Sectional Curvature Comparison II

MAT 569

May 5, 2020

Critical Point Theory

- We will first talk about critical point theory.
- This will be used throughout the chapter.
- We wish to understand gradients of distance functions which might not be smooth.

We begin with the following observation.

Lemma: Let $f : M \longrightarrow \mathbb{R}$ be a smooth proper function. Let $M_{\leq r} := f^{-1}([-\infty, r])$ for each $r \in \mathbb{R}$. Suppose that f has no critical values in the interval $[a, b]$. Then $M_{\leq a}$ and $M_{\leq b}$ are diffeomorphic. Also, there is a deformation retraction from $M_{\leq b}$ to $M_{\leq a}$, and hence the inclusion map

$$M_{\leq a} \hookrightarrow M_{\leq b}$$

is a homotopy equivalence.

Proof: The key idea of the proof is to "flow" $M_{\leq b}$ down to $M_{\leq a}$ using ∇f. Choose a smooth compactly supported bump function $\psi : M \longrightarrow \mathbb{R}$ equal to 1 along $f^{-1}([a, b])$. Define

$$X = \psi \cdot \frac{\nabla f}{|\nabla f|^2}.$$

Proof continued. Since X has compact support, its flow F^t is well defined. Our deformation retraction is then defined to be:

$$r_t : M_{\leq b} \longrightarrow M_{\leq a}$$

$$r_t(p) := \begin{cases} p & \text{if } p \in M_{\leq a} \\ F^{t(a-f(p))}(p) & \text{if } p \in M_{\leq b} - M_{\leq a}. \end{cases}$$

$\square$

Note that we used the fact that the function is proper in an essential way. If f wasn't proper then this lemma is false.

We wish to generalize the lemma above to functions which are not C^1. We will work exclusively with distance functions.

Lemma (Exercise). Suppose (M, g) is complete and let $K \subset M$ be a compact set. Then the distance function:

$$r : M \longrightarrow [0, \infty)$$

$$r(x) = d(x, K) = \min\{d(x, p) \ : \ p \in K\}$$

is proper.

Problem: The distance function r may not be smooth.

However, we wish to show that it still has a "gradient".

Definition:A *segment* is a continuous path $\sigma : [0, l] \longrightarrow M$ satisfying

$$\text{length}(\sigma) = d(\sigma(0), \sigma(l)).$$

Definition: For $x \in M$, we define $\Gamma(x, K)$ or $\Gamma(x)$ to be the set of unit vectors $v \in T_x M$ so that $-v$ is tangent to a segment starting at x and ending at a point in K and so that the length of this segment is $r(x)$.

Example: If x is smooth at x, then $\Gamma(x, K) = \{\nabla r\}$.

Definition: We say that r is *regular* or *non-critical* at x if $\Gamma(x, K)$ is contained inside an open hemisphere of the unit sphere in $T_x M$. In other words, if there exists a unit vector $v \in T_x M$ so that the angle between v and any vector in $\Gamma(x, K)$ is less than $\pi/2$.

Definition: Let $\alpha \geq 0$. We say that $x \in M$ is an *α-regular point of r* if there exists $v \in T_x M$ so that the angle between v and any vector in $\Gamma(x, K)$ is $< \alpha$.

Note that v may not necessarily be in $\Gamma(x, K)$.

Definition: We define $G_\alpha r(x)$ to be the set of such vectors v. I.e.

$$G_\alpha r(x) = \{v \in T_x M \ : \ \angle(v, w) < \alpha, \ \forall \ w \in \Gamma(x, K)\}.$$

Proposition: Let $K \subset M$ be compact. Then:

1. $\Gamma(x, K)$ is compact for each $x \in M$.
2. The set of α regular points is open in M.
3. $G_\alpha r(x)$ is convex for all $\alpha \leq \pi/2$.
4. Let $U \subset M$ be the set of α-regular points in M. Then there is a vector field X on U satisfying

$$X(x) \in G_\alpha r(x), \ |X(x)| = 1, \ \forall \ x \in U.$$

   Furthermore, if γ is an integral curve of X, then

$$r(\gamma(t)) - r(\gamma(s)) > \cos(\alpha)(t - s), \ \forall \ s < t.$$

Definition: We will call X a *gradient-like vector field for* r.

Proof of 1. We wish to show that $\Gamma(x, K)$ is compact. Let $(v_i)_{i\in\mathbb{N}}$ be elements of $\Gamma(x, K)$. Let $l = r(x)$. Let

$$\sigma_i : [0, l] \longrightarrow M,\ \ i \in \mathbb{N}$$

be a sequence of segments from x to K so that $-v_i = \dot{\sigma}_i$ for each i. Since the the unit sphere in $T_x M$ is compact, we have v_i converges to a vector $v \in T_x M$ (after passing to a subsequence). Let

$$\sigma_\infty : [0, l] \longrightarrow M, \quad \sigma_\infty(t) := \exp_x(-tv).$$

Then since exp is continuous,

$$\lim_{i\to\infty} \sigma_i(l) = \sigma_\infty(l).$$

Since K is closed, $\sigma_\infty(l) \in K$, which implies that $v \in \Gamma(x, K)$. Hence $\Gamma(x, K)$ is sequentially compact.

