Computing Symplectic Homology of Affine Varieties

(using Spectral Sequences)

http://www.math.stonybrook.edu/~markmclean/talks/spectralsequencealltogether.pdf

Related Projects (in progress)

- Diogo-Lisi
- Ganatra, Pomerleano
- Sheridan, Borman
- Hülya Argüz
- Joint work with Tehrani, Zinger.

Disclaimer: only one of the spectral sequences in this presentation has been constructed in detail (the second one). The details of the first one have not been worked out fully yet.

An Introduction to Spectral Sequences.

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- *"A spectral sequence is an algebraic object, like an exact sequence, but more complicated"* - J. F. Adams.

- *"After my article was published, John Harper sent me email and said that when he was a graduate student back in the 1960s, he personally asked Leray about the term 'spectral' and in particular asked whether it had something to do with the spectrum of an operator. Leray began his reply by saying, "Non"; unfortunately, before he could continue, some professors approached and interrupted the conversation."*
  -Source: Timothy Chow/ Mathoverflow.net

- We will talk about *homological* spectral sequences since the workshop is on symplectic *homology*.

A *spectral sequence* is a sequence of bigraded chain complexes. This is page 0.

$$\begin{array}{ccccccc}
E^0_{-1,2} & E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} & E^0_{4,2} & E^0_{5,2} \\
E^0_{-1,1} & E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} & E^0_{4,1} & E^0_{5,1} \\
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E^0_{-1,-1} & E^0_{0,-1} & E^0_{1,-1} & E^0_{2,-1} & E^0_{3,-1} & E^0_{4,-1} & E^0_{5,-1}
\end{array}$$

A *spectral sequence* is a sequence of bigraded chain complexes. This is page 1.

$$
\begin{array}{ccccccc}
 & \uparrow & & & & & \\
E^1_{-1,2} & E^1_{0,2} & E^1_{1,2} & E^1_{2,2} & E^1_{3,2} & E^1_{4,2} & E^1_{5,2} \\
 & | & & & & & \\
E^1_{-1,1} & E^1_{0,1} & E^1_{1,1} & E^1_{2,1} & E^1_{3,1} & E^1_{4,1} & E^1_{5,1} \\
 & | & & & & & \\
\leftarrow E^1_{-1,0} \text{---} & E^1_{0,0} \text{---} & E^1_{1,0} \text{---} & E^1_{2,0} \text{---} & E^1_{3,0} \text{---} & E^1_{4,0} \text{---} & E^1_{5,0} \rightarrow \\
 & | & & & & & \\
E^1_{-1,-1} & E^1_{0,-1} & E^1_{1,-1} & E^1_{2,-1} & E^1_{3,-1} & E^1_{4,-1} & E^1_{5,-1} \\
 & \downarrow & & & & &
\end{array}
$$

A *spectral sequence* is a sequence of bigraded chain complexes. This is page 2.

$$
\begin{array}{ccccccc}
E^2_{-1,2} & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & E^2_{3,2} & E^2_{4,2} & E^2_{5,2} \\
E^2_{-1,1} & E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & E^2_{3,1} & E^2_{4,1} & E^2_{5,1} \\
E^2_{-1,0} & E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & E^2_{3,0} & E^2_{4,0} & E^2_{5,0} \\
E^2_{-1,-1} & E^2_{0,-1} & E^2_{1,-1} & E^2_{2,-1} & E^2_{3,-1} & E^2_{4,-1} & E^2_{5,-1}
\end{array}
$$

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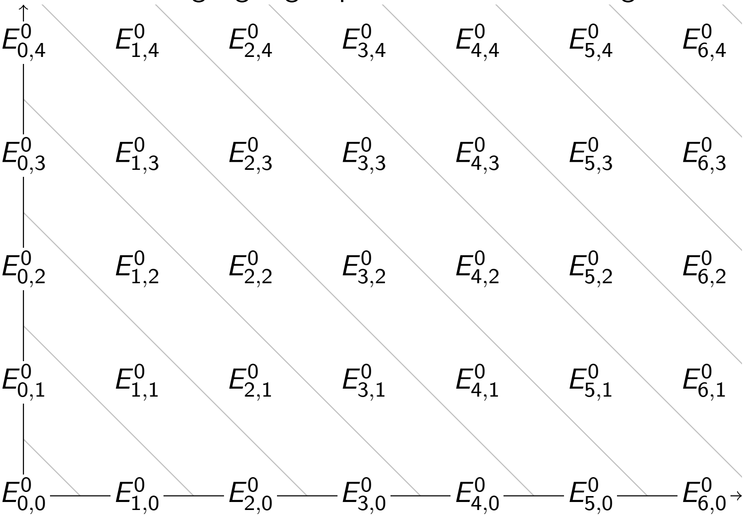
E^3_{-1,0} & E^3_{0,0} & E^3_{1,0} & E^3_{2,0} & E^3_{3,0} & E^3_{4,0} & E^3_{5,0} \\
E^3_{-1,-1} & E^3_{0,-1} & E^3_{1,-1} & E^3_{2,-1} & E^3_{3,-1} & E^3_{4,-1} & E^3_{5,-1}
\end{array}$$

The *total degree* is the sum of the degrees. Diagonal lines have been drawn to highlight groups of the same total degree.



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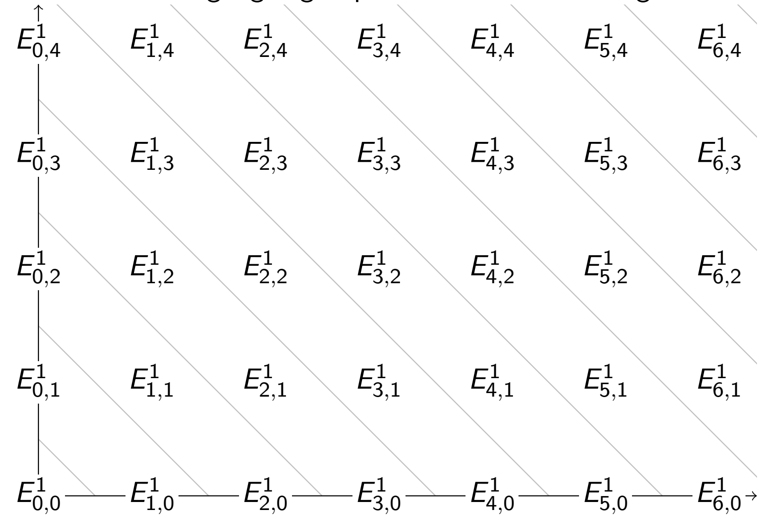

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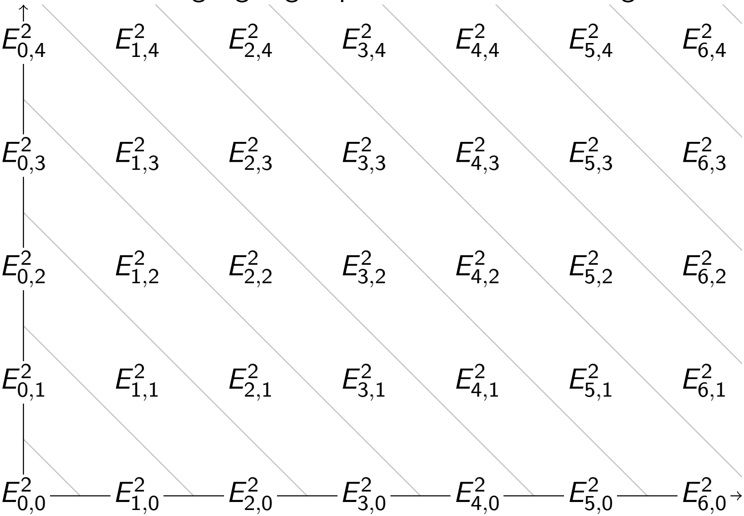

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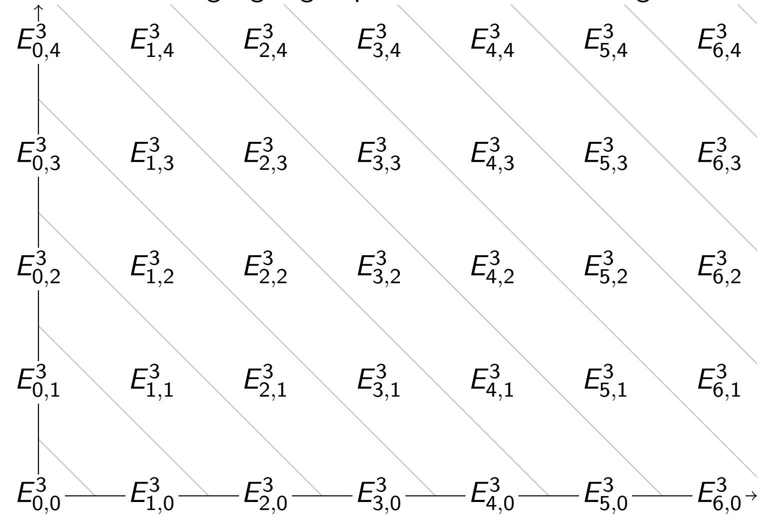

- The differential d on $E^r_{*,*}$ has degree $(-r, r-1)$. In other words, we have maps:

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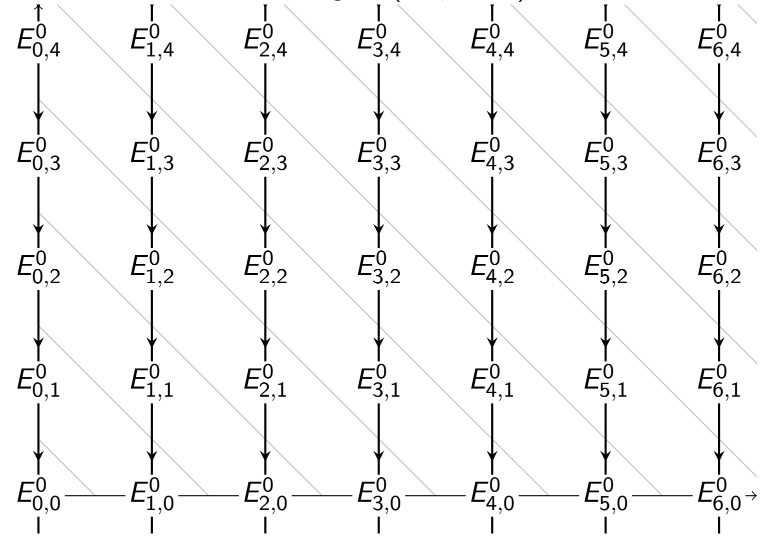
- Note that $d^r_{p,q}$ has total degree -1 since $(p-r)+(q+r-1) = p+q-1$.
- Also $E^{r+1}_{*,*}$ is the homology of the previous page $E^r_{*,*}$. In other words,

$$E^{r+1}_{p,q} = \ker(d^r_{p,q})/\mathrm{im}(d^{r-1}_{p+r,q-r+1}).$$

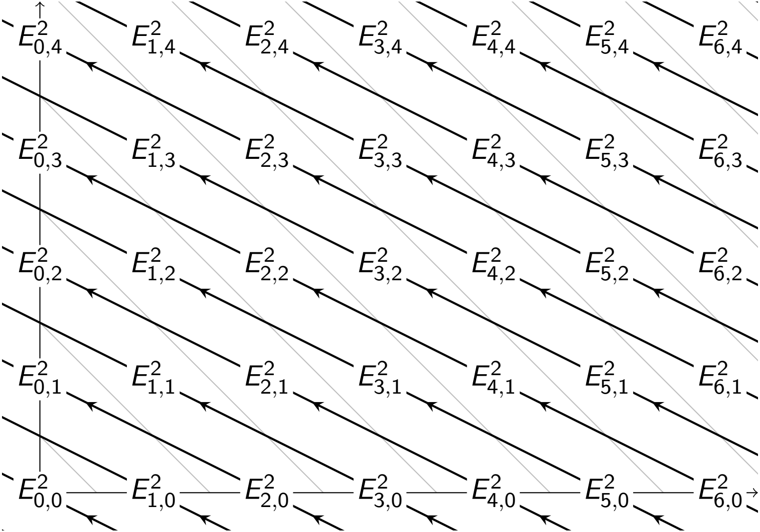
Here the differential has degree $(-0, 0-1)$.



