Norms and Convergence of Manifolds

MAT 569

April 9, 2020

Norms and Convergence

- We wish to have a norm which tells us how ‘flat a manifold is’.
- We also want subsets of complete flat manifolds to still have non-zero norm.
- We will use a family of norms depending on a ‘scale’ parameter r.
- We also want such norms to be defined for subsets of Riemannian manifolds as well.

Definition:Let A be a subset of M. The $C^{m,\alpha}$ *norm on the scale r* of A, denoted by

$$\|A \subset (M,g)\|_{C^{m,\alpha},r}$$

is $\leq Q$,if we can find charts:

$$\phi_s : U_s \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n,\ \ s \in I$$

so that

(n1) For each $p \in A$, there exists $s \in I$ so that $B_{\frac{1}{10}e^{-Q}r}(p) \subset U_s$.

(n2) $|D\phi_s|, |D\phi_s^{-1}| \leq e^Q$.

(n3) $\|\phi_s^{-1} \circ \phi_t\|_{C^{m+1,\alpha}} \leq f_3(n,Q,r)$.

(n4) $r^{|j|+\alpha}\|D^j((\phi_s^{-1})^* g)\|_\alpha \leq Q$ for all multi-indices j satisfying $0 \leq |j| \leq m$.

Note that the above definition only requires g to be $C^{m,\alpha}$. If $\alpha = 0$, then we replace $C^{m+1,\alpha}$ with C^{m+1} and $\|\cdot\|_\alpha$ with $\|\cdot\|_0$.

It turns out that (n2) + (n4) $\implies$ (n3), however we will keep (n3) as a property.

Example: Suppose (M, g) is a complete flat Riemannian manifold. Then $\|(M, g)\|_{C^{m,\alpha},r} = 0$ for each $r \leq \text{inj}(M, g)$.

In particular, $\|(\mathbb{R}^n, g_{\text{std}})\|_{C^{m,\alpha},r} = 0$ for each m, α, r.

Later on, we will show that if $\|(M, g)\|_{C^{m,\alpha},r} = 0$ for all m, α, r then $(M, g) = (\mathbb{R}^n, g_{\text{std}})$.

Next, we wish to describe $C^{m,\alpha}$ convergence of a sequence of Riemannian manifolds.

Definition: Let (M_i, g_i, p_i), $i \in \mathbb{N} \cup \{\infty\}$ be a sequence of pointed Riemannian manifolds of the same dimension. We say (M_i, g_i, p_i) *converges to* $(M_\infty, g_\infty, p_\infty)$ *in the pointed* $C^{m,\alpha}$*-topology*, written as:

$$(M_i, g_i, p_i) \stackrel{C^{m,\alpha}}{\longrightarrow} (M_\infty, g_\infty, p_\infty),$$

if for each $R > 0$, there exists a domain $\Omega \subset B_R(p_\infty)$ and embeddings $F_i : \Omega \longrightarrow M_i$, $i \gg 1$, so that

1. $F_i(\Omega) \supset B_{R,p_i}$ for each i,
2. $F_i^* g_i \stackrel{C^{m,\alpha}}{\longrightarrow} g$.
3. $F_i(p_\infty) = p_i$ for each i.

When the manifolds are closed, we have that F_i is a diffeomorphism for R large enough. In this case, one can show that the choice of basepoints p_i does not matter, and so we can also talk about unpointed convergence.

Proposition: Let $A \subset M$ be precompact. Then

1. $\|A \subset (M,g)\|_{C^{m,\alpha},r} = \|A \subset (M,\lambda^2 g)\|_{C^{m,\alpha},\lambda r}$ for each $r > 0$.
2. The function $r \longrightarrow \|A \subset (M,g)\|_{C^{m,\alpha},r}$ is continuous.
3. If $(M_i, g_i, p_i) \xrightarrow{C^{m,\alpha}} (M_\infty, g_\infty, p_\infty)$, then for every precompact domain $A_\infty \subset M_\infty$, there exists precompact domains $A_i \subset M_i$ for each $i \in \mathbb{N}$ so that
$$\|A_i\|_{C^{m,\alpha},r} \longrightarrow \|A_\infty\|_{C^{m,\alpha},r}, \ \forall \ r > 0.$$
If M_i is closed for each $i \in \mathbb{N} \cup \{\infty\}$ then we can assume $A_i = M_i$ for each $i \in \mathbb{N} \cup \{\infty\}$.