Proof of 2. We wish to show that the set U of α-regular points is open. We will do this by showing that $M - U$ is closed. Let $(x_i)_{i\in\mathbb{N}}$ be points in $M - U$, and suppose $x_i \to x_\infty$ for some $x_\infty \in M$. We wish to show $x_\infty \notin U$. More precisely, we need to show that for each $w \in T_{x_\infty}M$, there exists $v \in \Gamma(x, K)$ so that $\angle(v, w) \geq \alpha$.

Therefore, fix $w \in T_{x_\infty}M$. Choose $w_i \in T_{x_i}M$ so that $w_i \to w$ as $i \to \infty$. Then, there exists $v_i \in \Gamma(x, K)$ so that $\angle(v_i, w_i) \geq \alpha$. After passing to a subsequence, $v_i \to v \in T_{x_\infty}M$. Also,

$$\angle(v, w) \geq \liminf_{i\to\infty} \angle(v_i, w_i).$$

Finally, (by a similar proof to 1.), $\cup_{x\in M}\Gamma(x < K)$ is closed in TM, and hence $v \in \Gamma(x_\infty, K)$.

Proof of 3.

We wish to show that $G_\alpha r(x)$ is convex if $\alpha \leq \pi/2$. The open cone

$$C_\alpha(w) = \{v \in T_x M \ : \ \angle(v, w) < \alpha\}$$

is convex for each $w \in T_x M$. Hence

$$G_\alpha r(x) = \cap_{w \in \Gamma(x,K)} C_\alpha(w)$$

is convex.

Proof of 4. Let $U \subset M$ be the set of α-regular points of r. We wish to show that there exists a vector field X over U so that $X(x) \in G_\alpha r(x)$ for each $x \in U$ and so that

$$r(\gamma(t)) - r(\gamma(s)) > \cos(\alpha)(t - s), \ \forall \ s < t$$

for each integral curve γ of X.

For each $p \in U$, choose $v_p \in G_\alpha r(x)$. Choose a vector field V_p on M so that $V_p|_p = v_p$. Since $\cup_{x \in U} G_\alpha r(x)$ is open in the unit sphere bundle of M (by a similar proof to 2.), there is a neighborhood $U_p \subset M$ of p so that $V_p(x) \in G_\alpha r(x)$ for each $x \in U_p$. Let $(U_{p_i})_{i \in \mathbb{N}}$ be a locally finite subcover of $(U_p)_{p \in U}$. Let $(\lambda_i)_{i \in \mathbb{N}}$ be a partition of unity subordinate to this subcover. Define

$$V_1 = \sum_{i \in \mathbb{N}} \lambda_i V_{p_i}.$$

$$X := V_1(x)/|V_1(x)|.$$

Then since $G_\alpha r(x)$ is convex for each $x \in U$ by 3., $X(x) \in G_\alpha r(x)$ for each $x \in U$.

Proof of 4. continued. We now need to show

$$r(\gamma(t)) - r(\gamma(s)) > \cos(\alpha)(t-s), \ \forall \ s < t \tag{1}$$

for each integral curve γ of X. We only need to prove this for $s = 0$ and $t \leq 0$ small.

Let $\check{\gamma} : [0, l] \longrightarrow M$ be a segment from x to K of length $l = r(x)$. Let $\gamma : [-\epsilon, 0] \longrightarrow M$ be an integral curve of X. Then

$$r(\check{\gamma}(t)) - r(\check{\gamma}(s)) \geq t - s, \ \forall \ 0 \leq s < t.$$

Also,

$$d(\gamma(-t), \check{\gamma}(t)) > \sin(\alpha)(t)$$

for $t \leq 0$ sufficiently small since $\angle(\dot{\gamma}(0), -\dot{\check{\gamma}}(0)) < \alpha$. Equation (1) with $s = 0$, $t \leq 0$ now follows from the triangle inequality.

Corollary: Let $K \subset M$ be a compact submanifold and suppose (M, g) is complete. Suppose $r(x) = d(x, K)$ is regular on $M - K$. Then M is diffeomorphic to the normal bundle of K. In particular, if $K = \{p\}$, then M is diffeomorphic to $\mathbb{R}^n$.

