parallel transport maps are well defined for (E'', π'', θ''), hence $V \subset E''$. We define

$$\theta'_f := \theta_{Q'} + g\kappa' + d(gR').$$

This form extends over V because $\theta'_f = \theta'$ near V (where $g = 1$). The 1-form θ'_f makes π' into a Lefschetz fibration without boundary where each of the fibres have a convex symplectic structure. We define $\theta'_1 := \theta'_f$.

Step 2 Let $Q'' := \pi''^{-1}(0) \subset E''$. We also have that:

$$\theta'' = \theta_{Q''} + \kappa'' + dR''.$$

Here, κ'' is a 1-form on E'' satisfying $i^*\kappa'' = 0$ for all maps i where i is the inclusion map of a fibre of π'' into E'', and R'' is some function on $E'' \cap (\mathbb{C} \times W)$. Because $d\theta'' = d\theta'$, we have that $d\kappa' = d\kappa''$. This means that $\beta := \kappa' - \kappa''$ is a closed 1-form in E''. We can also show that β is exact as follows: Let $l : \mathbb{S}^1 \to E''$ be a loop. Because we are in dimension 4 or higher (dimension 2 is trivial and irrelevant for this thesis), we can perturb the loop so that it doesn't intersect the radial vanishing thimbles described above. We can then deform l using parallel transport to a loop l' contained in a smooth fibre F. We have $\beta|_F = 0$ which means that $\int_l \beta = \int_{l'} \beta = 0$. Hence $\beta = dL$ for some $L : E'' \to \mathbb{R}$. We define:

$$\theta''_f := \theta_{Q''} + g\kappa' + d(gL) + d(gR').$$

We have $d\theta''_f = d\theta'_f$, hence this makes (E'', π'') into a well defined symplectic subfibration of E'. We define $\theta''_1 := \theta''_f$.

Step 3 We can deform $\bar{f}$ through functions which are trivial at infinity to some $\bar{f}'$ where $\bar{f}' = 0$ outside some large compact set, and $(\bar{f}')^{-1}(1)$ contains an arbitrarily large compact set $K \subset F$. We can construct $f' : E' \to \mathbb{R}$ using $\bar{f}'$ in the same way that we constructed f from $\bar{f}$ and the deformation from $\bar{f}$ to $\bar{f}'$ induces a deformation from f to f'. We can choose a convex symplectic structure θ_S on the base so that $(E', \theta'_f + \pi'^*\theta_S)$ and $(E', \theta'_{f'} + \pi'^*\theta_S)$ are convex symplectic manifolds. Hence $\theta'_f + \pi'^*\theta_S$ is convex deformation equivalent to $\theta'_{f'} + \pi'^*\theta_S$. If we choose K large enough we get that $\theta'_{f'} + \pi'^*\theta_S$ is convex deformation equivalent to $(E', \theta' + \pi'^*\theta_S)$ by Lemma 10.2 and Lemma 10.3, and hence is convex deformation equivalent to (E', θ').

Because θ''_f is described in a very similar way to θ'_f, we can use exactly the same argument as above to show that (E'', θ'') is convex deformation equivalent to $(E'', \theta''_f + \pi^*\theta_{S,1})$. The 1-form $\theta_{S,1}$ is a convex symplectic structure on the base making $\theta''_f + \pi^*\theta_{S,1}$ into a convex symplectic structure.

□

Let X,D,M be as in Theorem 4.21. This means that Z is an irreducible divisor in X and $q \in (Z \cap M)$ is a point in the smooth part of Z. There is a rational function m on X which is holomorphic on M such that $\overline{m^{-1}(0)}$ is reduced and irreducible and $Z = \overline{m^{-1}(0)}$. We have $M' := \mathrm{Kalmod}(M, (Z \cap M), \{q\})$, and $M'' := M \setminus Z$. We also have $\dim_{\mathbb{C}} X \geq 3$.

Lemma 10.7. *There exist algebraic Lefschetz fibrations*

$$p' : M' \to \mathbb{C},\ p'' : M'' \to \mathbb{C}$$

such that p'' is a subfibration of p' (i.e $p' \circ (\text{inclusion}) = p''$). Also, if F' (resp. F'') is a page of p' (resp. p''), then F' is the proper transform of F'' in $\mathrm{Bl}_q X$. The singularities of p' are contained in M''.

Proof. Let Q be an effective ample line bundle on X with support equal to D and such that $Q'' := \overline{m^{-1}(0)} + Q$ is ample. Let s'', t'' be sections of Q'' such that ${s''}^{-1}(0) = \overline{m^{-1}(0)} + Q$. We choose t'' such that

$$p'' = \frac{t''}{s''} : M'' \to \mathbb{C}$$

is some algebraic Lefschetz fibration on M''. Let $\bar{F}''$ be the closure of one of the smooth fibres of p'' in M. We can move the point p to somewhere in the smooth part of $\bar{F}'' \cap Z$ as the smooth part of Z is connected (as Z is irreducible); M' is unchanged up to Stein deformation. NB here we use $\dim_{\mathbb{C}} X \geq 2$.