Proof of 1. If we replace g with $\lambda^2 g$, then we replace the charts ϕ_s by
$$\phi_s^\lambda : U_s \subset M \longrightarrow B_{\lambda r}(0) \subset \mathbb{R}^n, \ \phi_s^\lambda(x) := \lambda \phi_s(x).$$
The same conditions (n1)-(n4) still hold.

Proof of 2. We wish to show: the function $r \longrightarrow \|A \subset (M,g)\|_{C^{m,\alpha},r}$ is continuous. Instead of scaling r, we will replace ϕ_s with ϕ_s^λ (without changing the metric g).. This corresponds to replacing r with λr. So we only need to show continuous dependence on λ. Suppose

$$N(r) := \|A \subset (M,g)\|_{C^{m,\alpha},r} \leq Q.$$

Then

$$N(\lambda r) := \|A \subset (M,g)\|_{C^{m,\alpha},\lambda r} \leq \max\{Q + |\log(\lambda)|, Q \cdot \lambda^2\}.$$

This is enough to show continuous dependence on λ (Exercise).

Proof of 3. We need to show: If $(M_i, g_i, p_i) \stackrel{C^{m,\alpha}}{\longrightarrow} (M_\infty, g_\infty, p_\infty)$, then for every precompact domain $A_\infty \subset M_\infty$, there exists precompact domains $A_i \subset M_i$ for each $i \in \mathbb{N}$ so that

$$\|A_i\|_{C^{m,\alpha},r} \longrightarrow \|A_\infty\|_{C^{m,\alpha},r}, \ \forall\ r > 0. \tag{1}$$

First of all, we need to construct these domains A_i. By definition, there is a domain $\Omega \supset A$ so that for all large i we have smooth embeddings $F_i : \Omega \longrightarrow M_i$ satisfying $F_i^* g_i \longrightarrow g$ in $C^{m,\alpha}$ on Ω. We define $A_i := F_i(A_\infty)$. For $Q > \|A_\infty \subset (M_\infty, g)\|_{C^{m,\alpha},r}$, choose charts $\phi_s : U_s \subset M_\infty \longrightarrow B_r(0) \subset \mathbb{R}^n$ covering A_∞ satisfying (n1)-(n4). Define

$$\phi_{i,s} := \phi_s \circ F_i^{-1} : F_i(U_s) \subset M_i \longrightarrow B_r(0) \subset \mathbb{R}^n.$$

Then, $\phi_{i,s}$ satisfies (n1)-(n4)with A replaced by A_i and Q replaced by Q_i where $Q_i \longrightarrow Q$. (Exercise). This is enough to prove Equation (1). (Exercise).

- We now wish to have an Arzela-Ascoli type theorem using the norms $\|\cdot\|_{C^{m,\alpha},r}$ on manifolds.
- **Definition**: For $Q > 0$, $n \geq 2$, $m \geq 0$, $\alpha \in (0,1]$ and $r > 0$, define $\mathcal{M}^{m,\alpha}(n, Q, r)$ to be the class of complete, pointed Riemannian n-manifolds (M, g, p) satisfying $\|(M, g)\|_{C^{m,\alpha},r} \leq Q$. The manifolds here are $C^{m,\alpha}$-manifolds. I.e. transition functions are $C^{m,\alpha}$.
- **Theorem:** (Fundamental Theorem of Convergence Theory) $\mathcal{M}^{m,\alpha}(n, Q, r)$ is sequentially compact in the pointed $C^{m,\beta}$-topology for all $\beta < \alpha$.

Proof: The proof will proceed in 4 stages:

Step 1 Setup and comments on charts.

Step 2 $\mathcal{M}' := \mathcal{M}^{m,\alpha}(n, Q, r)$ is precompact in the pointed Gromov-Hausdorff topology.

Step 3 $\mathcal{M}'$ is closed in the pointed Gromov-Hausdorff topology.