Proof. Let X be the vector field on $U = M - K$ be as in 4. Also, since r is smooth near K, we can (by using bump functions) assume $X = \nabla r$ near K. Let

$$\nu(K) = \{v \in TM|_K \;:\; v \perp TK\}$$

be the normal bundle of K. Let

$$\exp : \nu(K) \longrightarrow M$$

be the exponential map.

Proof continued.
Then the curves

$$t \to \exp(tv), \quad v \in \nu(K)$$

are tangent to X near K. Therefore, for each unit vector $v \in \nu(K)$, there is a unique integral curve

$$\gamma_v : (0, \infty) \longrightarrow M$$

of X satisfying

$$\lim_{t \to 0} \dot{\gamma_v}(t) = v.$$

Define

$$F : \nu(K) \longrightarrow M,$$

$$F(v) := \begin{cases} p & \text{if } v = 0 \\ \gamma_{v/|v|}(|v|) & \text{if } v \neq 0 \end{cases} \qquad \forall\ v \in \nu(K) \cap T_pM,\ p \in K.$$

This is a diffeomorphism (Exercise).

We now wish to find conditions ensuring that r is regular.

Definition: A *hinge* consists of two geodesic segments

$$\sigma_1 : [0, l_1] \longrightarrow M, \quad \sigma_2 : [0, l_2] \longrightarrow M$$

satisfying $\sigma_1(l_1) = \sigma_2(0)$. All such segments will be parameterized by arc length. The *interior angle* α of this hinge is:

$$\alpha := \pi - \angle(\dot{\sigma_1}(l_1), \dot{\sigma_2}(0)).$$

Definition: A *triangle* consists of three segments that meet pairwise at three different points. All such segments will be parameterized by arc length.

Definition: A *generalized hinge/triangle* is defined in the same way as a hinge and a triangle above, except that segments are replaced by geodesics. (For instance, we could have a "triangle" consisting of three geodesic loops).

Definition: Let $(\check{M}, \check{g})$ be another Riemannian manifold. Suppose we have a hinge in (M, g). Then a *comparison hinge in* $(\check{M}, \check{g})$ is a hinge in $(\check{M}, \check{g})$ so that the corresponding segments have the same length and so that both hinges have the same interior angle.

Definition. Similarly, suppose we have a triangle in (M, g). Suppose that the segments of this triangle have lengths l_1, l_2, l_3 respectively.Then a *comparison triangle in* $(\check{M}, \check{g})$ is a triangle in $(\check{M}, \check{g})$ so that the three segments of this triangle have lengths l_1, l_2, l_3 respectively.

Lemma: Suppose (M, g) is a complete Riemannian manifold satisfying $\sec(M, g) \geq k$. Then every hinge in (M, g) has a comparison hinge in S_k^n. Similarly every triangle in (M, g) has a comparison triangle in S_k^n.

Proof. We will use Myers' theorem:

Theorem (Myers): If $k > 0$ then

$$\mathrm{diam}(M, g) \leq \pi/\sqrt{k} = \mathrm{diam}(S_k^n).$$

Since S_k^n has infinite diameter for $k \leq 0$, we have:

Corollary: $\mathrm{diam}(M, g) \leq \mathrm{diam}(S_k^n)$ for all k.

Proof continued. Now suppose we have a hinge in (M, g) consisting of two segments, σ_1 from p to q and σ_2 from q to r. Let α be its interior angle. Now the corollary above allows us to choose $\overline{p}, \overline{q} \in S_k^n$ so that $d(\overline{p}, \overline{q}) = d(p, q)$. Let $\overline{\sigma}_1$ be a unit speed geodesic from $\overline{p}$ to $\overline{q}$ in S_k^n. Also we can take the unique length $d(q, r)$ geodesic $\overline{\sigma}_2$ in S_k^n from q of angle α from $\overline{\sigma}_1$ at $\overline{q}$. Then $\overline{\sigma}_1$, $\overline{\sigma}_2$ is our comparison hinge, well, almost.

This is true if $k \leq 0$ or $\operatorname{diam}(M, g) < \operatorname{diam}(S_k^n)$. However, if $\operatorname{diam}(M, g) = \operatorname{diam}(S_k^n)$ and $k > 0$ then this hinge might be degenerate. However, in this case, we can use the following theorem instead:

Theorem (Cheng). Let (M, g) be a complete Riemannian manifold satisfying $\operatorname{Ric}(M, g) \geq (n-1)k > 0$ and $\operatorname{diam}(M, g) = \operatorname{diam}(S_k^n)$. Then (M, g) is isometric to S_k^n.

Note that if $\sec = k$ then $\operatorname{Ric} = (n-1)k$ (Exercise).