Remember b is the blowdown map $b : \mathrm{Bl}_q X \to X$. Let $s' := b^* s''$ and $t' := b^* t''$. Let Δ be the exceptional divisor $b^{-1}(p)$. The divisor ${s'}^{-1}(0)$ is equal to $\Delta + \text{other divisors}$. We can choose an effective divisor K' with support equal to the boundary divisor D' of M' in $\mathrm{Bl}_q X$ such that $K'' := K' - \Delta$ is ample. Hence, we can choose a meromorphic section h of K'' whose zero set is contained in D', and such that h has a pole of order 1 along the exceptional divisor and such that h is holomorphic away from $D' \cup \Delta$. Let L be the line bundle associated to K''. This means that $s' \otimes h \in H^0(\mathcal{O}(L \otimes b^* Q''))$ is non-zero away from D'. We let $p' := \frac{t' \otimes h}{s' \otimes h}$. This means that $p'|_{M''} = p''$. Because q is in the smooth locus of $\bar{F}'' \cap Z$ and $\bar{F}''$ is transverse to Z, we have that the closure of any smooth fibre of p' intersects each stratum of D' transversally.

We can choose holomorphic coordinates $z_1 \cdots z_n$ on an open set U of p and a holomorphic trivialisation of Q'' such that $s'' = z_1$ and $t'' = z_2$. We then blow up at the point p. Locally around p, we have a subvariety of $U \times \mathbb{P}^n$ defined by $X_i x_j = X_j x_i$ where $X_1 \cdots X_n$ are projective coordinates for $\mathbb{P}^n$. We choose the chart $Z_1 = 1$. This has local holomorphic coordinates $z_1, Z_2, Z_3, \cdots, Z_n$. We can choose a trivialisation of K'' so that the section h

is equal to $1/z_1$. This means that locally $b^*s'' = Z_1$ and $b^*t'' = Z_2 z_1$. Hence locally, $s' = 1$ and $t' = Z_2$ which means that $p' = Z_2$. This means that p' has no singular points near Δ. Hence p' is also an algebraic Lefschetz fibration which coincides with p'' away from Δ and such all the singular points of p' are the same as the singular points of p''. □

Lemma 10.8. *Let F' (resp. F'') be a fibre of p' (resp. p''). Let K be a compact set in F''. There is a Stein structure J on F' (depending on K) such that any J-holomorphic $u : T \to F'$, where T is a compact Riemann surface with boundary, has the property that $u(T) \subset F''$ if $u(\partial T) \subset F''$.*

Proof. Let G be the closure of F'' in M'. Then F' is biholomorphic to $\mathrm{kalmod}(G, G \cap Z, \{q\})$. Let ϕ_G be a Stein function for G. The compact set K is contained in $\phi_G^{-1}(C)$ for some large C. Let q' be a point outside $\phi_G^{-1}(C)$ which is contained in the smooth part of $G \cap Z$. Because $\dim_{\mathbb{C}} G \cap Z \geq 2$, we can assume that q and q' are in the same irreducible component U of $G \cap Z$. This is where we use the assumption that $\dim_{\mathbb{C}} X \geq 3$. The manifold $G' := \mathrm{kalmod}(G, G \cap Z, \{q'\})$ is naturally an Stein manifold by example 2.9. By the comment after Lemma 10.1, we have that G' is Stein deformation equivalent to F' such that it also induces a Stein deformation on M''. The Stein deformation is induced from moving q' smoothly down to q inside the smooth part of $U \subset G \cap Z$ (Note: U is irreducible, hence the smooth part of U is connected). This induces a Stein homotopy of M' and M'' which in turn induces a Stein homotopy of F' and F''.

From now on we assume that the symplectic structures on F' and G' are complete by [34, Lemma 6]. We can also ensure that the above Stein deformation between F' and G' is complete and finite type by the same lemma, hence by [34, Lemma 5] we have a symplectomorphism $h : F' \to G'$ induced by this Stein deformation. Let $J_{G'}$ be the natural complex structure on G'. Let $J := h^* J_{G'}$. Then if T is a J-holomorphic curve in F' with boundary inside K then $h(T)$ is a holomorphic curve in G' with boundary in $h(K)$. We can blow down this curve to give a holomorphic curve T' in G with boundary in $b(h(K))$. If T passes through the blowup of q, then T' passes through q'. This means that $\phi_G \circ T'$ has an interior maximum outside $\phi_G^{-1}(C)$, but this is impossible. Hence the curve T must be contained in F''. □

We can now apply the above lemmas to prove 4.20. We can apply theorem 10.6 to p' and p'' to get symplectic fibrations (M', p', θ_1') and (M'', p'', θ_1'').