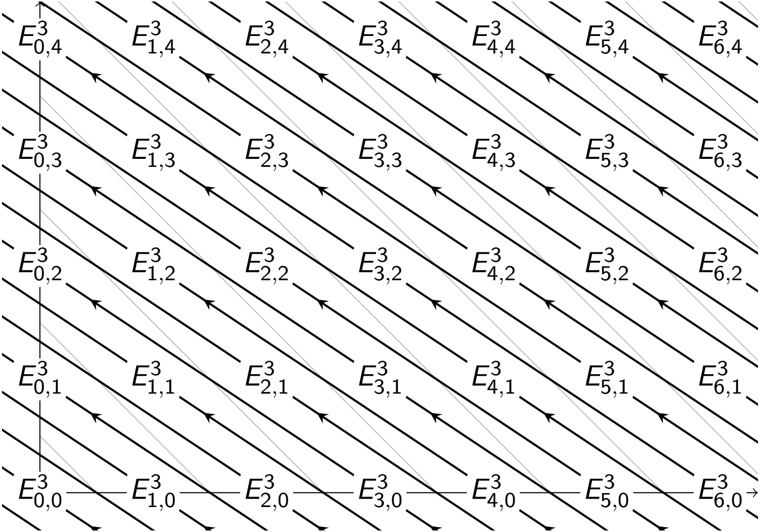
Here the differential has degree $(-1, 1-1)$.

$$
\begin{array}{ccccccccccccccc}
\uparrow & & & & & & & & & & & & & \\
E^1_{0,4} & \longleftarrow & E^1_{1,4} & \longleftarrow & E^1_{2,4} & \longleftarrow & E^1_{3,4} & \longleftarrow & E^1_{4,4} & \longleftarrow & E^1_{5,4} & \longleftarrow & E^1_{6,4} & \longleftarrow \\
E^1_{0,3} & \longleftarrow & E^1_{1,3} & \longleftarrow & E^1_{2,3} & \longleftarrow & E^1_{3,3} & \longleftarrow & E^1_{4,3} & \longleftarrow & E^1_{5,3} & \longleftarrow & E^1_{6,3} & \longleftarrow \\
E^1_{0,2} & \longleftarrow & E^1_{1,2} & \longleftarrow & E^1_{2,2} & \longleftarrow & E^1_{3,2} & \longleftarrow & E^1_{4,2} & \longleftarrow & E^1_{5,2} & \longleftarrow & E^1_{6,2} & \longleftarrow \\
E^1_{0,1} & \longleftarrow & E^1_{1,1} & \longleftarrow & E^1_{2,1} & \longleftarrow & E^1_{3,1} & \longleftarrow & E^1_{4,1} & \longleftarrow & E^1_{5,1} & \longleftarrow & E^1_{6,1} & \longleftarrow \\
E^1_{0,0} & \longleftarrow & E^1_{1,0} & \longleftarrow & E^1_{2,0} & \longleftarrow & E^1_{3,0} & \longleftarrow & E^1_{4,0} & \longleftarrow & E^1_{5,0} & \longleftarrow & E^1_{6,0} & \longleftarrow
\end{array}
$$

Here the differential has degree $(-2, 2-1)$.



Here the differential has degree $(-3, 3-1)$.



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- It is the set of elements which 'survive' forever.
- In our case, all the pages $E^r_{p,q}$ for $r > 0$ will be finite dimensional and they decrease in dimension as r increases.
- Therefore, for each p, q there is a constant $C_{p,q}$ so that $E^{r+1}_{p,q} = E^r_{p,q}$ for all $r \geq C_{p,q}$. Hence we can define $E^\infty_{p,q}$ to be $E^r_{p,q}$ for $r = C_{p,q}$.

- **Definition:** We say that a spectral sequence $(E^r_{p,q})$ *converges* to a graded group H_* if there is a filtration

$$\cdots F_{-1} \subset F_0 \subset F_1 \subset F_2 \subset \cdots \subset H_*$$

so that

$$E^\infty_{p,q} = F_p \cap H_{p+q} / F_{p-1} \cap H_{p+q}.$$

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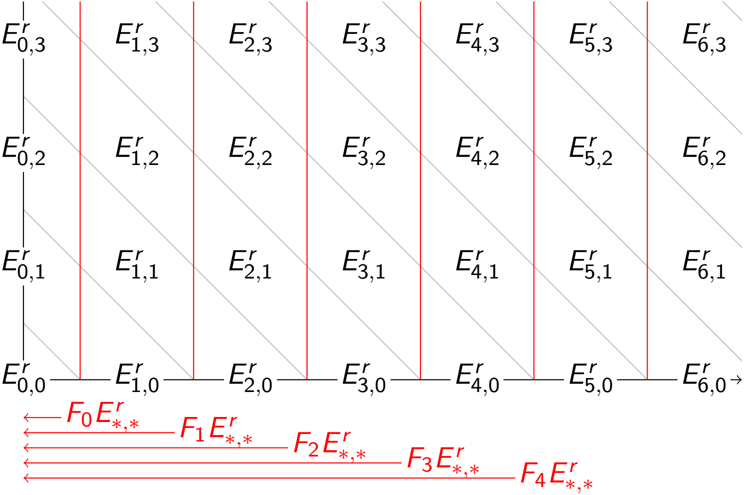
- In our case the filtration will be nice enough so that if the above spectral sequence converges then $H_n = \oplus_p E^\infty_{p,n-p}$.

Spectral Sequence from a filtered complex

- **Theorem:** Suppose we have a nice filtration $\cdots F_{-1}C_* \subset F_0C_* \subset F_1C_* \subset F_2C_* \subset \cdots \subset C_*$ of a chain complex (C_*, ∂). Then there is a spectral sequence converging to $H_*(C_*, \partial)$ with E^1 page equal to:

$$E^1_{p,q} = H_*(F_pC_{p+q}/F_{p-1}C_{p+q}, \partial).$$

- The filtration for us will be the action filtration.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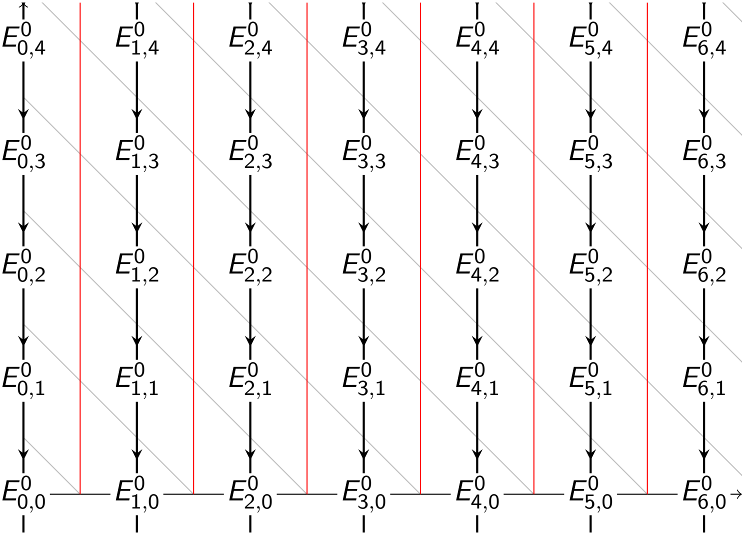
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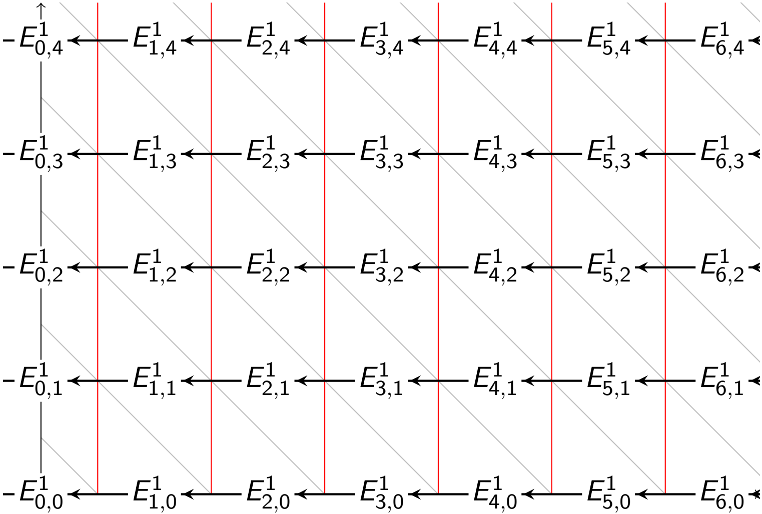
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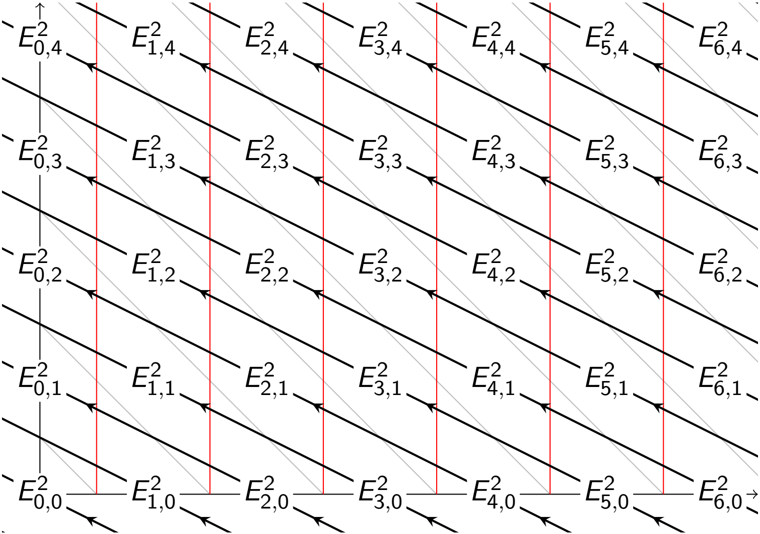
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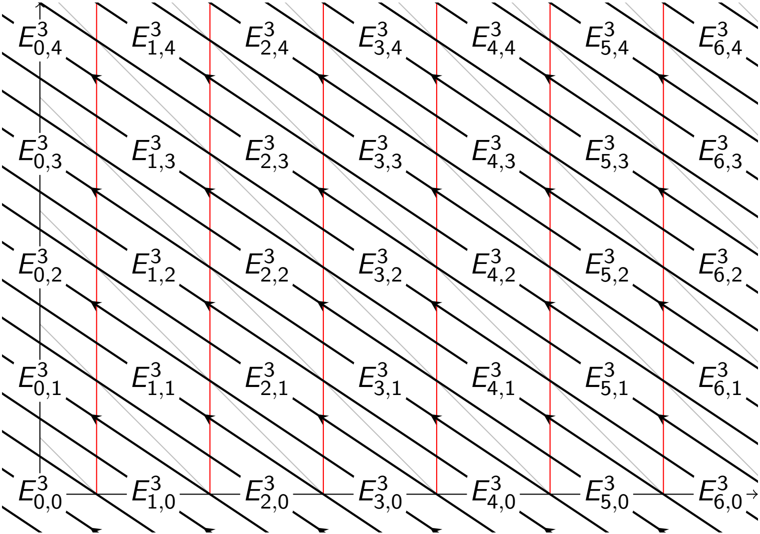
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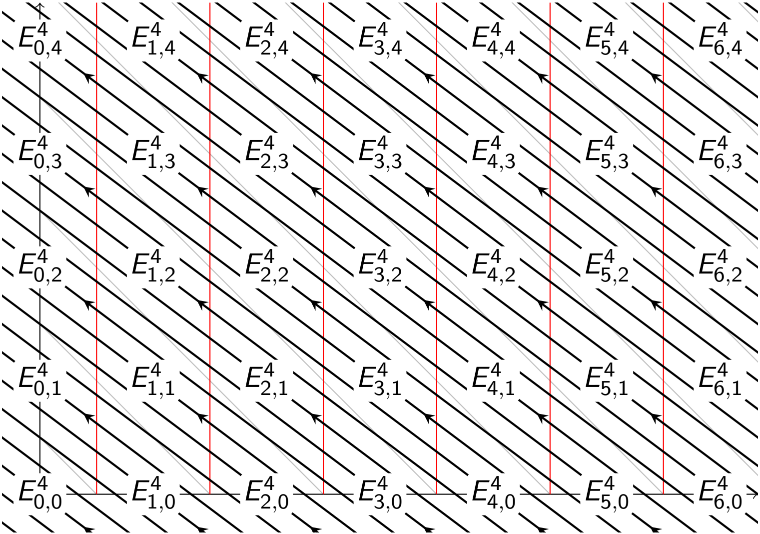
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3. Hope that the differentials that we are interested in vanish, or at least are understandable. For instance, if we wish to show that $H_n \neq 0$ then it is sufficient for us to find p, q so that $p + q = n$ and the differentials $d^r_{p,q}$ and $d^r_{p+r,q-r+1}$ vanish for all $r \geq 1$.