Proof continued. Now let us consider the triangle case. Suppose we have a triangle in (M, g) with vertices p, q, r. The corollary above allows us to choose $\overline{p}, \overline{q} \in S_k^n$ so that $d(\overline{p}, \overline{q}) = d(p, q)$. Now consider the two spheres

$$\partial B_{d(p,r)}(\overline{p}), \quad \partial B_{d(q,r)}(\overline{q}).$$

Hence these two spheres intersect by the triangle inequality. We let $\overline{r}$ be an intersection point these spheres, this gives us our comparison triangle $\overline{p}, \overline{q}, \overline{r}$. Again, there is the possibility that this triangle is degenerate, but this only happens when $k > 0$ and $\operatorname{diam}(M, g) = \operatorname{diam}(S_k^n)$, and Cheng's theorem above deals with this.

$\square$

Theorem (Toponogov, 1959). Let (M, g) be a complete Riemannian manifold satisfying $\sec(M, g) \geq k$. Then

1. For any hinge in (M, g) with vertices p, q, r and any comparison hinge in S_k^n with vertices $\overline{p}, \overline{q}, \overline{r}$, we have $d(p, r) \leq d(\overline{p}, \overline{r})$.
2. For any triangle T in (M, g) and any comparison triangle T' in S_k^n, we have that the interior angles of T are $\geq$ the interior angles of T'.

Before we prove this theorem, we need a preliminary proposition and lemma.

Proposition (Law of cosines). Let T be a triangle in S_k^n with side lengths a, b, c. Let α be the angle opposite to a. Then

1. If $k = 0$,
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
2. If $k = -1$, then
$$\cos(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\alpha).$$
3. If $k = 1$, then
$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha).$$

Proof: The key idea is the same in all three cases. We start with a point $p \in S_k^n$ and a unit speed segment

$$\sigma : [0, c] \longrightarrow S_k^n.$$

We then investigate the distance functions

$$r : S_k^n \longrightarrow \mathbb{R}, \quad r(x) := d(p, q)$$

and

$$r \circ \sigma : [0, c] \longrightarrow \mathbb{R}.$$

Proof of 1. I.e. $k = 0$ case. We consider the function

$$\phi : [0, c] \longrightarrow \mathbb{R}, \quad \phi(t) := \frac{1}{2} r(\sigma(t))^2 = \frac{1}{2} |p - \sigma(t)|^2.$$

We now need to compute the first and second derivatives of ϕ.

Proof of 1. continued.
First we compute:

$$\nabla \frac{1}{2} r^2 = r \nabla r.$$

$$\mathrm{Hess} \frac{1}{2} r^2 = \sum_i dx^i dx^i.$$

Hence

$$\phi'(t) = g(\dot{\sigma}, \nabla \frac{1}{2} r^2).$$

$$\phi''(t) = \mathrm{Hess} \frac{1}{2} r^2 (\dot{\sigma}, \dot{\sigma}) = 1.$$

We let $b = d(p, \sigma(0))$ and α the interior angle between σ and the segment joining p and $\sigma(0)$. Then

$$\cos(\pi - \alpha) = -\cos(\alpha) = g(\dot{\sigma}, \nabla r).$$

Proof of 1. continued. Now we integrate $\phi'' = 1$, giving us:

$$\phi(t) = \phi(0) + \phi'(0)t + \frac{1}{2}t^2$$

$$= \frac{1}{2}b^2 - b\cos(\alpha)t + \frac{1}{2}t^2.$$

We now get the law of cosines by setting $t = c$.

To recap, we proved this identity by solving the initial value problem:

$$\phi'' = 1$$

$$\phi(0) = b$$

$$\phi'(0) = -b\cos(\alpha)$$

Proof of 2. This is the case $k = -1$. In this case, we consider the function:

$$\phi : [0, c] \longrightarrow \mathbb{R}, \quad \phi(t) = \cosh(r \circ \sigma(t)) - 1.$$

Then

$$\phi'(t) = \sinh(r \circ \sigma(t)) g(\nabla r, \dot{\sigma}),$$

$$\phi''(t) = \cosh(r \circ \sigma(t)) = \phi(t) + 1.$$

We now solve the initial value problem

$$\phi'' - \phi = 1$$

$$\phi(0) = \cosh(b) - 1$$

$$\phi'(0) = -\sinh(b) \cos(\alpha).$$

Solving this set of equations and setting $t = 1$ and $a = d(p, \sigma(c))$ gives us our identity.

