Minimal Log Discrepancy of Isolated Singularities and Reeb Orbits

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- $A \subset \mathbb{C}^N$ affine variety of dimension $n$ with an isolated singularity at 0.
- $L_A = A \cap S_\epsilon$ where $S_\epsilon = \{z \in \mathbb{C}^N | |z| = \epsilon\}$.
- Here $L_A$ is a real $2n - 1$ dimensional $C^\infty$ manifold called the **link** of $A$ at 0 for $\epsilon$ small enough.
- A $C^\infty$ manifold diffeomorphic to $L_A$ is said to be **Milnor fillable by** $A$. 
Examples:

\[ L_{\mathbb{C}^n} = S^{2n-1} \]

\[ L \{ x^2 + y^2 + z^2 = 0 \} = \mathbb{RP}^3. \]

A is called **differentiably smooth** if \( L_A \) is diffeomorphic to \( L_{\mathbb{C}^n} = S^{2n-1} \).

**Question**: Which singularities are differentiably smooth?
Theorem (Mumford)
Let \( A \) be a normal surface singularity with link diffeomorphic to \( S^3 \). Then \( A \) is smooth at 0.

This is false in higher dimensions (Brieskorn):

\[
L\{x^2 + y^2 + z^2 + w^3 = 0\} = S^5
\]

Need more structure on the link.
Introduction to Contact Geometry

Let $C$ be a real $2n - 1$ dimensional $C^\infty$ manifold and let $\xi \subset TC$ be a hyperplane distribution. For simplicity, assume $\xi = \ker(\alpha)$ for some 1-form $\alpha$ on $M$.

(Frobenious Integrability Theorem): $\xi$ is the tangent space to a foliation iff $d\alpha|_\xi = 0$. 
Definition: \( \xi \) is a **Contact structure** if \( d\alpha|_\xi \) is a non-degenerate 2-form at every point.

A contact structure is the opposite of a Foliation!

Equivalently: \( \xi \) is a **Contact structure** iff \( \alpha \wedge (d\alpha)^{n-1} \neq 0 \) at every point.
We will call \((C, \xi)\) a contact manifold.

Any 1-form \(\alpha\) satisfying \(\text{Ker}(\alpha) = \xi\) is a contact form associated to \(\xi\).

Two contact manifolds are contactomorphic if there is a diffeomorphism preserving the respective hyperplane distributions.

(Gray’s stability theorem). If I have a smooth family of contact structures on a compact manifold, then they are all contactomorphic.
Example:

\[(\mathbb{R}^{2n-1}, \ker(dz - \sum_{j=1}^{n-1} y_j dx_j))\]

where \((x_1, y_1, \cdots, x_{n-1}, y_{n-1}, z)\) are the natural coordinates.
The Reeb vector field of $\alpha$ is the unique vector field $R$ on $C$ satisfying $i_R d\alpha = 0$, $i_R \alpha = 1$.

Intuition: Think of $C$ as the level set of a Hamiltonian, and $R$ is the Hamiltonian flow inside that level set. I.e. some dynamical system in some fixed energy level.

$R$ is uniquely determined by $\alpha$, but $R$ is not an invariant of $\xi$. If I replace $\alpha$ with $f\alpha$ for some $f : C \to \mathbb{R} \setminus \{0\}$, the associated Reeb vector field changes a lot.

A periodic Reeb orbit of period $L$ is a map $\mathbb{R}/L\mathbb{Z} \to C$ tangent to $R$. 
Example: Reeb vector field of

$$dz - \sum_{j=1}^{n-1} y_j \, dx_j$$

is

$$\frac{\partial}{\partial z}.$$
Let $A \subset \mathbb{C}^N$ have an isolated singularity at 0 with link $L_A = A \cap S_\epsilon$ as before. Let $i : T(A \setminus \{0\}) \to T(A \setminus \{0\})$ be complex multiplication.

Define: $\xi_A := TL_A \cap iTL_A$.

Lemma (Varchenko): For all $\epsilon > 0$ small enough, $(L_A, \xi_A)$ is a contact manifold and is an invariant of the germ of $A$ at 0 up to contactomorphism.
Conjecture (Seidel) If $A$ is normal and $(L_A, \xi_A)$ is contactomorphic to $(L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then $A$ is smooth at 0.

Seidel observed that this is true for hypersurface singularities using work by Eliashberg, Gromov, McDuff.
Let \((C, \xi)\) be a general contact manifold with \(\xi = \ker(\alpha)\).

Choose a complex structure \(J\) on the bundle \(\xi\) compatible with the symplectic form \(d\alpha|_{\xi}\). We define \(c_1(\xi) := c_1(\xi, J)\).

We will assume \(H^1(C; \mathbb{Q}) = 0, c_1(\xi) = 0\).
These topological conditions tell us that for each periodic Reeb orbit $\gamma$, we get an index: $\text{CZ}(\gamma) \in \mathbb{Q}$ called the **Conley-Zehnder index**.

Intuition: $\text{CZ}(\gamma)$ describes how many times the Reeb flow ‘wraps’ around $\gamma$. 