These fibrations are Lefschetz without boundary. We can cut down the fibres to Stein domains $\bar{F}'$ and $\bar{F}''$ where $\bar{F}''$, $\bar{F}'$ are large enough so that the support of all the monodromy maps of (M', p', θ_1') are contained in $\bar{F}''$ and $\bar{F}'' \subset \bar{F}'$. We can also remove the cylindrical end from the base. This will make p' and p'' into Lefschetz fibrations (E', π') and (E'', π'') respectively. Note that if we have a holomorphic curve T in $\bar{F}'$ with boundary in $\bar{F}''$, Lemma 10.8 implies that it is contained in $F'' \cap \bar{F}'$. The Stein maximum principle [25, Lemma 1.5] ensures that T is contained in $\bar{F}''$. Hence we get that Theorem 4.20 is a consequence of Lemma 10.6 and Lemma 10.8.

11. Appendix B: Stein structures and cylindrical ends

The problem with Stein structures is that the complex structure associated with them does not behave well with respect to cylindrical ends. Cylindrical ends here means that near infinity, the convex symplectic manifold is exact symplectomorphic to $(\Delta \times [1,\infty), r\alpha)$ where $r \in [1,\infty)$ and α is a contact form on Δ. The almost complex structure is convex with respect to this cylindrical end if $dr \circ J = -\alpha$. We will deal with this problem in this section.

Let (M, J, ϕ) be a complete finite-type Stein manifold with $\theta = -d^c\phi$ and $\omega = d\theta$. Let $c \gg 0$ be greater than the highest critical value of ϕ.

Theorem 11.1. *There exists a complete finite type convex symplectic structure (M, θ_1) with the following properties:*

1. *It has a cylindrical end with an almost complex structure J_1 which is convex at infinity.*
2. *$J_1 = J$ and $\theta_1 = \theta$ in the region $\{\phi \leq c\}$.*
3. *Any J_1 holomorphic curve with boundary in $\{\phi = c\}$ is contained in $\{\phi \leq c\}$.*
4. *It is convex deformation equivalent to (M, θ) via a convex deformation (M, θ_t) where $\theta_t|_{\{\phi \leq c\}} = \theta|_{\{\phi \leq c\}}$ for $t \in [0,1]$.*

Proof. Let $\lambda := \nabla\phi$ and $\Delta := \phi^{-1}(c+1)$. We define $G : M \to \mathbb{R}$, $G = \frac{1}{\|\nabla\phi\|^2}$ where $\|.\|$ is the norm defined using the metric $\omega(\cdot, J\cdot)$. Let $\lambda' := G\lambda$. Let $F_t : M \to M$ be the flow of λ'. This exists for all time because ϕ is unbounded and $\mathcal{L}_{\lambda'}\phi = 1$ which implies that ϕ increases linearly with t ($\mathcal{L}$ here means Lie derivative). We have an embedding $\Phi : \Delta \times [1,\infty) \to M$ defined by $\Phi(a, r) = F_{\log r}(a)$ where $a \in \Delta \subset M$ and $r \in [1,\infty)$. Also,

$\mathcal{L}_{\lambda'}\theta = G\theta$. Hence, $\Phi^*(\theta) = f\alpha$ where $f : \Delta \times [1,\infty) \to \mathbb{R}$, $f(a,r) := 1 + \int_0^r (G \circ \Phi)(a,t)dt$ and α is the contact form $\theta|_\Delta$ on Δ.

We will now deform the 1-form $f\alpha$ to a 1-form $f'\alpha$ such that $f' = f$ near $r = 1$ and $f' = r$ near infinity. We define θ_1 to be equal to $f'\alpha$ in this cylindrical end and equal to θ away from this end. This means that for r large, we have a cylindrical end with 1-form $f'\alpha = r\alpha$. If we have a function $g : \Delta \times [1,\infty) \to \mathbb{R}$, then $d(g\alpha)$ is non-degenerate if and only if $\frac{\partial g}{\partial r} > 0$. Also, the Liouville vector field associated to $g\alpha$ is $(g/\frac{\partial g}{\partial r})\frac{\partial}{\partial r}$, and hence we have that this Liouville vector field is transverse to every level set $\{r = \text{const}\}$ and pointing outwards. If $(g/\frac{\partial g}{\partial r})$ is bounded above by any polynomial, then the respective Liouville vector field is complete. We define $f' : \Delta \times [1,\infty) \to \mathbb{R}$ such that $f' = f$ near $r = 1$, $f' = r$ near infinity and $\frac{\partial f'}{\partial r} > 0$. This gives a complete finite type convex symplectic structure θ_1 on M as we can extend $f'\alpha$ outside M as $f' = f$ near $r = 1$. We can join f to f' via a smooth family of functions with f_t ($t \in [0,1]$) where $\frac{\partial f_t}{\partial r} > 0$ and such that $f_t = f$ near $r = 1$. This gives us a convex deformation from θ to θ'.