Step 4 Showing that any Gromov-Hausdorff convergent sequence in $\mathcal{M}'$ in fact convergences in the $C^{m,\beta}$-topology (after passing to a subsequence).

We will now give some details of the proof. However there will be many other details that we will skip.

Proof of Step 1. Setup. Write $\mathcal{M}' := \mathcal{M}^{m,\alpha}(n, Q, r)$. Fix $K > Q$. We will call a chart for a manifold in $\mathcal{M}'$ satisfying (n1)-(n4) with Q replaced by K an (n1)-(n4)-chart. It is sufficient for us to show that limit spaces exists for each $K > Q$.

Claim 0: Every (n1)-(n4)-chart $\phi : U \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n$ satisfies

a. $d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \leq e^K |x_1 - x_2|$,

b. $d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \geq \min(e^{-K}|x_1 - x_2|, e^{-K}(2r - |x_1| - |x_2|))$

where d is the distance measured in M, and $|\cdot|$ is the Euclidean norm in the chart.

Proof of **Claim 0**: The condition (n2): $|D\phi^{-1}| \leq e^K$, together with the convexity of $B_r(0)$ gives us a. If there is a line segment in U of length equal to $d(\phi^{-1}(x_1), \phi^{-1}(x_2))$ joining $\phi^{-1}(x_1)$ and $\phi^{-1}(x_2)$, then the condition (n2): $|D\phi^{-1}| \leq e^{-K}$ tells us

$$d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \leq e^{-K}|x_1 - x_2|$$

proving b.

Proof continued:

Now suppose that a segment $\sigma : [0,1] \longrightarrow M$ leaves U. Choose $t_1 < t_2$ in $(0,1)$ so that $\sigma|_{[0,t_1)}$ and $\sigma|_{(t_2,1]}$ lie in U and so that $\sigma(t_i) \notin U$ for $i = 1, 2$. Then

$$d(\phi^{-1}(x_1), \phi^{-1}(x_2)) = L(\sigma) \geq L(\sigma|_{[0,t_1)}) + L(\sigma|_{(t_2,1]})$$

$$\overset{(n2)}{\geq} e^{-K}(L(\phi \circ \sigma|_{[0,t_1)}) + L(\phi \circ \sigma|_{(t_2,1]}))$$

$$\geq e^{-K}(2r - |x_1| - |x_2|)$$

This proves b.

Proof of Step 2. Recall that we now wish to prove that $\mathcal{M}'$ is precompact in the pointed Gromov-Hausdorff topology. We will use Gromov's Theorem to do this. Before we prove this, we need to prove some claims.

Claim 1: Let $\delta = \frac{1}{10}e^{-K}r$. Then each δ-ball in M can be covered $N = N(n, K, r)$ balls of radius $\delta/4$ for each $M \in \mathcal{M}'$.

Proof of Claim 1: Every δ-ball is contained in some (n1)-(n4)-chart $\phi : U \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n$. We have bounds on the derivatives of ϕ, ϕ^{-1} and hence the metric on U and the flat metric on $B_r(0)$ are not too different. This gives us our bound N (Exercise: fill in details). *QED for* **Claim 1**.

Claim 2: Every ball $B(x; l \cdot \delta/2) \subset M$, $l \in \mathbb{N}$ can be covered by N^l balls of radius $\delta/4$.

Proof of Claim 2: We prove this by induction on l. By **Claim 1**, this is true for $l = 1$. Now suppose (by induction), $B_{l \cdot \delta/2}(x)$ is covered by $B_{\delta/4}(x_1), \cdots, B_{\delta/4}(x_{N^l})$. Then $B_{(l+1)\delta/2}(x) = B_{l\delta/2+\delta/2}x$ is covered by $B_{\delta}(x_1), \cdots, B_{\delta}(x_{N^l})$ (by the triangle inequality). Each $B_{\delta}(x_i)$ is covered by N balls of radius $\delta/4$, and hence $B_{(l+1)\delta/2}(x)$ is covered by $N \cdot N^l = N^{l+1}$ balls of radius $\delta/4$. *QED for* **Claim 2**.

We will now use **Claim 2** combined with Gromov's compactness result to show that $\mathcal{M}'$ is precompact in the pointed Gromov-Hausdorff topology.