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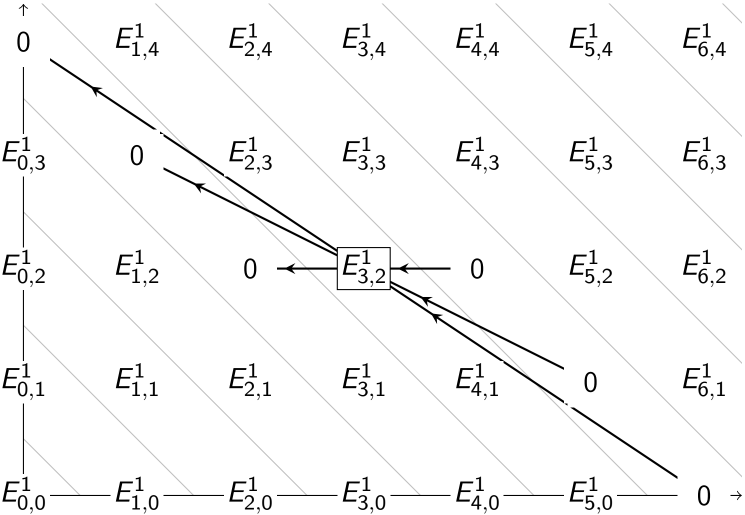
4. Compute $H_n = \oplus_p E^\infty_{p,n-p}$ (the direct sum of everything along the diagonal line containing $(n, 0)$).



Here $H_{3+2} = H_5$ is non-zero.

"... the behavior of this spectral sequence ... is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage strewn with corpses and only one actor left alive (namely the one who has to speak the last few lines)" - J. F. Adams.

A Spectral Sequence for Symplectic Homology.

- We will construct a spectral sequence converging to $SH_*(A)$ (symplectic homology of A) where A is a smooth affine variety of dimension n with $c_1(A) = 0$ (there is also a similar spectral sequence when $c_1(A)$ is torsion but we will not focus on that).

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- Choose a non-zero section κ_A of the *canonical bundle* $K_A \equiv \wedge^n T^*A$ of A.
- Such a section (up to homotopy) gives $SH_*(A)$ a $\mathbb{Z}$-grading.

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- **Theorem** (Hironaka) Every smooth affine variety A is isomorphic to $X - \cup_{i \in S} D_i$ for some X, $(D_i)_{i \in S}$ as above. From now on fix this notation.

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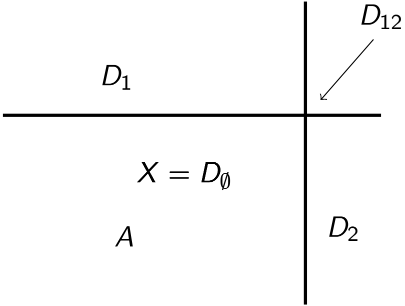
- **Theorem** (Hironaka) Every smooth affine variety A is isomorphic to $X - \cup_{i\in S} D_i$ for some X, $(D_i)_{i\in S}$ as above. From now on fix this notation.
- For any $I \subset S$, define $D_I \equiv \cap_{i\in I} D_i$. Here, $D_\emptyset = X$.



E.g. $A = \mathbb{C}^2$

$$X = \mathbb{CP}^1 \times \mathbb{CP}^1$$
$$D_1 = \mathbb{CP}^1 \times \{\infty\}$$
$$D_2 = \{\infty\} \times \mathbb{CP}^1$$
$$D_{12} = \{\infty\} \times \{\infty\}$$

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- Choose an ample line bundle L on X and a holomorphic section s_A of L so that s_A restricted to A is non-zero and $D = s_A^{-1}(0)$.
- We define the **wrapping number** w_i of D_i to be *minus* the order of $s_A^{-1}(0)$ along D_i.

- **Definition:** For each $I \subset S$ let ND_I be a small tubular neighborhood of D_I so that $ND_I \cap D_{I'}$ is a tubular neighborhood of $D_{I \cup I'}$ for all $I' \subset S$. Also ∂ND_I should intersect $D_{I'}$ transversally for all $I' \subset S$.

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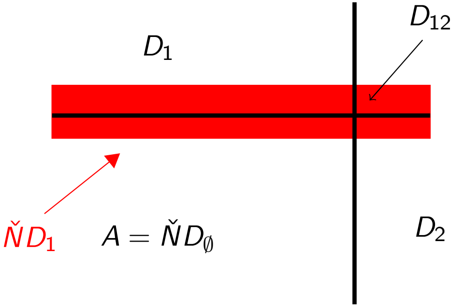
- Define $\check{N}D_I \equiv ND_I - \cup_{i \in S} D_i$. This as a bundle over $\check{V}_I \equiv D_I - \cup_{i \in S-I} D_i$ with fiber a product of punctured disks.



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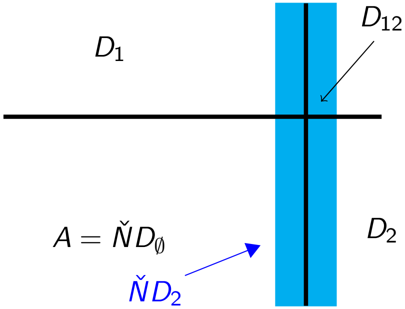

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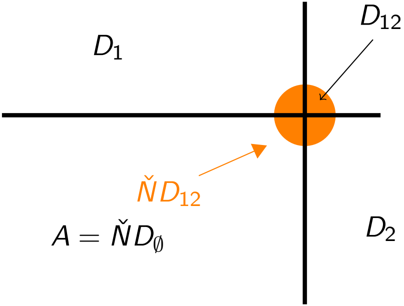

Theorem (M - in progress):
There is a spectral sequence converging to $SH_*(A)$ with E^1 page

$$E^1_{p,q} = \bigoplus_{\{(k_i)\in\mathbb{N}^S \,:\, \sum_i k_i w_i = -p\}} H^{n-p-q-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}})$$

where $\mathbb{N}^S$ is the set of tuples of non-negative integers indexed by S and $I_{(k_i)} = \{i \in S \;:\; k_i \neq 0\}$.

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- The future work of Diogo-Lisi and Ganata-Pomerleano hopefully should give better descriptions of the differentials in some cases.

Other Grading Conventions

- There are other grading conventions.
- You might need to replace (p, q) with $(-p, n-q)$ or $(-p, -q)$ and your spectral sequence differentials will go in the other direction (this would be a cohomological spectral sequence).

Sanity Check

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0	0	0	0	0	0	0
0	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Case: $n = 3$

$$E^2_{p,q} = 0 \text{ for all } p, q$$

$$SH_*(A) = 0$$

Example

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- Computations using ideas from Milnor's paper "On simply connected 4-manifolds". See also https://amathew.wordpress.com/2012/03/05/the-cohomology-of-projective-hypersurfaces/

$\mathbb{Z}$	0	0	0	0	0	0
0	0	0	0	0	0	0
$\mathbb{Z}^{64}$	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0
0	$\mathbb{Z}^{12}$	0	0	0	0	0
0	$\mathbb{Z}^{12}$	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0
0	0	0	0	0	0	0
0	0	$\mathbb{Z}$	0	0	0	0
0	0	$\mathbb{Z}^{12}$	0	0	0	0
0	0	$\mathbb{Z}^{12}$	0	0	0	0
0	0	$\mathbb{Z}$	0	0	0	0
0	0	0	0	0	0	0
0	0	0	$\mathbb{Z}$	0	0	0
0	0	0	$\mathbb{Z}^{12}$	0	0	0
0	0	0	$\mathbb{Z}^{12}$	0	0	0
0	0	0	$\mathbb{Z}$	0	0	0
0	0	0	0	0	0	0
0	0	0	0	$\mathbb{Z}$	0	0
0	0	0	0	$\mathbb{Z}^{12}$	0	0
0	0	0	0	$\mathbb{Z}^{12}$	0	0

Therefore

$$SH_*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 2 \\ \mathbb{Z}^{64} & \text{if } * = 0 \\ \mathbb{Z} & \text{if } * < -1 \text{ and } * = 0 \text{ or } 1 \mod 4 \\ \mathbb{Z}^{12} & \text{if } * < -1 \text{ and } * = 2 \text{ or } 3 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

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- $H^*(\check{N}D_\emptyset) = H^*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 1 \\ \mathbb{Z}^{150} & \text{if } * = 2 \\ 0 & \text{otherwise} \quad . \end{cases}$.