Proof of 3. This is the case $k = 1$. In this case, we consider the function:

$$\phi : [0, c] \longrightarrow \mathbb{R}, \quad \phi(t) = 2 - \cos(r \circ \sigma(t)).$$

Then we arrive at the initial value problem

$$\phi'' + \phi = 1$$

$$\phi(0) = 1 - \cos(b)$$

$$\phi'(0) = -\sin(b)\cos(\alpha).$$

Solving this set of equations and setting $t = 1$ and $a = d(p, \sigma(c))$ gives us our identity. □

Lemma: Let (M, g) be a complete Riemannian manifold satisfying $\sec(M, g) \geq k$. Let $p \in M$ and $r(x) = d(x, p)$. Then:

1. If $k = 0$ then $\mathrm{Hess} r_0 \leq g$ where $r_0 = \frac{1}{2}r^2$. (in the support sense).
2. If $k = -1$ then $\mathrm{Hess} r_{-1} \leq (\cosh(r))g = (r_{-1} + 1)g$ where $r_{-1} = \cosh(r) - 1$.
3. If $k = 1$ then $\mathrm{Hess} r_1 \leq \cos(r)g = (-r_1 + 1)g$ where $r_1 = 1 - \cos(r)$.

Proof. Omitted. See Lemma 57 of Peterson.

Let us now prove Toponogov's theorem part 1. Here is the statement we wish to prove:

Theorem (Toponogov, 1959). Let (M, g) be a complete Riemannian manifold satisfying $\sec(M, g) \geq k$. Then

1. For any hinge in (M, g) with vertices p, q, r and any comparison hinge in S_k^n with vertices $\overline{p}, \overline{q}, \overline{r}$, we have $d(p, r) \leq d(\overline{p}, \overline{r})$.

Setup: Let

$$\sigma : [0, l] \longrightarrow M$$

$$\overline{\sigma} : [0, l] \longrightarrow S_k^n$$

be the unit speed segments from q to r and $\overline{q}$ to $\overline{r}$ respectively.

Proof of Toponogov Theorem part 1. continued.
Define

$$r : M \longrightarrow \mathbb{R}, \quad r(x) := d(p, x)$$

$$\bar{r} : M \longrightarrow \mathbb{R}, \quad r(x) := \overline{d}(\bar{p}, x)$$

where d and $\overline{d}$ are the distance functions on (M, g) and S^n_k respectively. It is sufficient for us to show

$$\phi'(t) \leq \overline{\phi}'(t)$$

for each $t \in [0, l]$ where

$$\phi(t) = f(r \circ \sigma(t)), \quad \overline{\phi} = f(\bar{r} \circ \sigma(t))$$

and where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an appropriate increasing function.

Proof of Toponogov Theorem part 1. continued.

This function f will be:

a) $f(x) = \frac{1}{2}x^2$ if $k = 0$.

b) $f(x) = \cosh(x) - 1$ if $k = -1$.

c) $f(x) = 1 - \cos(x)$ if $k = 1$.

Now suppose $k = 0$. Then the previous proposition and lemma tell us:

$$\phi'' \leq 1, \quad \overline{\phi}'' = 1.$$

$$\phi'(0) = \overline{\phi}'(0)$$

and hence $\phi'(t) \leq \overline{\phi}'(t)$ giving us our result for $k = 0$.

Note: Technically, ϕ may not be differentiable at $t = 0$. As a result, we might have perturb this segment slightly towards p. The details are on Peterson page 343.

Proof of Toponogov Theorem part 1. continued.
If $k = -1$, then a similar argument holds.

$$\phi'' \leq \phi + 1, \quad \overline{\phi}'' = \overline{\phi} + 1.$$

$$\phi'(0) = \overline{\phi}'(0)$$

When $k = 1$, there are smoe additional difficulties, I think due to the fact that ϕ may not be differentiable everywhere. The details are resolved on page 345 of Peterson. I think (!) the same argument would work if we assumed that ϕ is differentiable. □

We now wish to apply the ideas above.

Theorem (Berger 1962, Grove-Shiohama 1977) Suppose (M, g) is closed, $\sec(M, g) \geq 1$ and $\mathrm{diam}(M, g) > \pi/2$. Then M is homeomorphic to a sphere.

Proof: It is sufficient for us to show that $M - \{\text{point}\}$ is $\mathbb{R}^n$. Choose points p, q so that $d(p, q) > \pi/2$. Let

$$r : M \longrightarrow \mathbb{R}, \quad r(x) := d(p, x).$$

We will show that the only critical point of r is q (away from p).By the previous results, this will show that $M - \{q\} = \mathbb{R}^n$ since we will have a gradient like vector field for r away from p and q.

Proof continued.
Let $x \in M - \{p, q\}$ and let α be the angle between any two geodesics from x to p and q.

Claim: $\alpha > \pi/2$.

Proof of **Claim:** Suppose not, then set $b = d(p, x)$ and $c = d(x, q)$ and $d = d(p, q)$. The hinge version of Toponogov's theorem along with the law of cosines tells us:

$$0 > \cos(d) \geq \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$

$$\geq \cos(b)\cos(c).$$

Hence $\cos(b)\cos(c)$ have opposite signs. If $\cos(b) \in (0, 1)$, then $\cos(d) > \cos(c)$ and so $c > d = \mathrm{diam}(M)$ giving us a contradiction. Similarly if $\cos(c) \in (0, 1)$. QED for **Claim**.

