Symplectic geometry of Stein manifolds and affine varieties

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- Suppose we have some physical system with position coordinates $q_1, \cdots, q_n$ and momentum coordinates $p_1, \cdots, p_n$.
- We have some function $H(q_1, \cdots, q_n, p_1, \cdots, p_n)$ called the Hamiltonian which tells us the energy of the system at each state.
- The system given by a path $(q_1(t), \cdots, q_n(t), p_1(t), \cdots, p_n(t))$ satisfies:

$$\frac{\partial H}{\partial q_1} = -\frac{dp_1}{dt}$$
$$\vdots$$
$$\frac{\partial H}{\partial q_n} = -\frac{dp_1}{dt}$$
$$\frac{\partial H}{\partial p_1} = \frac{dq_1}{dt}$$
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$$\frac{\partial H}{\partial p_n} = \frac{dq_n}{dt}$$

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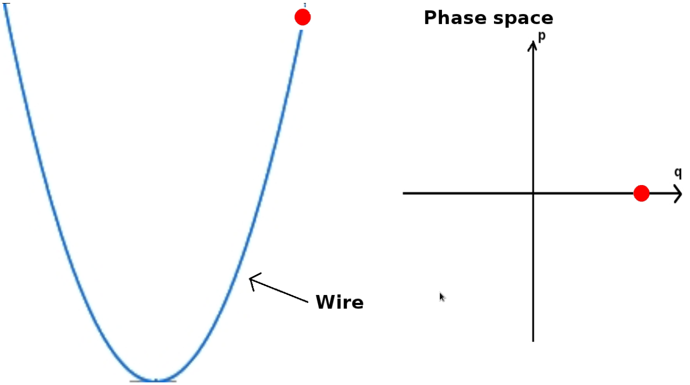
$$\begin{pmatrix} H_{q_1} \\ \vdots \\ H_{q_n} \\ H_{p_1} \\ \vdots \\ H_{p_n} \end{pmatrix} = - \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array}\right) \begin{pmatrix} \frac{dq_1}{dt} \\ \vdots \\ \frac{dq_n}{dt} \\ \frac{dp_1}{dt} \\ \vdots \\ \frac{dp_n}{dt} \end{pmatrix}$$

Example: A single bead on a wire



q(t) = horizontal component of position at time t
p(t) = horizontal component of momentum at time t
Bead

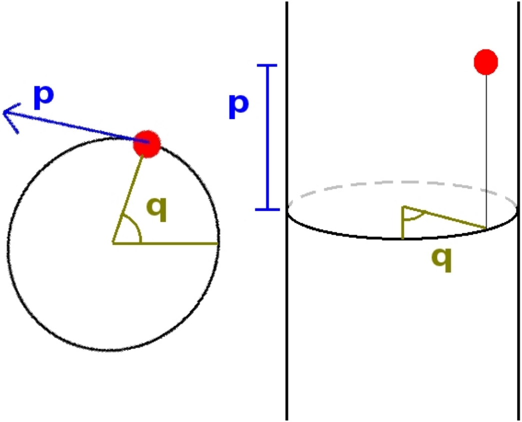
Click to Animate http://math.mit.edu/~mclean/beadonwire/manybeadsonawire. html

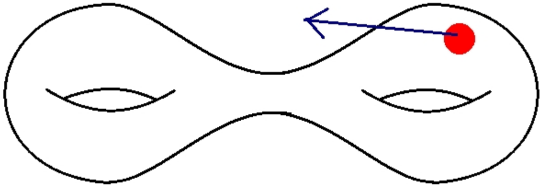
- A *symplectic manifold* is a manifold with a 2-form ω which locally looks like:

$$\sum_{j=1}^{n} dp_j \wedge dq_j = \left(\begin{array}{ccc|ccc} & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \\ \hline -1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & -1 & & & \end{array}\right)$$

- (Darboux) Equivalently it is a manifold with a closed non-degenerate 2-form.