We now need to construct our almost complex structure J_1. We have $\mathcal{L}_{\lambda'}\phi = Gd\phi(\nabla\phi) = 1$. This means that $\Phi^*(\phi) = \log r$ so the level sets of ϕ coincide with the level sets of $\log r$. By abuse of notation we will just write J for the pullback Φ^*J and we will write ϕ for $\log r$. We have two orthogonal symplectic vector subbundles of the tangent bundle $\Phi^*(TM) = T(\Delta \times [1,\infty))$ whose direct sum is the entire tangent bundle (the symplectic structure we are dealing with here is θ_1). These are: $V_1 := \text{Ker}(\theta_1) \cap \text{Ker}(dr)$ and $V_2 := \text{Span}(\frac{\partial}{\partial r}, X_r)$ where X_r is the Hamiltonian flow of r. The problem is that J is not necessarily compatible with $d\theta_1$, so we need to deform it so that it is. However, near $r = 1$, J is in fact compatible with $d\theta_1$ because $\theta = \theta_1$ in some region $\Xi := \{r \leq 1+\epsilon\}$. Inside Ξ, we have that: $J|_{V_1}$ and $J|_{V_2}$ are holomorphic subbundles of $\Phi^*(TM)$. There exists a complex structure J_{V_1} (resp. J_{V_2}) on the vector bundle V_1 (resp. V_2) compatible with $d\theta_1|_{V_1}$ (resp. $d\theta_1|_{V_2}$) such that, $J_{V_1} = J|_{V_1}$ (resp. $J_{V_2} = J|_{V_2}$) when restricted to Ξ. Because $V_1 \bigoplus V_2 = \Phi^*(TM)$, this gives us an almost complex structure J_1 on $\Phi^*(TM)$ compatible with $d\theta_1$ which is equal to J in the region Ξ. We can choose J_{V_1} and J_{V_2} so that $J_{V_2}(\frac{\partial}{\partial r}) = -\frac{1}{r}X_r$ for $r \gg 0$ and J_{V_1} is invariant under the flow of $\frac{\partial}{\partial r}$ for $r \gg 0$. This ensures that J_1 is convex at infinity. Also, we have that r is plurisubharmonic with respect to J_1 hence any J_1 holomorphic curve with boundary in $\{r = 1\}$ is contained in $\{r \leq 1\}$. Hence property (3) is satisfied. $\square$

12. Appendix C: Transfer maps and handle attaching

The purpose of this section is to show that symplectic homology is additive as a ring under end connect sums. This was already done by Cieliebak in [6] but without taking into account the ring structure. Throughout this section, let (M, θ), (M', θ') be compact convex symplectic manifolds such that M' is an exact submanifold of M of codimension 0. We let $C := N \times [1, \infty)$ be a cylindrical end of M where $\theta = r\alpha$, α is a contact form on N, and r is the coordinate for $[1, \infty)$. Similarly we have a cylindrical end C' of M'. Let $H : M \to \mathbb{R}$ be an admissible Hamiltonian with an almost complex structure J, convex at infinity. Let $SH_*^{(-\infty,a)}(M, H, J)$ be the group generated by orbits of action $< a$. For $b \geq a$, we define

$$SH_*^{[a,b)}(M, H, J) := SH_*^{(-\infty,b)}(M, H, J)/SH_*^{(-\infty,a)}(M, H, J).$$

12.1. Weak cofinal families.

Definition 12.1. *We say that the pair (H, J) is* **weakly admissible** *if there exists an $f : N \to \mathbb{R}$ and a constant b such that for $r \gg 0$, $H = re^{-f} + b$ and $d(re^{-f}) \circ J = -\theta$.*

Every admissible pair (H, J) is weakly admissible with $f =$ const. Symplectic homology $SH_*(M)$ is defined as a direct limit of $SH_*(M, H, J)$ with respect to admissible pairs ordered by $\leq$. We wish to replace "admissible" with "weakly admissible". The reason why we wish to do this is because in section 12.3 we carefully construct a cofinal family of weakly admissible pairs to show that symplectic homology behaves well under end connect sums. We construct a partial order $\leq$ on weakly admissible pairs as follows: $(H_0, J_0) \leq (H_1, J_1)$ if and only if $H_0 \leq H_1$. We will show that:

$$SH_*(M) := \varinjlim_{(H,J)} SH_*(M, H, J)$$

where the direct limit is taken over weakly admissible pairs (H, J) ordered by $\leq$. Note that a family of weakly admissible Hamiltonians (H_s, J_s) is cofinal with respect to $\leq$ if the corresponding functions $f_s : N \to \mathbb{R}$ tend uniformly to $-\infty$ as s tends to ∞. In order to ensure that this direct limit exists, we will show that if $(H_0, J_0) \leq (H_1, J_1)$, then there is a natural map of rings $SH_*(M, H_0, J_0) \to SH_*(M, H_1, J_1)$.