To show that $\mathcal{M}'$ is precompact, it is sufficient to show that the radius R balls:

$$B_R(p) \subset M, \quad (M, g, p) \in \mathcal{M}'$$

are precompact with respect to the Gromov-Hausdorff topology. By Gromov's theorem, it is sufficient to show that for each $\epsilon > 0$, there exists $N(\epsilon) = N(\epsilon, R, K, r, n)$ with the property that each $B_R(p)$ as above contains at most $N(\epsilon)$ disjoint ϵ-balls. We will do this by considering volume. Let $B_\epsilon(x_1), \cdots, B_\epsilon(x_s)$ be disjoint balls in $B_R(p)$. Choose $l \in \mathbb{N}$ so that

$$l \cdot \delta/2 < R \leq (l+1) \cdot \delta/2.$$

Then

$$\begin{aligned} \mathrm{vol} B_R(p) &\overset{\textbf{Claim 2}}{\leq} (N^{l+1}) \cdot (\text{max volume of a } \delta/4 - \text{ball}) \\ &\leq (N^{l+1}) \cdot (\text{max volume of a (n1)-(n4)-chart}) \\ &\leq N^{l+1} e^{nK} \mathrm{vol} B_R(0) \\ &\leq V(R) = V(R, n, K, r). \end{aligned}$$

Proof continued. Now if $\epsilon < r$, then each $B_\epsilon(x_i)$ lies inside some (n1)-(n4)-chart $\phi : B_R(0) \longrightarrow U \subset M$. Hence $\phi^{-1}(B_\epsilon(x_i))$ contains an $e^{-K}\epsilon$-ball in $B_R(0)$. Hence

$$\mathrm{vol} B_\epsilon(x_i) \geq e^{-2nK} \mathrm{vol} B_\epsilon(0).$$

Hence

$$\begin{aligned} V(R) &\geq \mathrm{vol} B_R(p) \\ &\geq \sum_i \mathrm{vol} B_\epsilon(x_i) \\ &\geq s e^{-2nK} \mathrm{vol} B_\epsilon(0). \end{aligned}$$

Rearranging the above equation gives:

$$s \leq N(\epsilon) := V(R) e^{2nK} (\mathrm{vol} B_\epsilon(0))^{-1}.$$

Hence $\mathcal{M}'$ is precompact in the pointed Gromov-Hausdorff topology.

Proof of Step 3: We now wish to show $\mathcal{M}' := \mathcal{M}^{m,\alpha}(n, Q, r)$ is closed in the pointed Gromov-Hausdorff topology. Let (M_i, g_i, p_i) be a sequence in $\mathcal{M}'$ converging in the Gromov-Hausdorff topology to $(M_\infty, g_\infty, p_\infty)$. We wish to show $(M_\infty, g_\infty, p_\infty)$ is in fact a pointed Riemannian manifold inside $\mathcal{M}'$. First of all, we will construct continuous maps

$$\phi_{\infty s} : U_{\infty s} \subset M_\infty \longrightarrow B_R(0) \subset \mathbb{R}^n, \ s \in \mathbb{N}$$

satisfying (n3):

$$\phi_{\infty s}^{-1} \circ \phi_{\infty t}$$

is a $C^{m,\alpha}$-map satisfying

$$\|\phi_{\infty s}^{-1} \circ \phi_{\infty t}\|_{C^{m+1,\alpha}} \leq f_3(n, K, r).$$

After that we will show that these are (n1)-(n4)-charts.

Proof of Step 3 continued. We will in fact construct $\phi_{\infty s}^{-1}$ first. To construct $\phi_{\infty s}^{-1}$, choose a countable collection of (n1)-(n4)-charts

$$\phi_{is} : U_{is} \subset M_i \longrightarrow B_r(0) \subset \mathbb{R}^n,\ s \in \mathbb{N}$$

covering M_i for each $i \in \mathbb{N}$. By (n2), ϕ_{is}^{-1} has bounded Lipschiz constant and hence ϕ_{is}^{-1}, $i \in \mathbb{N}$ are equicontinuous - in the Gromov-Hausdorff sense. Hence, after passing to a subsequence, $\phi_{is}^{-1} \xrightarrow{d_{\mathrm{GH}}} \phi_{\infty s}^{-1}$ for some maps:

$$\phi_{\infty s}^{-1} : B_R(0) \longrightarrow M_\infty,\ s \in \mathbb{N}.$$

by the Gromov-Hausdorff extension of Arzela-Ascoli stated earlier. (Some details are missing here - for instance **Claim 0** a. must be used here.)

