Symplectic homology

August 3, 2009

Liouville domain

A Liouville domain is a compact symplectic manifold (M, ω) with boundary and a vector field X satisfying:

- $\mathcal{L}_X \omega = \omega$
- X is transverse to ∂M and pointing outwards.

We can define a 1-form θ by $\theta(\cdot) = \omega(X, \cdot)$. This is called the *Liouville form*.
We can create X and ω using the 1-form θ. We have that $\omega = d\theta$ and X is defined uniquely by $\theta(\cdot) = \omega(X, \cdot)$.

Examples of Liouville domains

- Take $\mathbb{C}^n$ with the standard symplectic form $\sum_j dx_j \wedge dy_j$ where $z_j = x_j + iy_j$ are the standard complex coordinates for $\mathbb{C}^n$. Choose $X = \sum_j r_j \frac{\partial}{\partial r_j}$. Here (r_j, θ_j) are polar coordinates for z_j.
- More generally let C be any properly embedded complex submanifold of $\mathbb{C}^n$. We define a 1-form $\theta := \sum r_j^2 d\theta_j$. Let B be a ball of radius $R > 0$ intersecting C transversally. Then $B \cap C$ is a Liouville domain with Liouville form $\theta|_{B\cap C}$. This is called a *Stein domain*.

Problems concerning Liouville domains

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- (*Weinstein conjecture*) The boundary of M is a contact manifold with contact form $\alpha := \theta|_M$. Does it have Reeb orbits? (i.e. are there maps $\psi : S^1 \to \partial M$ satisfying $\frac{d\psi}{dt} = R$ where R is a vector field satisfying $\alpha(R, Y) = 0$ for all vectors Y and $\alpha(R) = 1$).

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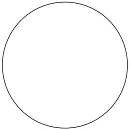
- We can complete M to form $\widehat{M}$ by attaching a cylindrical end $[1, \infty) \times \partial M$ along ∂M and extending θ by $r\alpha$ where r is the coordinate for $[1, \infty)$.
  Is there another Liouville domain N such that $\widehat{N}$ is diffeomorphic to $\widehat{M}$ but not symplectomorphic to $\widehat{M}$?

Attaching a cylindrical end

Liouville domain M



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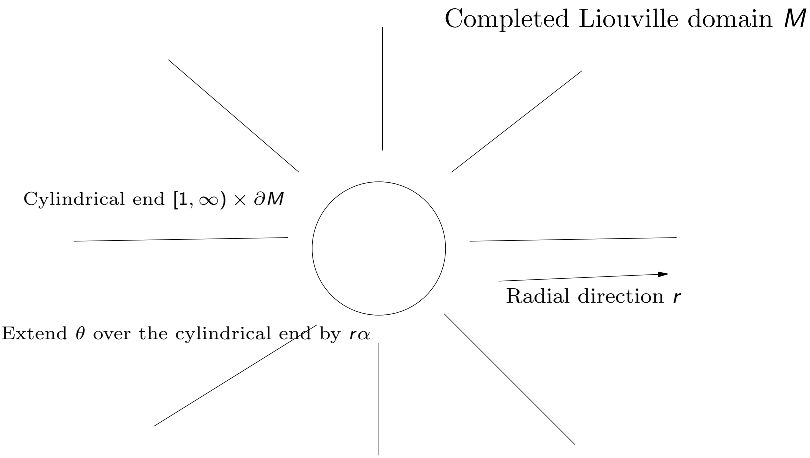


Symplectic homology definition

- We start with the completion $\widehat{M}$ of the Liouville domain M.
- Technical assumtion: If we have a Reeb orbit $\psi : S^1 \to \partial M$ then its length is the integral of $\psi^*\alpha$ over S^1. We assume that the set of Reeb orbit lengths in $\mathbb{R}$ is discrete.
- Let $H : \widehat{M} \to \mathbb{R}$ be a Hamiltonian such that $H = kr$ near infinity. We say H is an *admissible Hamiltonian*.
- We can choose k so that H has no periodic orbits near infinity.
- Let $J : T\widehat{M} \to T\widehat{M}$ be an almost complex structure compatible with ω. (i.e. $J^2 = -1$, $\omega(\cdot, J\cdot)$ is a Riemmanian metric and $\omega(JX, JY) = \omega(X, Y)$).
- J is cylindrical at infinity.
- $c_1(M) = 0$.

We will first define $SH_*(M, H, J)$

- We will be dealing for simplicity with coefficients in $\mathbb{Z}/2\mathbb{Z}$. But we can have coefficients over $\mathbb{Z}$.
- Let C be the $\mathbb{Z}/2\mathbb{Z}$ vector space generated by 1-periodic orbis of H.
- Each orbit has an index associated to it called the *Conley-Zehnder index*. This makes C into a graded vector space $\bigoplus C_k$.

Conley-Zehnder index

Basic idea: Let $\psi : S^1 \to M$ be a 1-periodic orbit.

- Trivialize the symplectic bundle $\psi^* TM \cong S^1 \times \mathrm{Sp}(2n)$.
- The derivative of the Hamiltonian flow ϕ_H^t of H induces a path of symplectic matrices $P : [0, 1] \to \mathrm{Sp}(2n)$ under this trivialization.
- There is a recipe for assigning an index to a path of symplectic matrices. This is the Conley-Zehnder index. (See work by Robbin and Salamon for a good recipe)

the differential

$\partial : C_k \to C_{k-1}$.
For an orbit x of index k we define
$\partial(x) := \sum_{\text{orbits } y \text{ of index k-1}} \sharp(\mathcal{M}(x,y)/\mathbb{R})y$.