This map will be a continuation map. In order for a continuation map to be well defined, we need a family of Hamiltonians T_s joining H_0 and H_1 such that solutions of the parameterised Floer equation $\partial_s u + J_t \partial_t u = \nabla^{g_t} T_s$

joining orbits of H_0 and H_1 stay inside some compact set. To ensure this, we flatten H_i so that it is constant outside some large compact set, and so that all the additional orbits created have very negative action. We do this as follows:

Let D be a constant such that any orbit of H_0 or H_1 has action greater than D. Then $SH_*(M, H_i, J_i) \cong SH_*^{[D,\infty)}(M, H_i, J_i)$. Near infinity, we have that $H_i = R_i + b_i$ where $R_i = re^{-f_i}$. We wish to create a new Hamiltonian K_i such that $K_i = H_i$ on $R_i \leq B$ where $B \gg 0$, and such that K_i is constant in $\{R_i > B + 1\}$ where all the additional orbits have action less than D. We assume that all the orbits of H_0 and H_1 lie in a compact set. We have a cylindrical end $C_i := N_i \times [K, \infty)$ where N_i is the contact manifold $\{re^{-f_i} = 1\}$ with contact form $\theta|_{N_i}$ and R_i is the coordinate for $[K, \infty)$. So, H_i is linear with slope 1 on this cylindrical end. Because all the orbits of H_i lie in a compact set, there are no Reeb orbits of length 1 in the contact manifold N_i. Choose $\epsilon > 0$ such that the length of any Reeb orbit of N_0 or N_1 is of distance more than ϵ from 1. We assume that B is large enough so that H_i is linear with respect to the cylindrical end C_i in $R_i \geq B$ and such that $B\epsilon > -b_i - D$ for $i = 0, 1$. Finally we let K_i be equal to H_i in the region $\{R_i \leq B\}$, and $K_i = k_i(R_i)$ where k_i is constant for $R_i \geq B + 1$, and $k_i = R_i + b_i$ near B and $k_i' \leq 1$. This means that the orbits of K_i in $\{R_i \leq B\}$ are the same as the orbits of H_i, and the orbits of K_i in $\{R_i > B\}$ have action

$$R_i k_i' - k_i < R_i(1 - \epsilon) - R_i - b_i < -B\epsilon - b_i < D.$$

Hence all the orbits of K_i of action greater than D are the same as the orbits of H_i.

Let $V > 0$ be a constant which is greater than the maximum action difference between any two orbits of K_0 or K_1. By [26, Lemma 1], there exists a $B_1 > B$ such that any J_i-holomorphic curve crossing both $R_i = B+1$ and $R_i = B_1$ has area greater than $2V$. We now wish to create an almost complex structure J_i^0 as follows: we let $J_i^0 = J_i$ for $R_i \leq B_1$ and for $R_i \gg B_1$, we let J_i^0 be convex with respect to the cylindrical end C (i.e. $dr \circ J = -\theta$). Let u be a cylinder or pair of pants satisfying Floer's equation ([32, Formula 8.1]) with respect to (K_i, J_i^0) such that each cylindrical end of u limits to a periodic orbit (or multiple of a periodic orbit in the pair of pants case) inside $\{R_i \leq B\}$. By [26, Lemma 1], u is contained in $\{R_i \leq B_1\}$. Then using [25, Lemma 1.5] we have that u is contained in $\{R_i \leq B\}$. Hence

$$SH_*(M, H_i, J_i) \cong SH_*^{[D,\infty)}(M, H_i, J_i^0) \cong SH_*^{[D,\infty)}(M, K_i, J_i^0).$$

We wish to create a continuation map:

$$SH_*^{[D,\infty)}(M, K_0, J_0^0) \to SH_*^{[D,\infty)}(M, K_1, J_1^0).$$

There exists a family of Hamiltonians A_s connecting K_0 and K_1, and such that A_s is monotonically increasing and A_s is constant at infinity. We now need to carefully construct a family of almost complex structures J_s^0 connecting J_0^0 and J_1^0. The continuation map is defined by counting solutions of:

$$\partial_s u(s,t) + J\partial_t u(s,t) = \nabla^t A_s$$

where ∇^t is the gradient with respect to an $\mathbb{S}^1$ family of metrics of the form $\omega(\cdot, J_s^0 \cdot)$. We assume that J_s^0 is convex with respect to the cylindrical end C for all s. We get compactness as usual from the maximum principle in [25, Lemma 1.5] because $u(s,t)$ is holomorphic where all the A_s's are constant. Note that a standard parameterised continuation map argument shows that this map is independent of choices of A_s.

If f_0 and f_1 are equal to 0, then we have a monotone increasing family of admissible Hamiltonians H_s^1 joining H_0 and H_1, and almost complex structures J_s joining J_0 and J_1. The standard continuation map $SH_*(M, H_0, J_0) \to SH_*(M, H_1, J_1)$ involves counting solutions of a parameterised Floer equation with respect to (H_s, J_s). We wish to show that this map is the same as the above continuation map from K_0 to K_1. The Hamiltonians H_s^1 are of the form $h_s(r)$ for $r \geq P$ where h_s' is constant. We assume that the almost complex structures J_s are convex with respect to the cylindrical end C for $r \geq P$. Hence [25, Lemma 1.5] ensures all the Floer trajectories with respect to (H_s, J_s) stay inside the compact set $r \leq P$. Also there is a constant P' such that K_i is a function of r for $r \geq P' \geq P$. The definition of K_i depends on a parameter B which can be arbitrarily large. We can choose B large enough so that $K_i = H_i$ in the region $r \leq P'$. We choose the functions A_s joining K_0 and K_1 so that A_s is a function of r for $r \geq P'$. We can also assume that $J_s^0 = J_s$. In order to show that the maps are the same, we need to show that any Floer trajectory associated to (A_s, J_s^0) connecting orbits inside $\{r \leq P\}$ is contained in $\{r \leq P\}$. This follows from [25, Lemma 1.5].