$\mathbb{Z}$	0	0	0	0	0	0
$\mathbb{Z}$	0	0	0	0	0	0
$\mathbb{Z}^{150}$	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	$\mathbb{Z}^2$	0	0	0	0	0
0	$\mathbb{Z}^{52}$	0	0	0	0	0
0	$\mathbb{Z}^{50}$	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	$\mathbb{Z}^8$	0	0	0	0
0	0	$\mathbb{Z}^{64}$	0	0	0	0
0	0	$\mathbb{Z}^{56}$	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	$\mathbb{Z}^{14}$	0	0	0
0	0	0	$\mathbb{Z}^{76}$	0	0	0
0	0	0	$\mathbb{Z}^{62}$	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	$\mathbb{Z}^{20}$	0	0
0	0	0	0	$\mathbb{Z}^{88}$	0	0
0	0	0	0	$\mathbb{Z}^{68}$	0	0

Therefore

$$SH_*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 1 \text{ or } 2 \\ \mathbb{Z}^{150} & \text{if } * = 0 \\ \mathbb{Z}^{2+3(-*-2)/2} & \text{if } * < -1 \text{ and } * = 2 \mod 4 \\ \mathbb{Z}^{52+3(-*-1)} & \text{if } * < -1 \text{ and } * = 1 \mod 4 \\ \mathbb{Z}^{50+3(-*-4)/2} & \text{if } * < -1 \text{ and } * = 0 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Weinstein Conjecture

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- **Definition**: A cooriented contact manifold (C, ξ) satisfies the Weinstein conjecture if every contact form α compatible with ξ has a Reeb orbit.
- Which contact manifolds satisfy the Weinstein conjecture?

- Recall that positive symplectic homology $SH_*^{>0}(M)$ of a Liouville domain M has a chain complex freely generated by two copies of each Reeb orbit on ∂M. In other words, we do not consider critical points of the Hamiltonian in the interior.

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- **Lemma:** If M satisfies the algebraic Weinstein conjecture then ∂M satisfies the Weinstein conjecture.

- Recall that positive symplectic homology $SH_*^{>0}(M)$ of a Liouville domain M has a chain complex freely generated by two copies of each Reeb orbit on ∂M. In other words, we do not consider critical points of the Hamiltonian in the interior.
- **Definition**: M satisfies the **algebraic Weinstein conjecture** if $SH_*^{>0}(M) \neq 0$.
- **Lemma:** If M satisfies the algebraic Weinstein conjecture then ∂M satisfies the Weinstein conjecture.
- **Question:** Which smooth affine varieties satisfy the algebraic Weinstein conjecture?

- $X =$ smooth projective variety and $A = X - \cup_i D_i$ where $(D_i)_{i \in S}$ is a smooth normal crossing divisor.

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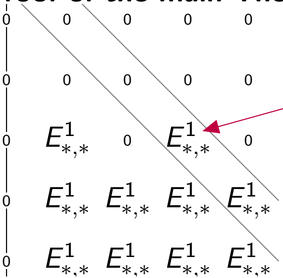
- **Theorem:** Suppose that the discrepancy a_i of D_i is ≤ -1 for all $i \in S$. Then A satisfies the algebraic Weinstein conjecture.

- X = smooth projective variety and $A = X - \cup_i D_i$ where $(D_i)_{i \in S}$ is a smooth normal crossing divisor.
- **Theorem:** Suppose that the discrepancy a_i of D_i is ≤ -1 for all $i \in S$. Then A satisfies the algebraic Weinstein conjecture.
- **Proof of the main Theorem**:



This is the highest non-zero E^1 term on the highest diagonal.

This term exists since $a_i \leq -1, \ \forall i$ and it survives to the E^∞ page since all differentials connecting this term have source or target 0. $\square$

An Additional Grading.

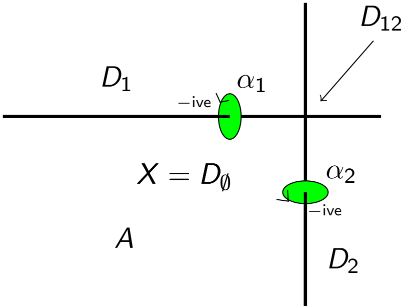
- We have a direct sum decomposition

$$SH_*(A) = \bigoplus_{\alpha \in H_1(A)} SH_{*,\alpha}(A)$$

where $SH_{*,\alpha}(A)$ is the subgroup generated by periodic orbits representing α.

- This grading can be seen in our spectral sequence.

- The H_1-**class associated to** D_i is a class $\alpha_i \in H_1(A)$ represented by the boundary of a small disk in X intersecting D_i once transversely and negatively at 0 and intersecting no other D_j's.



- For each $\alpha \in H_1(A)$, there is a spectral sequence converging to $SH_{*,\alpha}(A)$ with E^1 page

$$E^1_{p,q} = \bigoplus_{\left\{(k_i)\in\mathbb{N}^S \;:\; \substack{\sum_i k_i w_i = -p, \\ \alpha_{(k_i)} = \alpha}\right\}} H^{n-p-q-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}}).$$

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  where $\alpha_{(k_i)} \equiv \sum_i k_i \alpha_i$.
- Our original spectral sequence is the direct sum of the above ones over all $\alpha \in H_1(A)$.

- **Simple Corollary.** Our spectral sequence degenerates at the E^1 page when the affine variety A is one dimensional and not equal to $\mathbb{C}$.

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- Therefore if $A = C - \{p_1, \cdots, p_l\}$ where C is a Riemann surface and $p_1, \cdots, p_l$ distinct points then

$$SH_*(A) = H^{1-*}(C) \oplus \bigoplus_{i=1}^{l} \left(\oplus_{k \geq 1} H^{1-*-2k(a_i+1)}(S^1) \right)$$

Here a_i is the discrepancy of the divisor p_i, which isn't unique. The only constraint is $\sum_i a_i = -\chi(C)$.

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- **Proof:**
The spectral sequence computing $SH_{*,\alpha}(A)$ is non-zero only in one column for each $\alpha \in H_1(A)$. □

- **Theorem**

  The spectral sequence degenerates at the E^1 page when A is the complement of $\geq n+2$ generic linear hypersurfaces in $\mathbb{CP}^n$. I.e.

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$$SH_*(A) = \bigoplus_{(k_i)\in\mathbb{N}^S} H^{n-*-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}}).$$

- **Proof:**
  We have that $H_1(A)$ is the quotient of the free abelian group generated by $(\alpha_i)_{i\in S}$ quotiented out by the relation $\sum_{i\in S}\alpha_i = 0$ where α_i is the H_1-class associated to D_i. This means that for each $\alpha \in H_1(A)$, there is *at most* one representation of α of the form $\sum_{i\in I} k_i\alpha_i$ where $|I| \leq n$ and $k_i \geq 0$.
  Therefore the E^1 page of the spectral sequence computing $SH_{*,\alpha}(A)$ is contained in at most one column and hence must degenerate.
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- Does it detect the dual graph of these hypersurfaces?

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- Convergence is defined in the same way, except that the filtration has to respect the product structure on the algebra H_*.

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- For all $I, J \subset S$, define:

$$\begin{aligned} P_{IJ} : H^*(\check{N}D_I) \otimes H^*(\check{N}D_J) &\longrightarrow H^*(\check{N}D_{I\cup J}) \\ a \otimes b &\longrightarrow \iota^*_{I\cup J,I} a \ \cup \ \iota^*_{I\cup J,J} b. \end{aligned}$$

Conjecture

The spectral sequence above is in fact a spectral sequence of algebras converging to $SH_{n+*}(A)$ with the pair of pants product. The product structure

$$E^1_{p,q} \otimes E^1_{p',q'} \longrightarrow E^1_{p+p',q+q'}$$

on the E^1 page

$$E^1_{p,q} = \bigoplus_{\left\{(k_i)\in\mathbb{N}^S \,:\, \sum_i k_i w_i = -p\right\}} H^{-p-q-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}}).$$

is induced by the maps P_{IJ} above.

Recall: α_i is the H_1-class associated to D_i.

Theorem (assuming conjecture): Suppose that $\alpha_i \neq \alpha_j$ for all $i \neq j$ and $\alpha_i \neq 0$ for all $i \in S$ and suppose the union of all images of the restriction maps $P_{iI} : H^*(\check{N}D_i) \longrightarrow H^*(\check{N}D_I)$ for all $i \in I$ generate $H^*(\check{N}D_I)$ as an algebra for all $I \subset S$. Then the spectral sequence above degenerates on the first page. Hence there is a filtration on $SH_{n+*}(A)$ whose associated graded algebra is:

$$\bigoplus_{(k_i) \in \mathbb{N}^S} H^{-*-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}}),$$

graded by $\sum_i k_i$.

- **Folklore Theorem(?)** If a degree n or $n-1$ element in the $p=0$ page is killed then the affine variety A is ruled by lines $\mathbb{C}$.

  Related to the work of Diogo-Lisi and Ganata-Pomerleano.

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- **Why?**
  Because one should be able to make the Hamiltonian H defining $SH_*(A)$ equal to 0 and then a limiting argument produces a family of curves isomorphic to $\mathbb{C}$ passing through every point of a real hypersurface and hence through every point of A (since the space of such curves has even real dimension).

Other Floer Cohomology Groups

1. **Floer homology** $HF_*(\phi)$ **of a symplectomorphism** $\phi : M \longrightarrow M$. The chain complex here is generated by fixed points of ϕ and the differential counts holomorphic strips $u : \mathbb{R} \times [0,1] \longrightarrow M$ satisfying $\phi(u(s,1)) = u(s,0)$ for all $s \in \mathbb{R}$.