Proof continued. To construct our gradient-like vector field, it is sufficient to show that $M - \{p, q\}$ consists of $\pi/2$-regular points of r (by an earlier theorem in these slides).

In other words, the set $\Gamma(x, \{p\})$, which is the set of unit vectors $v \in T_x$ tangent to the endpoint of a segment from p to x, has angle at most $\pi/2$ with some vector $v_x \in T_x M$.

However, by the **Claim** above, if we choose v_x to be tangent to the initial point of any geodesic from x to q, then v_x has the property we want.

$\square$

The Soul Theorem

The goal of this section is to show:

Theorem (Cheeger-Gromoll-Meyer, 1969,1972). Let (M, g) be a complete non-compact Riemannian manifold satisfying $\sec(M, g) \geq 0$. Then M contains a totally geodesic submanifold $S \subset M$ so that M is diffeomorphic to the normal bundle of S. If $\sec(M, g) > 0$ then S is a point, and so M is diffeomorphic to $\mathbb{R}^n$.

Definition: The *soul* of (M, g) is the submanifold S above.

Before we prove this theorem, we will use a less ambitious result with a similar proof.

Lemma (Gromov's critical point estimate 1981). Suppose (M, g) is a complete Riemannian manifold of non-negative sectional curvature and let $p \in M$. Let $r(x) = d(x, p)$ be the distance function from p. Then there exists a constant $R > 0$ so that every point outside $B_R(p)$ is a regular point of r. As a result, (M, g) is homotopic to a compact manifold with boundary.

Proof. Suppose (for a contradiction) r has a sequence of critical points p_k, $k \in \mathbb{N}$ satisfying $p_k \to \infty$. After passing to a subsequence, we can assume

$$r(p_{k+1}) \geq 2r(p_k), \quad \forall\ k. \tag{2}$$

Let σ_k be a segment from p to p_k of length $r(p_k)$.

Claim: The angle between any two segments σ_k and σ_{k+l} is $\geq 1/6$.

Proof of **Claim**. Suppose (for a contradiction) there exists k, l so that the angle between σ_k and σ_{k+l} is $\geq 1/6$. The hinge version of Toponogov's theorem together with the law of cosines tells us:

$$d(p_k, p_{k+l})^2 < (d(p, p_{k+l}))^2 + (d(p_k, p))^2 - 2d(p, p_{k+l})d(p_k, p)\cos(1/6)$$

$$\leq \left(d(p, p_{k+l}) - \frac{3}{4}d(p_k, p)\right)^2.$$

Since p_k is not a critical point of r, we can find a segment s from p_k to p_{k+l} and possibly change the segment σ_k so that the angle between σ_k and s is $\leq \pi/2$.

Proof of **Claim** *continued.*
Hence by Toponogov's theorem and the law of cosines:

$$d(p_k, p_{k+l})^2 < (d(p_k, p))^2 + (d(p, p_{k+l}))^2$$

$$\leq (d(p_k, p))^2 + \left(d(p, p_{k+l}) - \frac{3}{4}d(p_k, p)\right)^2.$$

$$= \frac{25}{16}(d(p_k, p))^2 + (d(p, p_{k+l}))^2 - \frac{3}{2}d(p, p_{k+l})d(p_k, p)$$

Hence

$$d(p, p_{k+l}) \leq \frac{25}{16}(d(p_k, p))^2.$$

But this gives us a contradiction, since by Equation 2, we have

$$d(p, p_{k+1}) \geq d(p, p_{k+1}) \geq 2d(p_k, p).$$

QED for **Claim**.

Proof of Gromov's lemma continued.

Now the **Claim** gives us a contradiction as follows: Each vector $\dot{\sigma}_k \in T_pM$ is a unit vector and the angle between any two of them is $\geq 1/6$. Hence we can cover the unit sphere in T_pM with an infinite collection of disjoint balls of radius $1/6$. Contradiction. $\square$

Definition: A subset $A \subset M$ is *totally convex* if every geodesic starting and ending on A is contained in A.

Definition: A function $f : M \longrightarrow \mathbb{R}$ is *concave* if the Hessian is non-positive everywhere (in the weak sense).

Lemma: If $f : M \longrightarrow \mathbb{R}$ is a concave, then every superlevel set

$$M_{\geq a} := \{x \in M \ : \ f(x) \geq a\}$$

is totally convex.