12.2. **Transfer maps.** In this section we will construct a natural ring homomorphism: $SH_*(M) \to SH_*(M')$. We say that H is called **transfer admissible** if $H \leq 0$ on M'. We have: $SH_*(M) = \varinjlim_{(H,J)} SH_*(M,H,J)$ where the direct limit is taken over transfer admissible Hamiltonians ordered by $\leq$.

Lemma 12.2. *We have an isomorphism of rings,*

$$\varinjlim_{(H,J)} SH_*^{[0,\infty)}(M,H,J) \cong SH_*(M')$$

where the direct limit is taken over transfer admissible Hamiltonians.

Proof. We construct a particular cofinal family of transfer admissible Hamiltonians H_i and show the above isomorphism of rings. We can embed $\widehat{M'}$ into $\widehat{M}$ by Lemma 2.6. Our cylindrical end C' is then a subset of M. We assume that the action spectrum $\mathcal{S} := \mathcal{S}(\partial M')$ is discrete and injective. Let $k : \mathbb{N} \to \mathbb{R} \setminus \mathcal{S}$ be a function such that $k(i)$ tends to infinity as i tends to infinity. Let $\mu : \mathbb{N} \to \mathbb{R}$ be defined by $\text{dist}(k(i), \mathcal{S})$ ($\text{dist}(a, B)$ is the shortest distance between a and B). From now on we just write k instead of $k(i)$, and similarly for μ.

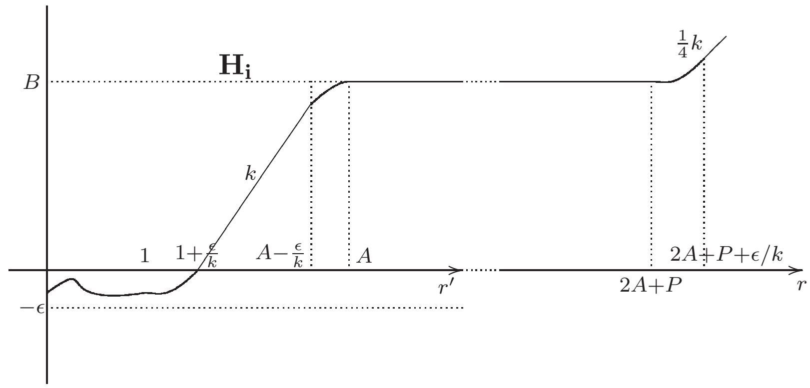
Define:

$$A = A(i) := 6k/\mu > k > 1.$$

We can assume that $A > k > 1$ because we can choose $k(i)$ to make $\mu(i)$ arbitrarily small whilst $k(i)$ is large. We also let $\epsilon := \epsilon(i)$ tend to 0 as i tends to infinity. We assume that $H_i|_{M'} \leq 0$, and has slope $k(i)$ on $1 + \epsilon/k \leq r' \leq A - \epsilon/k$. We also assume that on $1 \leq r' \leq A$, $H_i = h(r')$ for some function h where h has non-negative derivative $\leq k$. For $A \leq r' \leq 2A$, we assume that H_i is constant. Let B be this constant. B is arbitrarily close to $k(A-1)$. We can assume that $B \notin \mathcal{S}$. We now describe H_i on the cylindrical end C. We keep H_i constant until we reach $r = 2A + P$ where P is some constant large enough so that $\{r' \leq 1\} \subset \{r \leq P\}$. This means that $\{r' \leq A\} \subset \{r \leq 2A + P\}$ as long as we embed C' in the same way as Lemma 2.6. We then let H_i be of the form $f(r)$ for $r \geq 2A + P$ where $f' < \frac{1}{4}k$ and has slope $\frac{1}{4}k$ for $r > 2A + P + \epsilon/k$.

Here is a picture of what we have:

Figure 12.3.



The action of an orbit on a level set $r' = a$ is $h'(a)a - h(a)$. The orbits near $r' = 1$ have positive action less than or equal to k. Let p be a point on an orbit o lying in the region $A - \epsilon/k \leq r' \leq A$. The slope $h'(r')$ of H_i at p is $\leq k - \mu$. Hence, the orbits near $r' = A$ have action $\leq (k-\mu)A - B = -\mu A + k \to -\infty$ as $i \to \infty$. So, we can assume that these orbits have negative action. Also all the orbits in $\{r' \geq 2A, r \leq 2A + P\}$ are fixed points, so have action $-B < 0$. Finally, the orbits in $r > 2A + P$ have action: $\leq \frac{1}{4}k.(2A+P) - B = -\frac{1}{2}kA + k + \frac{1}{4}Pk \to -\infty$ as $i \to \infty$. Hence, we can assume that all the orbits of H_i of non-negative action lie in $r' < 1+\epsilon/k$.