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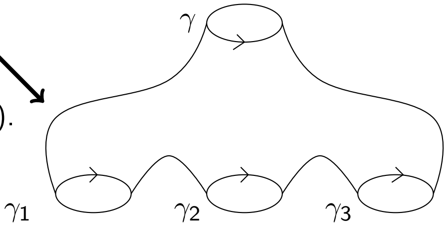
2. **Full contact homology** $CH_*(C,\xi)$ of a $2n-1$-contact manifold (C,ξ) indexed by Conley-Zehnder index $+(n-3)$. Chain complex is the free supercommutative algebra generated by Reeb orbits of a compatible contact form λ. The differential is:

Number of holomorphic
in the symplectization is
the γ coefficient of $\partial(\gamma_1\gamma_2\gamma_3)$.



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- A **positive slope perturbation** of ϕ is a C^∞ small perturbation to $\check{\phi}$ so that ϕ is the time 1 flow of the Hamiltonian δr_M near ∂M where $\delta > 0$ is small (I.e. $\check{\phi}$ is the time δ Reeb flow near ∂M).

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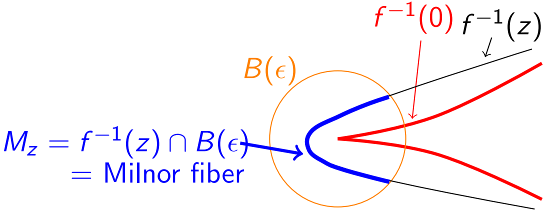

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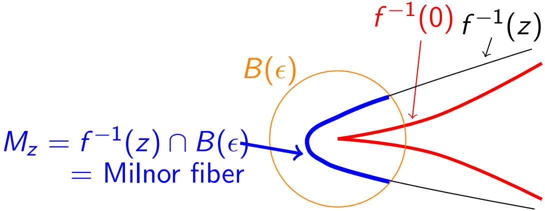

- The monodromy map $\phi_f : M_\delta \longrightarrow M_\delta$ around the loop $\epsilon e^{it},\ t \in [0, 2\pi]$ can be deformed to an exact symplectomorphism as above. It has a grading induced from $\mathbb{C}^{n+1}$.

- **Defintion:** A *log resolution* of $(\mathbb{C}^{n+1}, f^{-1}(0))$ is a proper map $\pi : Y \longrightarrow \mathbb{C}$ so that

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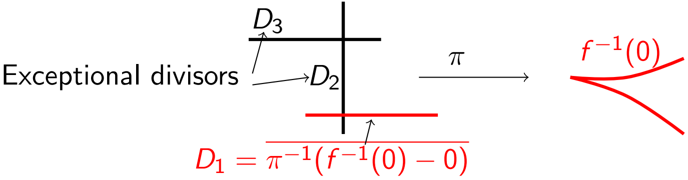
- The hypersurfaces $(D_i)_{i\in S}$ are called *resolution divisors* and the hypersurfaces $(D_i)_{i\in \widehat{S}}$, $\widehat{S} \subset S$ satisfying $\pi(D_i) = \{0\}$ are called *exceptional divisors*.



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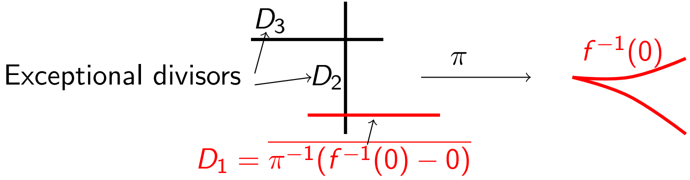

- Goal: compute (parts of) $HF_*(\phi_f, +)$ from a log resolution.

Simple example

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- The Milnor fiber is the ball of radius ϵ in $\mathbb{C}^n$.
- The monodromy map ϕ_f is the identity map, but the grading is non-trivial.
- $HF_*(\phi_f^m) = \begin{cases} \mathbb{Z} & \text{if } * = n + 2m \\ 0 & \text{otherwise} \end{cases}$.

- Define
  $\Lambda(\phi_f^m) := \sum_{j=0}^{\infty}(-1)^j \mathrm{Tr}((\phi_f)_*^m : H_j(M_f;\mathbb{Z}) \longrightarrow H_j(M_f;\mathbb{Z}))$.

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- **Theorem** (A'Campo)

$$\Lambda(\phi_f^m) = \sum_{\{i \in \widehat{S}\ :\ m_i\ |\ m\}} m_i \chi(D_i^o), \quad \forall m > 0.$$

- **Definition** Let s_Y be a meromorphic section of an ample line bundle on Y with a pole of order w_i along D_i for all $i \in \widehat{S}$ and which is non-zero and holomorphic away from $\pi^{-1}(0)$. The **wrapping number** of D_i is defined to be w_i.

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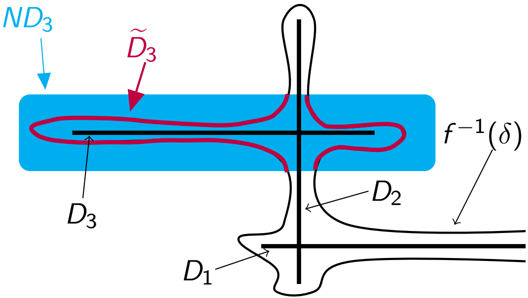
- **Definition** Choose a holomorphic coordinate chart $x_1, \cdots, x_{n+1}$ centered at some point of D_i. The **discrepancy** a_i of D_i is the order of the Jacobian of $\pi(x_1, \cdots, x_n)$ along D_i.

- **Definition** A **multiplicity** m **separating resolution** $\pi : Y \longrightarrow \mathbb{C}^{n+1}$ is a log resolution as above so that $m_i + m_j > m$ for all $i, j \in S$ satisfying $D_{ij} \neq 0$. I.e. the sum of the multiplicities of adjacent resolution divisors is greater than m.

- **Definition** For each $i \in S$ let ND_i be a small tubular neighborhood of D_i with boundary transverse to all of the strata of $\cup_i D_i$. Define $\widetilde{D}_i \equiv f^{-1}(\delta) \cap ND_i$ for $\delta > 0$ sufficiently small.

  This is homotopic to an m_i-fold cover of D_i^o.



- **Theorem** (M - 98% done):
  Fix $m > 0$, and let $\pi : Y \longrightarrow \mathbb{C}^{n+1}$ be a multiplicity m separating resolution. Then there is a spectral sequence converging to $HF_*(\phi_f^m)$ with E^1 page

$$E^1_{p,q} = \bigoplus_{\left\{i \in \widehat{S} \;:\; \substack{m_i | m \\ \frac{m}{m_i} w_i = p}\right\}} H_{n+p+q-2m\left(\frac{a_i+1}{m_i}\right)}(\widetilde{D}_i)$$

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- The Euler characteristic of the right hand side is naturally equal to $(-1)^n$ times the right hand side of A'Campo's formula above. Similarly the left hand side of A'Campo's formula is $(-1)^n \chi(HF_*(\phi_f^m))$.

Simplest Example, m=1.

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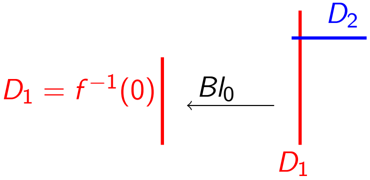

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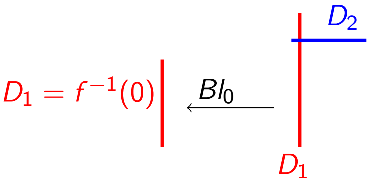

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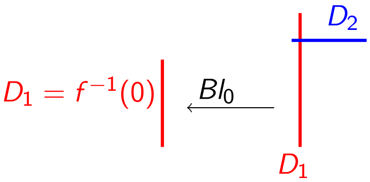

- $m_1 = 1, m_2 = 1, w_2 = 1$ and $a_2 = n - 1$.
- Our spectral sequence degenerates and we get:

$$HF_*(\phi_f) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

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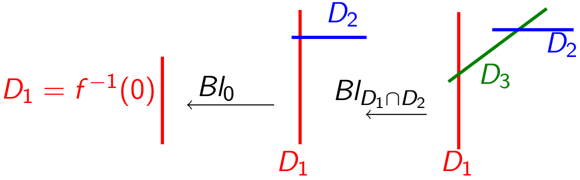

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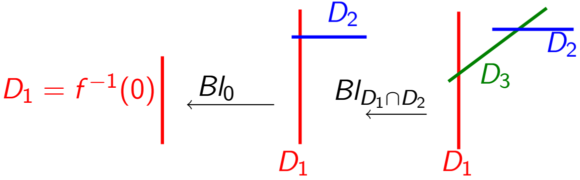
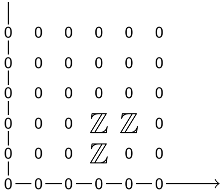


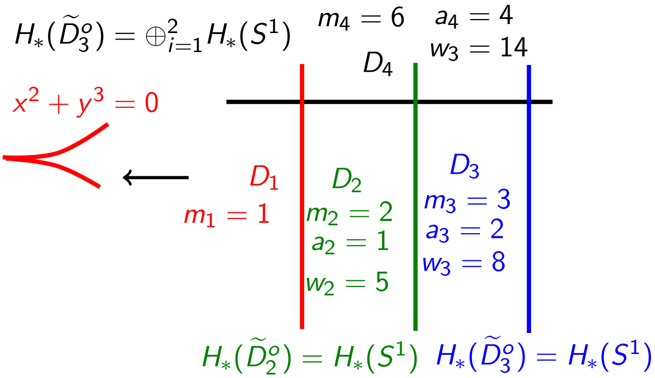
- $w_2 = 2, w_3 = 3$, $m_2 = 1, m_3 = 2$, $a_1 = 1, a_2 = 2$ and $H_*(\widetilde{D}_2^o) = H_*(\text{pt})$ and $H_*(\widetilde{D}_3^o) = H_*(S^1)$.



Case: $n = 2$

$$HF_*(\phi_f^2) = \left\{ \begin{array}{cc} \mathbb{Z} & \text{if } * = 4 \\ 0 & \text{otherwise.} \end{array} \right\}$$

Example 2, x^2+y^3, $m=1,2,3,4,5$



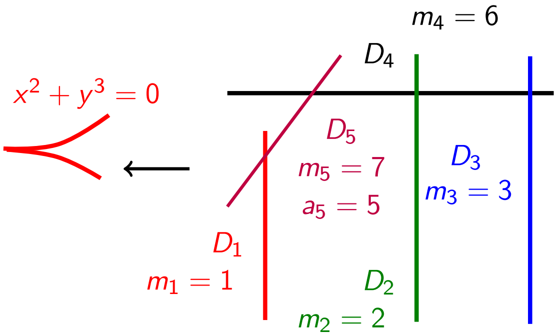
1. $HF_*(\phi_f^m)=0$ if $m=1,5$,
2. $HF_*(\phi_f^2)=H_{*-2}(S^1)$,
3. $HF_*(\phi_f^3)=H_{*-4}(S^1)$ and
4. $HF_*(\phi_f^4)=H_{*-6}(S^1)$.

$f(x,y) = x^2 + y^3$

Having said that, we cannot use the spectral sequence to compute $HF_*(\phi_f^6)$ since our E^1 page is:

$$
\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}^3 & \mathbb{Z} & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0
\end{array}
$$

- But, ϕ_f^7 is fine.