![Diagram of a Reeb flowline with a nearby Reeb flowline.]
Let $\phi_t : C \to C$ be the Flow of the Reeb vector field $R$ of $\alpha$.

This flow preserves $\xi$ (i.e. $D\phi_t(\xi) = \xi$).

The **linearized return map** of $\gamma : \mathbb{R}/L\mathbb{Z} \to C$ is the natural map $D\phi_L|_{\xi_\gamma(0)} : \xi_\gamma(0) \to \xi_\gamma(L) = \xi_\gamma(0)$.
For simplicity, we will define CZ(γ) under the following conditions:

1. $D\phi_t|_\xi$ is $J$ holomorphic for some compatible almost complex structure $J$ on $\xi$.
2. $D\phi_L|_{\xi\gamma(0)} = \text{id}$.
3. $c_1(\xi) = 0$.

Choose a trivialization of the complex vector bundle $\gamma^*\xi$ with complex structure $J$. 
Using this trivialization and the above properties, the map $t \rightarrow (\phi_t|_{(\xi)_{\gamma(0)}})$ is viewed as a map from $Q : \mathbb{R}/L\mathbb{Z} \rightarrow U(n - 1)$. We define $\text{CZ}(\gamma)$ to be twice the degree of the map $\det(Q) : \mathbb{R}/L\mathbb{Z} \rightarrow U(1)$. 

\[ \gamma(0) = \gamma(L). \]
Define
\[ \text{ISFT}(\gamma) := \text{CZ}(\gamma) - \frac{1}{2} \dim \ker(D\phi_L|_{\xi|_{\gamma(0)}} - \text{id}) + (n - 3). \]

For any \( \alpha \) such that \( \ker(\alpha) = \xi \), define the \textbf{minimal index of} \( \alpha \) as \( \text{mi}(\alpha) := \inf(\text{ISFT}(\gamma)) \).

Define the \textbf{highest minimal index} \( \text{hmi}(C, \xi) := \sup_{\alpha} \text{mi}(\alpha) \) where the supremum is taken over all \( \alpha \) such that \( \ker(\alpha) = \xi \).
Recall: $A$ is an isolated singularity and $L_A$ is its link.

Assume $c_1(TA|_{L_A})$ is torsion. Fact: $c_1(TA|_{L_A}) = c_1(\xi_A)$. Such a singularity is called numerically $\mathbb{Q}$-Gorenstein.

Fix some resolution $\pi: \tilde{A} \to A$ so that $\pi^{-1}(0)$ has smooth normal crossing exceptional divisors $E_1, \cdots, E_l$. 

Define: \( B_\varepsilon := \{ |z| \leq \varepsilon \} \), \( A_\varepsilon := B_\varepsilon \cap A \) and \( \tilde{A}_\varepsilon := \pi^{-1}(A_\varepsilon) \).

Note: \( \partial \tilde{A}_\varepsilon = \partial A_\varepsilon = L_A \).
\[ \exists c_1(\tilde{A}_\varepsilon, L_A; \mathbb{Q}) \rightarrow c_1(\tilde{A}_\varepsilon; \mathbb{Q}) \rightarrow c_1(L_A; \mathbb{Q}) \]

\[ H^1(L_A; \mathbb{Q}) \overset{0}{\longrightarrow} H^2(\tilde{A}_\varepsilon, L_A; \mathbb{Q}) \rightarrow H^2(\tilde{A}_\varepsilon; \mathbb{Q}) \rightarrow H^2(L_A; \mathbb{Q}) \]

\[ H_{2n-2}(\tilde{A}; \mathbb{Q}) \quad \text{freely generated by } [E_j] \]

So \( c_1(\tilde{A}_\varepsilon, L_A; \mathbb{Q}) = \sum_i a_i[E_i] \) for unique \( a_i \in \mathbb{Q} \).
Define $a_j$ to be the **discrepancy of** $E_j$.

Define **Minimal discrepancy** to be

\[
\text{md}(A) = \begin{cases} 
\min(a_j) & \text{if } \min(a_j) \geq -1 \\
0 & \text{otherwise}
\end{cases}
\]

Minimal discrepancy measures how singular $A$ is at 0.

**Examples:**

1. $\text{md}({\mathbb C}^n) = n - 1$.
2. $\text{md}(\{x^2 + y^2 + z^2 + w^3 = 0\}) = 1$.
3. $\text{md}(\{x^7 + y^{11} + z^{13} + w^{17} = 0\}) = -\infty$. 
Theorem: If \( A \) is numerically \( \mathbb{Q} \)-Gorenstein (i.e. \( c_1(\xi_A) \) is torsion) and \( H^1(L_A; \mathbb{Q}) = 0 \) then:

\[
\operatorname{hmi}(L_A, \xi_A) = \begin{cases} 
2 \operatorname{md}(A) & \text{if } \operatorname{md}(A) \geq 0 \\
< 0 & \text{otherwise.}
\end{cases}
\]
Shokurov’s Conjecture (Combined with work from: Boucksom, de Fernex, Favre, Urbinati): If $A$ is numerically $\mathbb{Q}$-Gorenstein with $\text{md}(A) = n - 1$ then $A$ is smooth at 0.