Proof: For any geodesic γ, we have that $f \circ \gamma$ has non-positive weak second derivative. Hence $f \circ \gamma$ is a concave function. On any compact interval, the minimum of this function is obtained at its endpoints. $\square$

Busemann functions.
Let $\gamma : [0, \infty) \longrightarrow M$ be a unit speed ray. Define

$$b_t(x) := d(x, \gamma(t)) - t.$$

Proposition:

1. For any $x \in M$, the function

$$t \longrightarrow b_t(x)$$

   is decreasing, and bounded in absolute value by $d(x, \gamma(0))$.
2. $|b_t(x) - b_t(y)| \leq d(x, y)$.
3. $\Delta b_t(x) \leq \frac{n-1}{b_t+t}$ everywhere.

The proof is found in Peterson section 3.4.

The proposition above tells us that the functions $b_t : M \longrightarrow \mathbb{R}$, $t \in \mathbb{R}$ are

- pointwise decreasing as t increases.
- equicontinuous.

Hence $b_t \xrightarrow{C^0} b_\gamma$ for some C^0 function

$$b_\gamma : M \longrightarrow \mathbb{R}.$$

Definition: b_γ is called the **Busemann function** associated to the ray γ.

Lemma: Let (M, g) be a complete non-compact Riemannian manifold satisfying $\sec(M, g) \geq 0$. Let γ_α, $\alpha \in I$ be the collection of all unit speed rays emanating from a point $p \in M$.Then

$$f = \inf_{\alpha \in I} b_{\gamma_\alpha}$$

is proper and concave.

We will skip the proof for reasons of time. This is Lemma 59 in Peterson.

Lemma: Let $A \subset M$ be totally convex. Then A can be written as a disjoint union $A = \mathrm{Int}A \sqcup \partial A$ so that $\mathrm{Int}A$ is a submanifold of M and so that the following property holds: For each $x \in \partial A$, there exists a vector $v_x \in T_x M$ satisfying the following property: Let $\gamma : [0, a] \longrightarrow A$ be a geodesic satisfying:

1. $\gamma(0) = x$,
2. $\gamma(a) \in \mathrm{Int}A$

Then $\angle(w, \dot{\gamma}(0)) < \pi/2$.

Definition: Int$A$ is called the *interior of* A and ∂A is called the *boundary of* A.

This is NOT the topological interior and boundary.

Proof of Lemma: We will use the following fact (from the convexity radius estimate in Chapter 6):

Fact: There is a positive function $\epsilon : M \longrightarrow (0, \infty)$, so that the distance function $r_p(x) = d(x, p)$ is smooth and strictly convex on $B_{\epsilon(p)}(x)$.

We will start be defining $\mathrm{Int}A$. Find the largest integer k so that A contains a submanifold of dimension k (not necessarily properly embedded). We define $\mathrm{Int}A$ to be the union of all such submanifolds and $\partial A := A - \mathrm{Int}A$.

Claim 1: Int$A$ is a submanifold of M.

Proof of **Claim 1**. Let $p \in \mathrm{Int}A$ and let N_p be a small submanifold in A containing p. We can assume $N_p \cap B_\delta(p) = N_p$ for some $0 < \delta < \epsilon(p)$ small (to be chosen later). Suppose (for a contradiction) $B_\delta(p) \cap A \neq N_p$. Let $q \in B_\delta(p) \cap A - N_p$. Choose $\delta > 0$ small enough so that $\delta < \mathrm{inj}_q$. Hence every point in N_p can be connected to q be a unique segment $B_\delta(p)$. The union of such segments is contained in A by definition. Also this union contains a $k+1$-dimensional submanifold giving us a contradiction. QED for **Claim.**

Claim 2: Int$A$ is dense in A. In fact, for each segment $\sigma : [0, l] \longrightarrow A$ satisfying $\sigma(0) \in \partial A$, $\sigma(l) \in \mathrm{Int}A$, we have $\sigma(0, l) \subset \mathrm{Int}A$.

Proof of **Claim 2:** Let $N \subset \mathrm{Int}A$ be a small $k-1$ dimensional submanifold orthogonal to $\sigma(l)$. Consider the union of all segments starting at $\sigma(0)$ and ending at a point in N. Then $N - \sigma(0)$ is a k-submanifold of A, giving us our conclusion. QED for **Claim 2**.

We now wish to show the *supporting hyperplane property:* For each $x \in \partial A$, there exists a vector $v_x \in T_x M$ satisfying the following property: Let $\gamma : [0, a] \longrightarrow A$ be a geodesic satisfying:

1. $\gamma(0) = x$,
2. $\gamma(a) \in \mathrm{Int}A$

Then $\angle(w, \dot{\gamma}(0)) < \pi/2$.

Proof continued.
Define the *tangent cone at* $x \in \partial A$ to be:

$$C_x A := \{v \in T_x M \ : \ \exp_x(tv) \in \mathrm{Int} A, \ \text{for some } t > 0\}.$$

This is a cone. We will show:

1. This cone is open inside some linear subspace $T \subset T_x M$.
2. This cone is not equal to T. In fact, once we have shown the openness property, we just need to show that T does not contain a line.