- To construct a cotangent bundle T^*M of some manifold M you think of a bead constrained to that manifold.
- If M is the circle S^1 then our phase space is a cylinder T^*S^1:





Definition

A smooth affine variety is a complex submanifold of $\mathbb{C}^N$ given by the zero locus of some polynomial equations. This has a symplectic form given by restricting the standard one $\sum_j dp_j \wedge dq_j$ on $\mathbb{C}^N = \mathbb{R}^{2N}$.

Examples

1. $\left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \,\middle|\, z_1^2 + z_2^2 + z_3^2 = 1 \right\} \underset{\text{symp}}{\cong} T^*(S^2)$.
2. $\left\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \,\middle|\, \begin{array}{l} z_1^2 + z_2^2 = 1 \\ z_3^2 + z_4^2 = 1 \end{array} \right\} \underset{\text{symp}}{\cong} T^*(T^2)$
3. $\left\{ \frac{(z_1 z_3 + 1)^2 - (z_2 z_3 + 1)^3}{z_3} = 0 \right\}$
   This smooth affine variety is not symplectomorphic to any cotangent bundle. It is in fact contractible.

- Smooth affine varieties and cotangent bundles are both examples of Stein manifolds.
- A *Stein manifold* is a properly embedded complex submanifold of $\mathbb{C}^N$. This has a symplectic form given by restricting the standard one $\sum_{j=1}^n dp_j \wedge dq_j$ on $\mathbb{C}^N = \mathbb{R}^{2N}$.

Some questions about Stein manifolds and affine varieties

- What are good ways of describing Stein manifolds symplectically?
- What is the relationship between algebraic/analytic properties of the affine variety and the symplectic structure?
- Dynamical questions.
  - e.g. how many 1-periodic orbits does a Hamiltonian system on this affine variety have?
- Mirror Symmetry questions.

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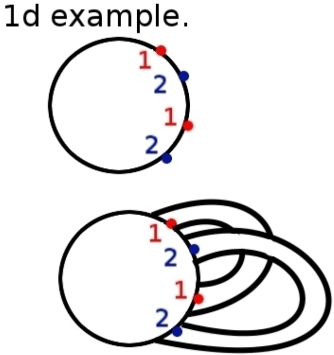
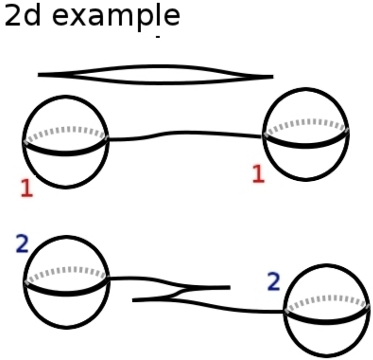
Theorem

(Weinstein). Every Stein manifold has an explicit handle decomposition. Each handle has an explicit symplectic structure on it.





Theorem

(Eliashberg) If there is a diffeomorphism between Stein manifolds A and B (satisfying an additional topological condition) then $\mathbb{C} \times A$ is symplectomorphic to $\mathbb{C} \times B$.

Theorem

(Seidel-Smith) There are at least two non-symplectomorphic smooth affine varieties diffeomorphic to $\mathbb{R}^{2n}$ for each $n \geq 3$.

Theorem

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Theorem

(Seidel-Smith, M) There are infinitely many pairwise non-symplectomorphic smooth affine varieties diffeomorphic to $\mathbb{R}^{2n}$ for each $n \geq 3$.

Theorem

(M). There is no algorithm telling us in general if two Weinstein handle presentations diffeomorphic to $\mathbb{R}^{2N}$ ($N > 7$) are symplectomorphic or not.

- There is an earlier result by Seidel where $\mathbb{R}^{2N}$ is replaced by a more complicated manifold.

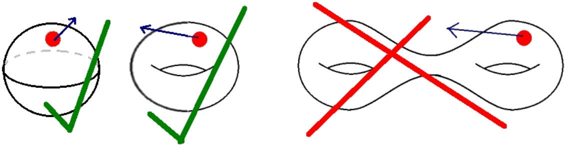
Cotangent bundles and smooth affine varieties

Theorem

(M) Most cotangent bundles are not symplectomorphic to smooth affine varieties.