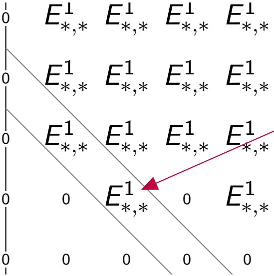
- $HF_*(\phi_f^7) = H^{*-11}(S^1)$.

- **Theorem 2** Fix $m > 0$. Let $\pi : Y \longrightarrow \mathbb{C}^{n+1}$ be a multiplicity m separating resolution with exceptional divisors $(D_i)_{i \in \widehat{S}}$ of multiplicity $(m_i)_{i \in \widehat{S}}$ and discrepancy $(a_i)_{i \in \widehat{S}}$. Define $S_m \equiv \{i \in \widehat{S} \ : \ m_i | m\}$. Then

$$\inf\{\alpha : HF_\alpha(\phi_f^m, +) \neq 0\} = \inf\left\{2m\left(\frac{a_i + 1}{m_i}\right) - n \ : \ i \in S_m\right\}.$$

In particular, $HF_*(\phi_f^m, +)$ vanishes if and only if m_i does not divide m for each $i \in \widehat{S}$.

Proof



This is the lowest non-zero E^1 term on the lowest diagonal.

This term exists since $\dim(\bigoplus_{p,q} E^1_{p,q}) < \infty$ and it survives to the E^∞ page since all differentials connecting this term have source or target 0.

□

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- **Lemma**: $\text{mult}_0 f = \min_{i \in \widehat{S}} m_i$.

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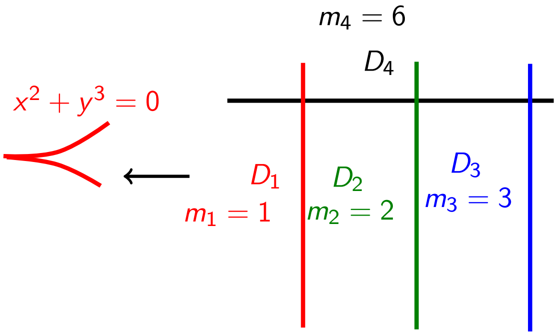

Here $\text{mult}_0(x^2 + y^3) = \min(m_2, m_3, m_4) = 2$.

Corollary of Theorem 2: $\mathrm{mult}_0(f) = \inf_m HF_*(\phi_f^m, +) \neq 0$.

This proves a conjecture by Seidel.

E.g.
If $f = x^2 + y^3$ then $HF_*(\phi_f, +) = 0$ but
$HF_*(\phi_f^2, +) = H_{*-2}(S^1) \neq 0$.

Log Canonical Threshold

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- Used for proving vanishing theorems in algebraic geometry (helpful in birational geometry).
- A version of lct has been used to prove certain Fano manifolds are non-rational (Corti, de Fernex, Ein, Mustata).

- **Lemma:** Let $\pi : Y \longrightarrow \mathbb{C}^{n+1}$ be a log resolution for $(\mathbb{C}^{n+1}, f^{-1}(0))$ with exceptional divisors $(D_i)_{i \in \widehat{S}}$ of multiplicity $(m_i)_{i \in \widehat{S}}$ and discrepancy $(a_i)_{i \in \widehat{S}}$. Then

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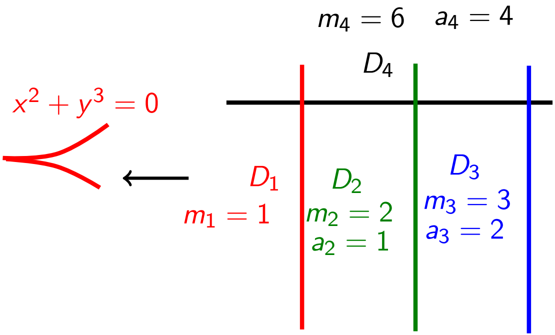
- **Corollary**: $\operatorname{lct}_0(f)$ is a rational number.

E.g.



$$\mathrm{lct}_0(x^2 + y^2) = \min\left(\frac{1+1}{2}, \frac{2+1}{3}, \frac{4+1}{6}\right) = \frac{5}{6}.$$

Aside: Counting Solutions Mod p^m

- Suppose $f \in \mathbb{Z}[z_0, \cdots, z_n]$ and let N_k be the number of solutions of $f = 0 \bmod p^k$ (p is a fixed prime).

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- Suppose $f \in \mathbb{Z}[z_0, \cdots, z_n]$ and let N_k be the number of solutions of $f = 0$ mod p^k (p is a fixed prime).
- **Theorem** (Igusa). $Z_p(f) := \sum_{k \in \mathbb{N}} p^{-kn} N_k z^k$ is a meromorphic function whose nearest pole is at $-\mathrm{lct}_0(f)$. (Proof uses p-adic integration).

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- This means that we know the radius of convergence of $Z_p(f)$ which can be used to estimate the growth of N_k.

- E.g. (ratio test):

$$\lim \frac{p^{-(k+1)n} N_{k+1}}{p^{-kn} N_k} |z| = \lim \frac{N_{k+1}}{p^n N_k} < 1 \quad \text{iff} \quad |z| < \mathrm{lct}_0(f).$$

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- Hence $\exists\ C_1, C_2$ such that

$$C_1 \left(\frac{p^n}{\mathrm{lct}_0(f)} \right)^k < N_k < C_2 \left(\frac{p^n}{\mathrm{lct}_0(f)} \right)^k \quad \forall k.$$

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- **Corollary of Theorem 2**:

$$\mathrm{lct}_0(f) = \inf\left\{\frac{\alpha+n}{2m} \;:\; HF_\alpha(\phi_f^m) \neq 0 \text{ or } \frac{\alpha+n}{2m} = 1\right\}$$

- **Lemma** (Varchenko) For all sufficiently small $\epsilon > 0$, $L_f \equiv f^{-1}(0) \cap S(\epsilon)$ is a contact submanifold of the ϵ-sphere $S(\epsilon) \subset \mathbb{C}^{n+1}$.

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- **Theorem** If f_1 and f_2 have contactomorphic embedded links then $HF_*(\phi_{f_1}^m) = HF_*(\phi_{f_2}^m), \ \forall\ m > 0$.

- **Zariski Conjecture**: Let $f_1, f_2 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ have isolated singularities at 0 and suppose that there is a diffeomorphism $S(\epsilon) \longrightarrow S(\epsilon)$ sending L_{f_1} to L_{f_2} then is the multiplicity of f_1 equal to the multiplicity of f_2?

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- **Corollary**: Suppose f_1 and f_2 have contactomorphic embedded links, then they have the same multiplicity and log canonical threshold at 0.
- For instance, if $f_1, f_2 \in \mathbb{Z}[z_0, \cdots, z_n]$ then the number N_k^1, N_k^2 of solutions of $f_1 = 0$ and $f_2 = 0$ mod p^k respectively satisfy

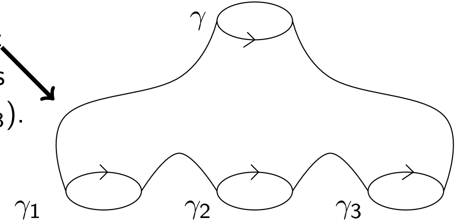
$$C_1 N_k^1 < N_k^2 < C_2 N_k^2$$

for some constants C_1, C_2.

Full Contact Homology

Full contact homology $CH_*(C,\xi)$ of a $2n-1$-contact manifold (C,ξ) indexed by Conley-Zehnder index $+(n-3)$. Chain complex is the free supercommutative algebra generated by Reeb orbits of a compatible contact form λ. The differential is:

Number of holomorphic
in the symplectization is
the γ coefficient of $\partial(\gamma_1\gamma_2\gamma_3)$.



- An **isolated singularity** $A \subset \mathbb{C}^N$ is the germ at 0 of an affine variety $A \equiv \{z \in \mathbb{C}^n \ : \ f_1 = \cdots = f_l = 0\}$ with an isolated singularity at 0, or is smooth at 0 (I.e. the matrix $\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}$ has constant rank on $U - \{0\}$ where U is a neighborhood of 0).

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- **Lemma** (Varchenko): $L_A \equiv A \cap S(\epsilon)$ is a contact manifold with contact structure $\xi_A \equiv TL_A \cap J_0 TL_A$ where $J_0 : T\mathbb{C}^N \longrightarrow T\mathbb{C}^N$ is the standard complex structure.

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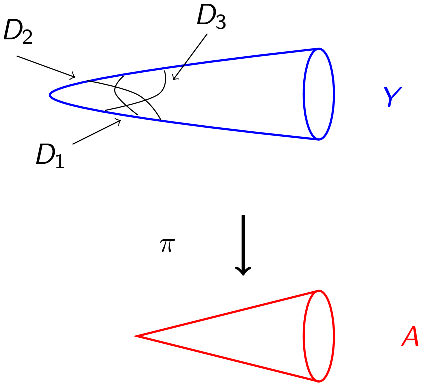
- We call (L_A, ξ_A) the **link** of A at 0.

- A **resolution** of A at 0 is a proper morphism $\pi : Y \longrightarrow A$ so that
  1. Y is smooth.
  2. $\pi|_{\pi^{-1}(U-\{0\})}$ is a biholomorphism onto its image for some small neighborhood U of 0,
  3. $\pi^{-1}(0)$ is a smooth normal crossing divisor $(D_i)_{i\in S}$.

  The divisors $(D_i)_{i\in S}$ are called **exceptional divisors**.



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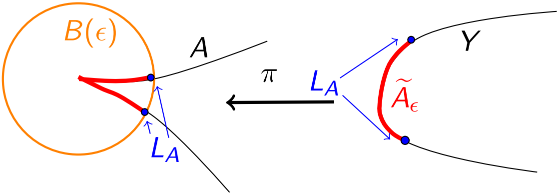
- The **discrepancy** a_i of an exceptional divisor D_i of a numerically $\mathbb{Q}$-Gorenstein singularity is defined as follows: Let $\widetilde{A}_\epsilon \equiv \pi^{-1}(A \cap B(\epsilon))$ where $B(\epsilon)$ is the closed ϵ ball. Then $\partial \widetilde{A}_\epsilon = L_A$. Also one can show that $c_1(\widetilde{A}_\epsilon; \mathbb{Q}) \in H^2(\widetilde{A}_\epsilon; \mathbb{Q})$ lifts to a unique class in $H^2(\widetilde{A}_\epsilon, L_A; \mathbb{Q})$. The Lefschetz dual of this class is a unique sum $\sum_i a_i[D_i] \in H_{n-2}(\widetilde{A}_\epsilon; \mathbb{Q})$. We define the discrepancy of D_i to be a_i.



- Let $\pi : Y \longrightarrow A$ be a resolution. Suppose that we have an ample line bundle and a meromorphic section which is non-zero away from $\pi^{-1}(0)$ and has a non-trivial pole of order w_i along D_i for each $i \in S$. Then the **wrapping number** w_i of D_i is the order of this pole along D_i.