Corollary. Suppose that Shokurov’s Conjecture is true. If $A$ is normal and $(L_A, \xi_A) \overset{\text{cont.}}{\cong} (L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then $A$ is smooth at 0.

(Markushevich, Reid, Kawamata), Shokurov’s conjecture is true in dimension $\leq 3$.

Corollary. For all $n \leq 3$, if $A$ is normal and $(L_A, \xi_A) \overset{\text{cont.}}{\cong} (L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then $A$ is smooth at 0.
Proof

➢ **Easier part:** Find some contact form $\alpha_A$ associated to $\xi_A$ so that:

$$\text{mi}(\alpha_A) = 2\text{md}(A)$$

This gives us a lower bound form $h\text{mi}(\xi)$.

➢ **Hard part:** For every compatible contact form, find a Reeb orbit $\gamma$ so that:

$$\text{ISFT}(\gamma) \leq \begin{cases} 
2\text{md}(A) & \text{if } \text{md}(A) \geq 0 \\
< 0 & \text{otherwise.}
\end{cases}$$

This gives us an upper bound form $h\text{mi}(\xi)$.
Proof in the case of cone singularities.

- Assume $A$ is the cone over a smooth projective $X \subset \mathbb{CP}^{N-1}$. E.g. $X = \mathbb{CP}^{n-1}$, $A = \mathbb{C}^n$.
- $\tilde{A} = \text{Bl}_0 A$ and let $\pi : \tilde{A} \to A$ be the blowdown map.
- We also have the $\mathcal{O}(-1)$ bundle $P : \tilde{A} \to X$. We identify $X$ with the zero section of $P$. 
Easier Part:

- \( A \subset \mathbb{C}^N \). Define \( \alpha_A := \sum_j x_j dy_j - y_j dx_j \big|_{L_A} \) where \( z_j = x_j + iy_j \).

- \( P : \tilde{A} \to X \) is a Hermitian line bundle \( \mathcal{O}_X(-1) \) with Hermitain form coming from the standard symplectic form on \( \mathbb{C}^N \).

- The Reeb flow uniformly rotates the fibers of \( P \). I.e. \( \phi_t(z) = e^{it}(z) \) (up to a time reparameterization).
So through each point $p$ in $L_A$ there are Reeb orbits of period $2k\pi$ wrapping $k$ times around $X$.

The lSFT index of such an orbit is $2k(a_1 + 1) - 2$ where $a_1$ is the discrepancy of $X \subset \tilde{A}$.

Hence $\text{mi}(\alpha_A) = 2a_1 = 2\text{md}(A)$. 

We now start with any contact form $\alpha$ associated to $\xi_A$. We wish to find an orbit $\gamma$ with the right bound on its index.

Compactify $\tilde{A}$ to $\bar{A} = \mathbb{P}(\mathbb{C} \oplus O_X(-1))$ and let $\bar{\pi} : \bar{A} \to X$ be the natural map.

Let $[F] \in H_2(\bar{A})$ be the class of the fiber of $\bar{\pi}$, then $GW_{[F],0}(\text{[pt]}) \neq 0$. Hence for any compatible almost complex structure $J$ on $\bar{A}$, there is a $J$-holomorphic curve: $u_J : \mathbb{P}^1 \to \bar{A}$ representing $[F]$. 
We now deform the symplectic form on $\overline{A}$ through symplectic forms to a new symplectic form $\omega$ so that we have an embedding $\iota : C \hookrightarrow \overline{A}$ so that $\iota^* \omega = d\alpha$.

We now choose a family $J_i$'s compatible with $\omega$ which ‘stretch’ along $C$.

The associated $u_{J_i}$'s ‘break’ and their ends converge to Reeb orbits $\gamma_1, \cdots, \gamma_k$.

Simple Example: $X$ is a point, so $\overline{A} = \mathbb{C}P^1$. Our degeneration is $\{x^2 + y^2 = t\} \subset \mathbb{C}P^2$ as $t \to 0$. The complex structure here stretches along the equator $\mathbb{R}P^1$. 
Schematic Picture

Reeb orbits

\( \gamma_1 \)

\( \gamma_2 \)

\( C \)

\( \bar{A} \)

Section at infinity

\( X \)

\( C' \)

\( u_i \)

\( u_\infty \)
The space of such broken maps $\overline{u}_\infty$ converging to $\gamma_1, \cdots, \gamma_k$ has dimension given by a formula involving the discrepancy and Reeb orbits.

This gives us an inequality:

$$2a_1 - \sum_j \text{ISFT}(\gamma_j) \geq 0$$

proving the hard part of the theorem.
Further directions

- What other parts of the resolution can we recover? E.g. Information from the dual graph? other invariants such as Log Canonical Threshold?
- Some of the holomorphic curves involved look like arcs. What is the relationship between these curves and the (short) arc space?
- Secretly our proof is showing that a group called Contact Homology has lowest non-zero degree equal to $\text{md}(A)$ or is $< 0$ depending on the sign of $\text{md}(A)$. What is the relationship between this group and the singularity?