We now need to show that any differential connecting two orbits of non-negative action is contained entirely in $r' < 1+\epsilon/k$. By [26, Lemma 1] there exists a $K > 0$ such that any J holomorphic curve which intersects $r' = A$ and $r' = 2A$ has area greater than KA. Any differential connecting two orbits of non-negative action must have area $\leq k < KA$ for i large enough. This means any differential connecting orbits of non-negative action must be contained in $r' \leq 2A$. By the maximum principle in [25, Lemma 1.5] we have that no trajectory or pair of pants can have a maximum in $1 \leq r' \leq 2A$. This means that we have maps $SH_*^{[0,\infty)}(M, H_i, J) \cong SH_*(M', H_i', J')$, where $H_i' : \widehat{M'} \to \mathbb{R}$ has slope k. Taking direct limits gives us a vector space isomorphism

$$\varinjlim_{(H,J)} SH_*^{[0,\infty)}(M, H, J) \cong SH_*(M').$$

We need to show it is also a ring isomorphism. The pants product gives us a map:

$$SH_*^{[0,\infty)}(M, H_i, J) \otimes SH_*^{[0,\infty)}(M, H_i, J) \to SH_*^{[0,\infty)}(M, 2H_i, J).$$

If we have a pair of pants u associated to the above product connecting orbits of non-negative action, then it has area $\leq 2k < KA$ for i large enough. In the region $A \leq r' \leq 2A$, u is holomorphic. This means that it cannot cross $r' = 2A$ by [26, Lemma 1], and a maximum principle as before means that in fact it cannot have a maximum in $r' > 1$. $\square$

This lemma enables us to define a transfer map:

$$SH_*(M) \cong \varinjlim_{(H,J)} SH_*(M, H, J) \to \varinjlim_{(H,J)} SH_*^{[0,\infty)}(M, H, J) \cong SH_*(M').$$

A Hamiltonian is called **weakly transfer admissible** if it is weakly admissible and is negative when restricted to M'. We can combine the above results with the results of section 12.1 to construct the above transfer map using a cofinal family of weakly transfer admissible Hamiltonians. We will need to construct a cofinal family of weakly transfer admissible Hamiltonians in section 12.3 to show that a particular transfer map is an isomorphism of rings.

Here is an application of the transfer map:

Lemma 12.4. *If $SH_*(M) = 0$, then $SH_*(M') = 0$.*

Proof. We have a commutative diagram:

$$\begin{tikzcd} H^{n-*}(M) \arrow[r, "a"] \arrow[d, "b"] & H^{n-*}(M') \arrow[d, "d"] \\ SH_*(M) \arrow[r, "c"] & SH_*(M') \end{tikzcd}$$

Suppose for a contradiction $SH_*(M') \neq 0$. Then [32, Section 8] says that the map d is non-zero in degree n. Also the map a is an isomorphism in degree n. Hence $d \circ a$ is non-zero, and so $c \circ b = d \circ a$ is non-zero. This means that $SH_*(M) \neq 0$ and we get a contradiction. $\square$

Corollary 12.5. *If M is subcritical and $SH_*(M') \neq 0$, then M' cannot be embedded in M as an exact codimension 0 submanifold. In particular, if $H_1(M') = 0$ then M' cannot be symplectically embedded into M.*

Proof. By one of the applications of [26], we have that $SH_*(M) = 0$ because M is subcritical. The result follows from the above lemma. $\square$

12.3. **Handle attaching.** We will now describe handle attaching in detail as in [6, Section 2.2]. The paper [10] or [7, Theorem 9.4] ensures that this construction corresponds to attaching a Stein 1-handle. We will define ϕ, p_i, q_i, X, ω, $\psi(x, y)$ as in [6, Section 2.2]. We will now remind the reader what these variables are: We set $k = 1$, so we are describing 1-handles only. We let $\mathbb{R}^{2n}$ have coordinates $(p_1, q_1, \ldots, p_n, q_n)$.

$$\omega := \sum_i dp_i \wedge dq_i,$$

$$\phi := \frac{1}{4}\sum_{i=1}^{n-1}\left(q_i^2 + p_i^2\right) + q_n^2 - \frac{1}{2}p_n^2,$$

$$X := \nabla\phi = \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\partial}{\partial q_i} + \frac{\partial}{\partial p_i}\right) + 2\frac{\partial}{\partial q_n} - \frac{\partial}{\partial p_n}.$$