- Let ND_I be a small tubular neighborhood of D_I whose boundary is transverse to the strata of $\cup_i D_i$ and so that $ND_I \cap D_{I'}$ is a tubular neighborhood of $ND_{I \cup I'}$ for all $I, I' \subset S$. Define $\check{N}D_I \equiv ND_I - \cup_{i \in S-I} D_i$.

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- This is a fiber bundle over $D_I - \cup_{i \in S-I} D_i$ with fiber $(\mathbb{D} - 0)^{|I|}$.
- Hence for each tuple $(b_i)_{i \in I}$ of integers, there is a $U(1)$ action on $\check{N}D_I$ preserving the fibers so that $\beta \in U(1)$ sends a point $(x_i)_{i \in I} \in (\mathbb{D} - 0)^{|I|}$ to $(\beta^{b_i} x_i)_{i \in I}$. Let $\overline{N}D_I^{/(b_i)}$ be the corresponding quotient.

Conjecture.
Let $\pi : Y \longrightarrow A$ be a resolution of an isolated numerically $\mathbb{Q}$-Gorenstein singularity A with exceptional divisors $(D_i)_{i\in S}$.
Define

$$A_{p,q} \equiv \bigoplus_{\left\{(k_i)\in\mathbb{N}^S \ : \ \sum_i k_i w_i = p\right\}} H_{p+q-2\sum_i k_i a_i}(\overline{N}D^{/(k_i)}_{I_{(k_i)}};\mathbb{Q})$$

where $I_{(k_i)} \equiv \{i \in S \ : \ k_i \neq 0\}$.
Then there is a spectral sequence converging to $CH_*(L_A, \xi_A)$ with E^1 page equal to the free supercommutative algebra generated by the bigraded vector space $A_{*,*}$. I.e.

$$E^1_{*,*} = \bigoplus_{n\geq 0} \mathrm{Sym}^n_{\mathbb{Q}}\left(A_{*,*}\right).$$

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- Work of de Fernex and Yu-Chao proves this conjecture when the tangent cone of A at 0 has a reduced component.

- **Theorem (assuming spectral sequence conjecture.)** If A is log canonical and numerically Q-Gorenstein then the smallest degree for which $CH_*(L_A, \xi_A)/\mathbb{Q}\langle \mathrm{id} \rangle$ is non-zero is $2\mathrm{md}_0(A)$. Here $\mathbb{Q}\langle \mathrm{id} \rangle$ is the subvector space spanned by the identity element.

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- **Proof idea:** Find the largest p satisfying $E^1_{p,q} \neq 0$ where $q = 2\mathrm{md}_0(A) - p$. This cannot kill or be killed by any differential $d^r_{p,q}$, $r > 0$.

- **Question:** Can $CH_*(L_A, \xi_A)$ detect whether A is log canonical or not? (I.e. whether $\mathrm{md}_0(A) = -\infty$ or not)?

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- This would reprove the theorem (M - 2014) that (L_A, ξ_A) detects smoothness of A at 0 assuming Shokurov's conjecture.

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- We say that (C, α) is **Morse-Bott** if every Reeb orbit sits inside a Morse-Bott submanifold.

- Now suppose that we have a Liouville domain (M, θ). Recall that the chain complex for $SH_*(M, \theta)$ consists of critical points of some Morse function in the interior of M plus two copies of each Reeb orbit *after* perturbing the Liouville form generically so that the contact form is non-degenerate.

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- This chain complex has a natural increasing filtration given by the length of these Reeb orbits and where the critical points are at the bottom of this filtration. We will call this the **action filtration.**

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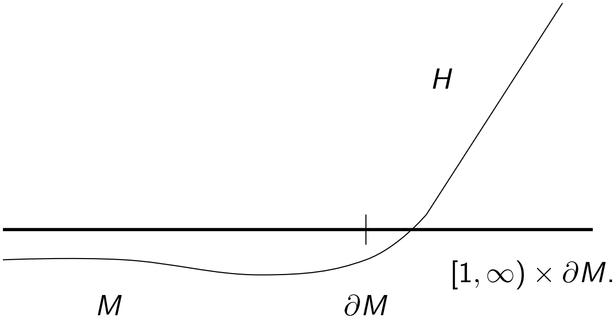
- Define $CF_*^{[a,b]}(H) := CF_*^{\leq b}(H) / CF_*^{\leq a}(H)$ and let $HF_*^{[a,b]}(H)$ be the homology of this chain complex.

- Define $SH_*^{[a,b]}(M,\theta) := \varinjlim_H HF_*^{[a,b]}(H)$ where our direct limit is taken over admissible Hamiltonians H satisfying $H|_M < 0$.



- **Lemma**: Let $a_i \in \mathbb{R}$, $i \in \mathbb{N}$ be an increasing sequence tending to infinity where a_i is not the length or a Reeb orbit or 0. There is a spectral sequence converging to $SH_*(M, \theta)$ with E^1 page

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$$E^1_{p,q} = SH^{[a_p,a_{p+1}]}_{p+q}(M,\theta).$$
- **Lemma**: Suppose that the set of Reeb orbits of length in $[a_p, a_{p+1}]$ is a finite union of connected Morse-Bott families $(B^j_p)_{j\in I_p}$ of Reeb orbits all of the same length. Then $SH^{[a_p,a_{p+1}]}_{p+q}(M,\theta) = \oplus_{j\in I_p} H^{p+q-CZ(B_p)}(B_p, \mathcal{L}_{B^j_p})$ where $\mathcal{L}_{B^j_p}$ is a certain local coefficient system.

- By the two lemmas above:

  Proposition: Suppose that (M, θ) has a Morse-Bott boundary and let $(B_k^j)_{k \in \mathbb{N}, j \in I_k}$ be the set of all of the Morse-Bott submanifolds so that

  1. they are connected,
  2. I_k is a finite set for all $k \in \mathbb{N}$,
  3. the length of B_k^j is the length of $B_k^{j'}$ for all $j, j' \in I_k$ and these lengths tend to infinity as $k \to \infty$ and
  4. the length of B_k^j is less than the length of $B_{k+1}^{j'}$ for all $k \in \mathbb{N}$, $j \in I_k$ and $j' \in I_{k+1}$.

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- **Definition**. An **isolated family of Reebs orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.

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- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying

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- **Definition** : A **minimally degenerate contact pair** a contact pair (C, α) so that every periodic orbit is contained inside a minimally degenerate subset.

- We have a similar spectral sequence in this case (this isn't proven yet, really).

Proposition: Suppose that the boundary of (M, θ) is a minimally degenerate contact pair and let $(B_k^j)_{k \in \mathbb{N}, j \in I_k}$ be the set of all of the minimally degenerate subsets so that

1. they are connected,
2. I_k is a finite set for all $k \in \mathbb{N}$,
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- In order to find a such a boundary, we need to construct a symplectically nice neighborhood of the divisor in question (resolution divisor or compactifying divisor).
- We need a purely symplectic notion of divisor. See 1011.2542 and work of M-Tehrani-Zinger.

- Let (X, ω) be a symplectic manifold. Let $(D_i)_{i \in S}$ be transversally intersecting codimension 2 symplectic submanifolds so that $D_I \equiv \cap_{i \in I} D_i$ is symplectic form all $I \subset S$.
- **Definition**: The **symplectic orientation of** D_I is the orientation on D_I induced by the symplectic structure.

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- We say that $(D_i)_{i \in S}$ is a **symplectic SNC divisor** if the symplectic orientation of D_I is equal to the intersection orientation of D_I for all $I \subset S$.

- **Example**:
  Let M be a Kähler manifold with Kähler form ω. Let $(D_i)_{i\in S}$ be smooth transversally intersecting complex hypersurfaces. Then $(D_i)_{i\in S}$ is a symplectic SNC divisor.

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- **Non-example** Let $M = T^*\mathbb{R}^2$ with the standard symplectic form. Let D_1 be the graph of the 1-form xdy and let D_2 be the graph of ydx. Then D_1, D_2 are transversely intersecting but they intersect negatively and hence cannot be a symplectic SNC divisor.

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- **Non-example 2**. There is a 3-dimensional example of three codimension two linear hypersurfaces D_1, D_2, D_3 in $\mathbb{R}^6$ in which the intersection orientation is equal to the symplectic orientation for $I = \{1,2\},\ \{2,3\},\ \{1,2,3\}$ but not for $I = \{1,3\}$.

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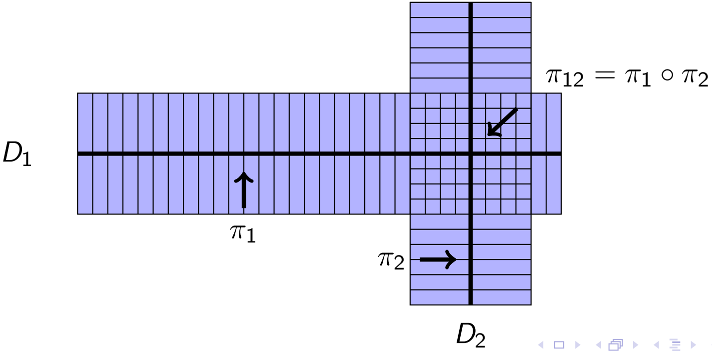
- **Definition:** Let $S \subset W$ be a submanifold of a manifold W. A **tubular fibration** is a smooth fibration $P : U_S \longrightarrow S$ where $U_S \subset W$ is a neighborhood of S in W so that the differential of P along S is the identity map.

- A **regularization** of a symplectic SNC divisor $(D_i)_{i \in S}$ inside (X, ω) consists of tubular fibrations $(\pi_I)_{I \subset S}$ of $(D_I)_{I \subset S}$ with symplectic fibers so that
  1. $\pi_{I_1 \cup I_2} = \pi_{I_1} \circ \pi_{I_2}$ on their common domain of definition for all $I_1, I_2 \subset S$ and
  2. the fibers of π_I are symplectomorphic to a product $\prod_{i \in I} \mathbb{D}(\epsilon)$ of ϵ disks and the associated symplectic connection has parallel transport maps rotating these disks giving us a $U(1)^{|I|}$ structure group.
  3. There should also be a particular almost complex structure but we won't need this.



- **Theorem** M (2011), M-Tehrani-Zinger (2014): Every symplectic SNC divisor is isotopic through symplectic SNC divisors to one which admits a regularization.

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- The proof first involves proving the Theorem in the linear case first and then using a Moser argument to extend this linear argument to the general non-linear case.

Proof idea in the Linear Case

- Let $D_i \subset \mathbb{C}^n$ be equal to $\mathbb{C}^{i-1} \times 0 \times \mathbb{C}^{n-i}$ and let ω be a linear symplectic form on $\mathbb{C}^n$ so that $(D_i)_{i=1}^n$ is a symplectic SNC divisor. Let $\check{D}_i \subset \mathbb{C}^n$ be the complementary subspace $0 \times \mathbb{C} \times 0$ where $\mathbb{C}$ is the ith $\mathbb{C}$ factor.