These two properties then tell us that the cone must be contained inside a half space:

$$\{v \in T_x M \ : \ L(v) > 0\}$$

where $L : T_x M \longrightarrow \mathbb{R}$ is some linear function and this will prove our supporting hyperplane property.

Proof continued. We define $T \subset T_xM$ to be the intersection of all linear subspaces of T_xM containing C_xA. Let $\exp : T \longrightarrow M$ be the exponential map. Let $C = \exp^{-1}(\mathrm{Int}A \cap B_\eta(x))$ where $\eta > 0$ is very small. Then C is a k-dimensional submanifold of T which is convex. Hence it must be an open subset of a linear subspace T'. The closure of C is $\exp^{-1}(A \cap B_\eta(x))$ by **Claim 2**. Hence $T = T'$ and C_xA is an open subset of T.

Finally we need to show that $C_xA \subset T_xM$ does not contain a line. If it did, then $\exp^{-1}(\mathrm{Int}A \cap B_\eta(x))$ would be an open subset of T containing 0. The submanifold exp(small neighborhood of 0 in T) would then be contained in A and so $x \in \mathrm{Int}A$ giving us a contradiction.

□

Lemma: Let (M,g) satisfy $\sec(M,g) \geq 0$. Let A be totally convex and let

$$r : M \longrightarrow \mathbb{R}, \quad r(x) = d(x, \partial A).$$

Then r is concave in A and strictly concave if $\sec > 0$.

We will not prove this because of time considerations. This is lemma 62 in Peterson.

We are now ready to prove the soul theorem. Here is the statement we wish to prove.

Theorem (Cheeger-Gromoll-Meyer, 1969,1972). Let (M, g) be a complete non-compact Riemannian manifold satisfying $\sec(M, g) \geq 0$. Then M contains a totally geodesic submanifold $S \subset M$ so that M is diffeomorphic to the normal bundle of S. If $\sec(M, g) > 0$ then S is a point, and so M is diffeomorphic to $\mathbb{R}^n$.

Proof. Let $f : M \longrightarrow \mathbb{R}$ be the concave function constructed from Busemann functions above. Let

$$C_1 := \{x \in M \ : \ f(x) = \max f\}.$$

Since f is proper and concave (by a lemma above), we have that C_1 is convex. Also by the previous lemma, C_1 must be a point if $\sec > 0$ (since f must be strictly concave in this case).

Proof continued.

We now construct a finite (or possibly infinite) sequence of subsets:

$$C_1 \supset C_2 \supset C_3 \cdots$$

by induction as follows: We have already constructed C_1. Suppose we have constructed $C_1 \supset \cdots \supset C_k$. If $\partial C_k = \emptyset$, we stop our sequence. Otherwise, we consider the concave function

$$f_k : C_k \longrightarrow \mathbb{R}, \quad f_k(y) := d(y, \partial C_k)$$

(this is concave by the previous lemma). We define C_{k+1} to be the maximum set of this function.

Proof continued.

Claim: $\dim \operatorname{Int} C_k > \dim \operatorname{Int} C_{k+1}$.

Proof of Claim. If not, then $\operatorname{Int} C_{k+1}$ is an open subset of $\operatorname{Int} C_k$. Let $p \in \operatorname{Int} C_{k+1}$ and consider a segment σ from p to ∂C_k of length $f_k(p)$. Now $f_k \circ \sigma|_{C_{k+1}}$ must be constant (as C_{k+1} is is the maximum set of f_k). However, this is impossible as the distance from $\sigma(t)$ to ∂C_k must decrease along this segment. Contradiction. QED for **Claim**.

As a result of the above claim, we have finite sequence:

$$C_1 \supset \cdots \supset C_m.$$

Since $\partial C_m = \emptyset$, we get that C_m is a closed submanifold of M.

Proof continued.

Consider the function:

$$h : M \longrightarrow \mathbb{R}, \quad h(y) = d(y, C_m).$$

Claim. h has no critical points outside C_m.

The proof of this claim will finish the proof of our theorem, with the Soul S equal to C_m.

Proof of Claim. (I am not 100 percent sure this argument is correct). Define $C_{m+1} = \emptyset$ and $C_0 = M$. Also define $f_0 := f$. We will show that h has no critical points inside $C_k - C_{k+1}$. Suppose we have a segment connecting a point $y \in M - C_m$ to C_m of length $h(y)$. Then since C_{k+1} is totally convex, we have that the distance function to C_{k+1} strictly decreases along this segment. Also the superlevel set $E := f_k^{-1}([f_k(y), \infty))$ is strictly convex inside C_k. Hence $\Gamma(y, C_m)$ is contained in the tangent cone $C_y E$ of E. This is an open cone contained in a half space (see earlier), hence y is a regular point.