ψ is a function of x and y where:

$$x := \sum_{i=1}^{n-1}\left(A_i q_i^2 + B_i p_i^2\right),$$

$$y := B_n p_n^2,$$

and $A_i, B_i > 0$ are constants. It satisfies $X.\psi > 0$ provided that:

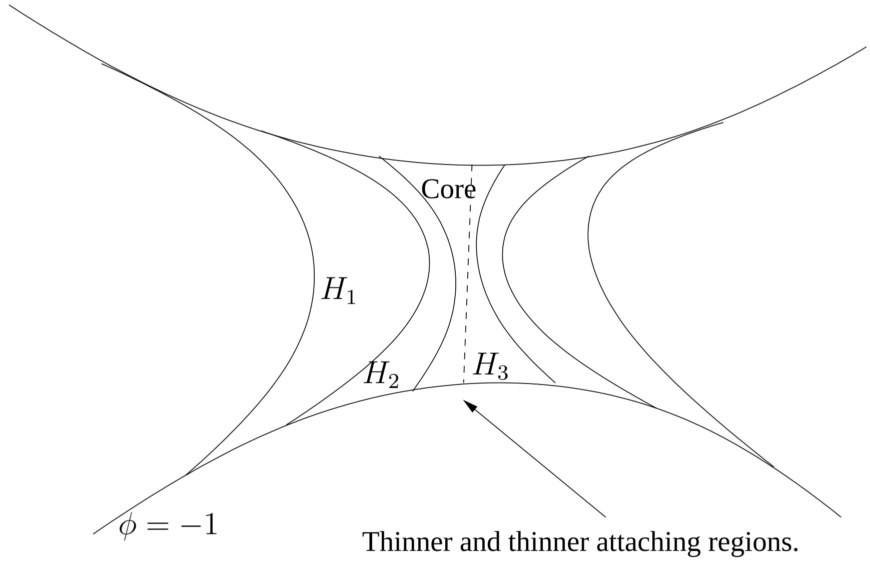
$$\frac{\partial\psi}{\partial x} \geq 0,\ \frac{\partial\psi}{\partial y} \leq 0,\ \frac{\partial\psi}{\partial x}(x, 0) > 0,\ \frac{\partial\psi}{\partial y}(y, 0) < 0,$$

and the partial derivatives are not simultaneously 0. We can choose ψ so that the level sets $\{\phi = -1\}$ and $\{\psi = 1\}$ agree outside some compact set. This ensures that when we glue the handle onto our convex symplectic manifold, it still has a smooth boundary so we don't have to smooth the handle once we have attached it.

The handle $H = H_1^{2n} := \{\phi \geq -1\} \cap \{\psi \leq 1\}$. We define $\partial^- H$ to be the boundary $\{\phi = -1\}$. We can ensure that the only 1-periodic orbit of ψ is the critical point at the origin by [6, Section 2.2]. We wish to construct a family of 1-handles (constructed in the same way as H) $(H_l)_{l\in\mathbb{N}}$ with the following properties:

(1) $H_{l+1} \subset H_l$
(2) The attaching region $\partial^- H_{l+1}$ is a subset of $\partial^- H_l$
(3) As l tends to infinity, H_l converges uniformly to the core of the handle.

This can be done by shrinking ψ.



Let M be a compact convex symplectic manifold. After a deformation, we can assume that the boundary of M has a region A which is contactomorphic to the attaching region $\partial^- H_1$. We can also ensure that the period spectrum of ∂M is discrete and injective (we might have to deform the region A and ϕ slightly and hence all the handles). We can use the region A to attach the handle H_l to M to create a new compact convex symplectic manifold $M_l := M \cup_{\partial^- H_l} H_l$. We have that $M_{l+1} \subset M_l$ and the boundary of each M_l is transverse to the Liouville vector field on M_1. Let K be an admissible Hamiltonian on $\widehat{M}$. We assume that K has slope S in a neighbourhood of ∂M. We choose l large enough so that the attaching region $P := \partial^- H_l$ has the property that a Reeb flowline outside P intersecting P twice has length greater than S. We can now extend the Hamiltonian K to a Hamiltonian $K' : M_l \to \mathbb{R}$ using the function $B\psi$ where B is some constant. Hence ∂M_l is a level set of K' and K' is linearly increasing on a neighbourhood of ∂M_l. Hence we can extend K' to an admissible Hamiltonian on $\widehat{M_l}$. The periodic orbits of K' are the same as the periodic orbits of K with an extra fixed point at the origin of the 1-handle. We can ensure that the index of the extra fixed point at the origin of the 1-handle has index strictly increasing as S increases (see the last part of the proof of Theorem 1.11 in [6, Section 3.4]). Because ∂M_l is transverse to the Liouville field of ∂M_1, we have that $\widehat{M_1} = \widehat{M_l}$ and K' is weakly admissible. Hence we have a cofinal family of weakly transfer admissible Hamiltonians K'. The only orbit outside $M \subset \widehat{M_1}$ has arbitrarily large index, hence these K''s induce a transfer isomorphism of rings $SH_*(M) \to SH_*(M_1)$.

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