What is $\mathcal{M}(x,y)$? It is the set of maps $u : \mathbb{R} \times S^1 \to M$ satisfying

- $\partial_s u + J\partial_t u = \nabla H$, where (s,t) are the coordinates for $\mathbb{R} \times S^1$. (the gradient is taken with respect to the metric $\omega(\cdot, J\cdot)$.
- $u(s,t) \to x(t)$ as $s \to -\infty$.
- $u(s,t) \to y(t)$ as $s \to \infty$.
- The $\mathbb{R}$ action is translation in the s direction.

It turns out that $\mathcal{M}(x, y)$ is a manifold of dimension $\mathrm{index}(x) - \mathrm{index}(y) - 1$ and can be compactified to a manifold with corners.

In our case, $\mathcal{M}(x, y)$ has dimension 0 and is a compact manifold. We set $\sharp(\mathcal{M}(x, y))$ to be the number of points of $\mathcal{M}(x, y)$.

Defining $SH_*(M)$

If $H_1 \leq H_2$ are admissible Hamiltonians, the there is a natural map

$$SH_*(M, H_1, J_1) \to SH_*(M, H_2, J_2).$$

On the chain level, first choose a non-decreasing family of admissible Hamiltonians $(H_s)_{s\in\mathbb{R}}$ joining H_1 and H_2. Similarly join J_1 and J_2 with a family J_s.
If x is an orbit of H_1, then our chain level map ϕ is of the form

$$\phi(x) := \sum_{\text{orbits } y \text{ of } H_2 \text{ of index k}} \sharp(\mathcal{M}(x,y))y.$$

Here $\mathcal{M}(x,y)$ counts solutions of

$$\partial_s u + J_s \partial_t u = \nabla_{\omega(\cdot, J_s \cdot)} H_s.$$

$$SH_*(M) := \varinjlim_{\text{admissible H}} SH_*(M, H, J).$$

- The direct limit is taken with respect to the ordering $\leq$.
- Note that all we need to do is to consider a family $H_1, H_2, \cdots$ of Hamiltonians tending to infinity.

$$SH_*(M) := \varinjlim_i SH_*(M, H_i, J).$$

- In fact we just need $H_1 \leq H_2 \cdots$ such that the slope of H_i tends to infinity. This is because the continuation map $SH_*(M, H, J) \to SH_*(M, H + \text{const}, J)$ is an isomorphism.

Properties of $SH_*(M)$:

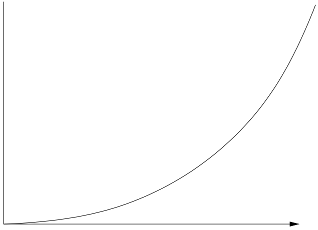
- It is an invariant of $\widehat{M}$ up to symplectomorphism if $H^1(M) = 0$.
- There is a natural map $H^{-*}(M) \to SH_*(M)$.
- If N is a codim 0 submanifold of M such that $\theta|_N$ is a Liouville form for N, then there is a map $SH_*(M) \to SH_*(N)$.
- The unit disk bundle D^*L of a Riemannian manifold L is a Liouville domain. We have $SH_*(D^*L) = H_*(\Omega L)$.
- It satisfies a Künneth formula: $SH_*(M \times N) = SH_*(M) \otimes SH_*(N)$.

A Calculation

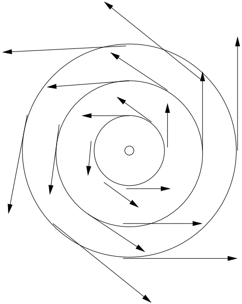
We will deal with $M = \mathbb{D}$ (the unit disk in $\mathbb{C}$), with $\theta := \rho^2 d\theta$ where (ρ, θ) are polar coords. We have $\widehat{M} = \mathbb{C}$.

$$\omega = dx \wedge dy.$$

An admissible Hamiltonian H of slope c has the form $c\rho^2$ near infinity.
Choose the cofinal family $H_k := (k\pi - 1)\rho^2$.
There is only 1 periodic orbit of H_k at 0,
This means the rank of $SH_*(\mathbb{D})$ is at most 1.
It turns out that the Conley-Zehnder index of this critical point is $2k + 1$.
This means the natural transfer map $SH_*(\mathbb{D}, H_k, J) \to SH_*(\mathbb{D}, H_{k+1}, J)$ is zero because it preserves the grading.
Hence we have a direct limit which is zero.



$H = (2k\pi - 1)\rho^2$ $\qquad$ r



An application

We will show that a Liouville domain of the form $N := \mathbb{D} \times M$ contains no exact Lagrangians.

Suppose for a contradiction that it does contain L. Then a neighbourhood of L is a Lioville subdomain equal to D^*L.
We have

$$SH_*(D^*L) = H_*(\Omega L).$$

and a commutative diagram:

An application

$$\begin{array}{ccc} H^{n-*}(N) & \xrightarrow{a} & H^{n-*}(D^*L) \\ \downarrow b & & \downarrow d \\ SH_*(N) & \xrightarrow{c} & SH_*(D^*L) \end{array}$$

The map d corresponds to the map $H^{n-*}(L) \cong H_*(L) \to H_*(\Omega L)$.
Hence in degree 0, we have $c \circ d \neq 0$. Which implies that $a \circ b \neq 0$. Hence $SH_*(N) \neq 0$.
But $SH_*(\mathbb{D}) = 0$ and the Künneth formula implies

$$SH_*(N) = SH_*(\mathbb{D} \times M) = 0.$$

Contradiction.