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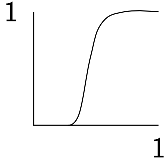
- Let $p_i : \mathbb{C}^n \longrightarrow D_i$, $\check{p}_i : \mathbb{C}^n \longrightarrow \check{D}_i$ be the natural projection maps. Let $\rho : [0,1] \longrightarrow \mathbb{R}$ be equal to:



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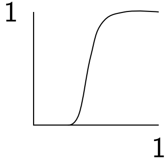

- Then $\omega_t := (1 - \rho(t))\omega + C\rho(t)\check{p}_i^*(\omega|_{\check{D}_i}) + \rho(t)p_i^*(\omega|_{D_i})$ is a smooth family of symplectic forms making $(D_i)_{i=1}^n$ into a symplectic SNC divisor for $C \gg 0$.

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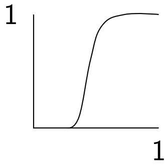

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- Repeat this process for all i until $\omega = \sum_i C_i \check{p}_i^*(\omega|_{\check{D}_i})$ for large $C_i > 0$ (this has a regularization).

- **Lemma:** If two symplectic SNC divisors are isotopic to each other through symplectic SNC divisors then their complements are naturally symplectomorphic (note that there may not be a symplectomorphism sending one divisor to the other though).

- We now need to know which divisors have a natural (concave or convex) contact neighborhood.

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- In algebraic geometry, if we have an effective ample divisor representing a Kähler form then it has a natural concave contact neighborhood. Conversely if we have an anti-effective ample divisor then it has a convex contact neighborhood.

- Let L be an ample line bundle on some smooth quasi-projective variety Y. Choose a Hermitian metric $|\cdot|$ on L so that $-i$ times its curvature form is a Kähler form ω.

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  is a Liouville submanifold for all large enough C.
- $Y - M$ is a neighborhood of $\cup_i D_i$ with concave boundary.

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- There is a similar construction when s has poles along D_i and no zeros, and then we get a convex neighborhood of $\cup_i D_i$.
- We need a symplectic version of this (anti-)ampleness condition so that we can mimic the above construction of a neighborhood with concave (or convex) boundary. We will do this by defining a purely symplectic notion of *wrapping number*.

- **Definition**: An **exact symplectic SNC divisor** $((D_i)_{i \in S}, \theta)$ in (X, ω) is a symplectic SNC divisor $(D_i)_{i \in S}$ and a 1-form $\theta \in \Omega^1(X - \cup_i D_i)$ satisfying $d\theta = \omega$.

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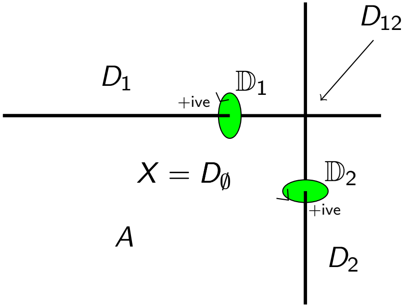
- **Definition**: Let $((D_i)_{i \in S}, \theta)$ be an exact symplectic SNC divisor. Let $\mathbb{D}_i \subset X$ be a small symplectic disk intersecting D_i once at 0 positively and not intersecting D_j for all $j \neq i$ with polar coordinates (r, ϑ). The **wrapping number of** D_i is the unique $w_i \in \mathbb{R}$ so that $\frac{w_i}{2\pi} d\vartheta \in \Omega^1(\mathbb{D}_i - 0)$ is cohomologous to $(\theta - \frac{1}{2} r^2)|_{\mathbb{D}_i - 0}$.



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- Then w_i are the unique numbers so that $-\sum_i w_i[D_i] \in H_{2n-2}(U)$ is the Lefschetz dual of

$$\Omega \in \Omega^2_c(U), \quad \Omega = \begin{cases} d(\rho\theta) & \text{outside } \cup_i D_i \\ \omega & \text{near } \cup_i D_i \end{cases}.$$

- Let $r_i : X \longrightarrow \mathbb{R}$ be the the distance from D_i with respect to some metric. A function $f : X - \cup_i D_i \longrightarrow \mathbb{R}$ is **compatible with** $(D_i)_{i \in S}$ if
$$f = \sigma + \sum_i c_i \log(r_i^2)$$
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for some constants $(c_i)_{i\in S}$ and a smooth function $\sigma : X \longrightarrow \mathbb{R}$.
- This is our 'symplectic version' of $\ln(|s|)$ mentioned earlier (s was our holomorphic section and $|\cdot|$ our Hermitian metric.).

- **Proposition M**. Let $(D_i)_{i \in S}$, w_i, θ be as above. Suppose that all of the wrapping numbers w_i are negative. Then there is a smooth function $g : X - \cup_i D_i \longrightarrow \mathbb{R}$ so that $df(X_{\theta+dg}) > 0$ near $\cup_i D_i$. In particular $(f^{-1}(-C), \theta + dg)$ is a concave contact boundary of a small neighborhood of $\cup_i D_i$ for $C \gg 1$.
- Also $(f^{-1}(-C, \infty), \theta)$ is a Liouville submanifold for all $C \gg 1$.

- **Proposition** The contactomorphism type of $(f^{-1}(-C), \theta + dg)$ does not depend on the choice of f or g although it does depend on the choice of 1-form θ.

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- **Definition:** We will call this contact manifold the **contact boundary of** $((D_i)_{i\in S}, \theta)$**.**
- Similarly if all the wrapping numbers are positive we can choose g so that $df(X_{\theta+dg}) < 0$ near D_i. Hence $f^{-1}(C)$ is convex contact boundary of a neighborhood of $\cup_i D_i$ also called the **contact boundary of** $((D_i)_{i\in S}, \theta)$**.**

- **Proposition:** If there is a smooth family of exact symplectic SNC divisors, $((D_i^t)_{i \in S}, \theta_t)$ $t \in [0,1]$ so that the wrapping numbers of D_i^t are all positive or all negative, then the contact boundaries of $(D_i^t)_{i \in S}$ are all naturally contactomorphic.
- Hence the contact boundary of $\cup_i D_i$ is an invariant up to isotopy.

- Now suppose that our symplectic SNC divisor $(D_i)_{i\in S}$ admits a regularization $(\pi_I)_{I\subset S}$ (recall, these are tubular fibrations of D_I) and that the wrapping numbers w_i of $((D_i)_{i\in S},\theta)$ are negative.

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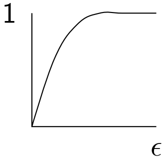
- Since the tubular fibrations $\pi_{\{i\}}$ have a natural $U(1)$ structure group, we have radial coordinates $r_i : \mathrm{Dom}(\pi_{\{i\}}) \longrightarrow \mathbb{R}$.
- Let $f = \sum_i \ln(\rho(r_i))$ where ρ is:



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  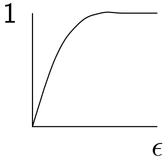

  Then f is compatible with $(D_i)_{i \in S}$.
- We can choose g so that $\theta + dg$ restricted to each fiber $\prod_{i \in I}(\mathbb{D} - 0)$ of D_I is $\sum_i (r_i^2 + \frac{w_i}{2\pi}) d\vartheta_i$ where (r_i, ϑ_i) are polar coordinates on the ith $\mathbb{D}$ factor.

- We have $df(X_{\theta+dg}) > 0$ near $\cup_i D_i$ and so $(f^{-1}(-C), \alpha_C := \theta + dg|_{f^{-1}(C)})$ is a contact boundary of $\cup_i D_i$ called a **regular contact boundary of** $\cup_i D_i$.

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- The contact form α_C is minimally degenerate.
- For each $I \subset S$ and each $(k_i)_{i\in I} \in \mathbb{N}^I_{>0}$ there is a minimally degenerate subset $B_{(k_i)_{i\in I}}$ of length $-\sum_i l_i(2\pi w_i + \epsilon)$ and all Reeb orbits are contained in one such subset.

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- $B_{(k_i)_{i \in I}}$ is diffeomorphic to a $U(1)^{|I|}$ fibration over $D_I - \cup_{i \in S-i} \mathrm{Dom}(\pi_i)$ and is homotopic to $\check{N} D_I$.

what about Conley-Zehnder index?

- Now suppose that $c_1(X - \cup_i D_i) = 0$. Then we can choose a (not necessarily unique) representative $\sum_i a_i[D_i] \in H_{2n-2}(X)$ Poincaré dual to $c_1(X)$.

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- a_i is called the **discrepancy of** D_i.
- It coincides with the definition of discrepancy earlier when X was projective.
- The Conley-Zehnder index of $B_{(k_i)_{i \in S}}$ is $-2\sum_i k_i(a_i + 1) - n - \frac{|I|}{2}$.

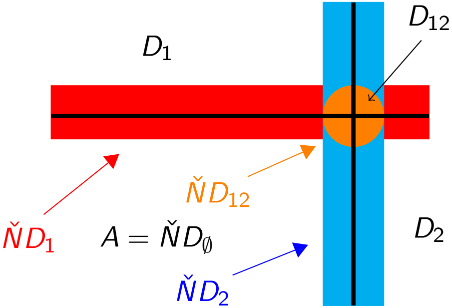
- **Main idea:**
  1. Deform the symplectic SNC divisor so that it has a regularization.
  2. This does not change the contact boundary of such a divisor up to contactomorphism.
  3. Then construct the regular contact boundary using this regularization as above.
- We will now use this technique for affine varieties.

Recall that we wish to prove the following:

There is a spectral sequence converging to $SH_*(A)$ with E^1 page

$$E^1_{p,q} = \bigoplus_{\{(k_i)\in\mathbb{N}^S \ : \ \sum_i k_i w_i = -p\}} H^{n-p-q-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}})$$

where $\mathbb{N}^S$ is the set of tuples of non-negative integers indexed by S and $I_{(k_i)} = \{i \in S \ : \ k_i \neq 0\}$.



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3. Therefore in order to compute $SH_*(A)$ we need to compute $SH_*(M, \theta)$ where $(M, \theta) \equiv (\phi^{-1}(-\infty, C], -d^c \phi)$ for some $C \gg 1$.

4. **Lemma:** The wrapping numbers of the exact symplectic SNC divisor $((D_i)_{i \in S}, -d^c\phi)$ are equal to the wrapping numbers defined in the first lecture using s (I.e. minus the order of $s^{-1}(0)$ along D_i). Also $(\partial M, \theta)$ is contactomorphic to the contact boundary of the symplectic SNC divisor $(D_i)_{i \in S}$.

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6. Since this regular contact boundary is contactomorphic to $\phi^{-1}(C)$ we can deform our Liouville domain (M, θ) so that it is minimally degenerate as described earlier.
7. The spectral sequence is then the associated Morse-Bott spectral sequence.