Here 'most' means that these cotangent bundles T^*Q have complicated topology.

- If $\pi_1(Q)$ grows exponentially.
- If $\pi_1(Q) = 0$ and the sum of the Betti numbers is greater than $2^{\dim Q}$.



The previous theorem is similar to a result by Kulkarni.

- Let Q be a compact manifold described by the zero locus of some polynomial equations with real variables:

$$\left\{ (x_1, \cdots, x_N) \in \mathbb{R}^N \;\middle|\; \begin{matrix} p_1(x_1, \cdots, x_N) = 0 \\ \vdots \\ p_k(x_1, \cdots, x_N) = 0 \end{matrix} \right\}$$

- Now replace the real coordinates x_i with complex ones z_i:

$$Q(\mathbb{C}) := \left\{ (z_1, \cdots, z_N) \in \mathbb{C}^N \;\middle|\; \begin{matrix} p_1(z_1, \cdots, z_N) = 0 \\ \vdots \\ p_k(z_1, \cdots, z_N) = 0 \end{matrix} \right\}$$

- **Theorem**

  (Kulkarni) Suppose that the inclusion $Q \hookrightarrow Q(\mathbb{C})$ is a homotopy equivalence. Then Q has nonnegative Euler characteristic.

The main tool used to prove this is called the *growth rate of symplectic cohomology*. This assigns to any Stein manifold M a number $\Gamma(M) := \{-\infty\} \cup [0, \infty]$.

Theorem

(M) For smooth affine varieties A, $\Gamma(A) < \mathit{dim}_{\mathbb{C}} A$.

Theorem

(Abbondandolo-Schwartz,Salamon-Weber,Viterbo)
$\Gamma(T^*Q) = \infty$ *for sufficiently complicated compact manifolds Q.*

Hamiltonian Floer cohomology

- For any Hamiltonian H, we get: $HF^*(H)$.
- Chain complex freely generated by fixed points of the time 1 Hamiltonian flow of H.
- The differential is given by a matrix (with respect to the basis of fixed points). Each entry is a count of solutions to: $\partial_s u + J\partial_t u = JX_H$

Defining growth rate

- For a Stein manifold $M \subset \mathbb{C}^N$, we choose a Hamiltonian $H = r^2$ where r is the distance from the origin in $\mathbb{C}^N$.
- The *growth rate* $\Gamma(M)$ (roughly) is the rate at which the rank of $HF^*(\lambda H)$ grows as λ tends to infinity.
- One can use other Hamiltonians H (satisfying certain properties) and still get the same invariant $\Gamma(M)$.

Bounding growth rate of affine varieties

For an affine variety A, we can find a nice Hamiltonian H so that the number of fixed points of λH grows like a polynomial of degree $\dim_{\mathbb{C}} A$.

This bounds $HF^*(\lambda H)$ and hence $\Gamma(A) \leq \dim_{\mathbb{C}}(A)$.

- X = compactification of A.
- $D = X \setminus A$ = smooth normal crossing.
- **log Kodaira dimension** $\overline{\kappa}(A)$ = rate at which $\rho_m := \text{rank}(H^0(m(K_X + D)))$ grows.

Theorem

(M)

- *Log Kodaira dimension is a symplectic invariant for acyclic smooth affine varieties of dimension* 2*.*
- *Partial results in dimension* 3*.*
- *(Work in progress): If A, B are symplectomorphic affine varieties and $\overline{\kappa}(A) < 1$ + technical conditions then $\overline{\kappa}(B) < 1$.*

Theorem

(M) Suppose A is **uniruled** *(i.e. there is a rational curve passing through every point) and B is symplectomorphic to A then B is also uniruled.*

We use Gromov-Witten invariants to prove these results:

- Embed A as an open subset of a projective variety X.
- Count holomorphic maps $\mathbb{P}^1 \to X$.
- Relate log Kodaira dimension to these counts of curves.

further directions

- What other algebraic structures are remembered by the symplectic structure? (rational connectedness?). What about log general type affine varieties?
- Relationship between dynamical properties and algebraic properties.
- What can symplectic/contact topology say about singularities?