Proof of Step 3 continued. Now **Claim 0** b. tells us $d(\phi_{is}^{-1}(x_1), \phi_{is}^{-1}(x_2)) \geq \min(e^{-K}|x_1 - x_2|, e^{-K}(2r - |x_1| - |x_2|))$ for each $i, s \in \mathbb{N}$. As a result, one can show (Exercise), that $\phi_{\infty s}^{-1}$ is an injective map for each $s \in \mathbb{N}$. Hence we have well defined maps:

$$\phi_{is} : U_{is} \subset M_i \longrightarrow B_r(0) \subset \mathbb{R}^n, \; s \in \mathbb{N}$$

This also means that we can talk about transition maps $\phi_{\infty s}^{-1} \circ \phi_{\infty t}$.

Proof of Step 3 continued. Since

$$\|\phi_{is}^{-1} \circ \phi_{it}\|_{C^{m+1,\alpha}} \leq f_3(n, K, r)$$

for each $i, s \in \mathbb{N}$ by (n3), we have by another compactness argument that $\phi_{\infty s}$ satisfies (n3) (stated earlier). Also

$$\phi_{is}^{-1} \circ \phi_{it} \xrightarrow{C^{m,\beta}} \phi_{\infty s}^{-1} \circ \phi_{\infty t}$$

for each $s, t \in \mathbb{N}$. (I have omitted some details here - see Peterson.). It is also fairly straightforward to show $\phi_{\infty s}$ satisfies (n1).

Proof of Step 3 continued. Hence we have shown $(M_\infty, d_\infty, p_\infty)$ is a $C^{m,\alpha}$ manifold with special charts $\phi_{\infty s}$, $s \in \mathbb{N}$. We now need to construct an appropriate Riemannian metric on this manifold. To do this, we consider the metric $g_{is} := (\phi_{is}^{-1})^* g_i$ for each $i, s \in \mathbb{N}$. Now, ϕ_{is} and g_{is} satisfy (n2) and (n4) respectively. Hence these derivative bounds ensure that

$$g_{is} \xrightarrow{C^{m,\beta}} g_{\infty s}$$

for some metric $g_{\infty s}$. These locally defined metrics patch together to give us a Riemannian metric g_∞ on M_∞. Also, locally, the distance metric induced by $g_{\infty s}$ should coincide with d_∞ (Exercise). Finally it is fairly straightforward to show that $\phi_{\infty s}$, $s \in \mathbb{N}$ satisfies (n1)-(n4), however we won't spell out the details here. QED for Step 3.

Proof of Step 4. We now need to show
$(M_i, g_i, p_i) \xrightarrow{C^{m,\beta}} (M_\infty, g_\infty, p_\infty)$.

Definition: We say two maps F_1, F_2 between subsets of M_∞ and M_i are $C^{m,\beta}$-*close* if all their coordinate compositions

$$\phi_{is} \circ F_1 \circ \phi_{\infty t}^{-1}, \quad \phi_{is} \circ F_2 \circ \phi_{\infty t}^{-1}$$

are $C^{m,\beta}$-close.

Define

$$f_{is} := \phi_{is}^{-1} \circ \phi_{\infty s} : U_{\infty s} \longrightarrow U_{is}$$

for each $s \in \mathbb{N}$. Then f_{is}, f_{it} converge to each other in the $C^{m,\beta}$-topology as $i \to \infty$ for each $s, t \in \mathbb{N}$. Also

$$f_{is}^* g_\infty|_{U_{\infty s}} \longrightarrow g_i|_{U_{is}}$$

in the $C^{m,\beta}$ sense (after pre and post composing with chart maps).

Proof of Step 4. continued.
Therefore, it is sufficient for us to construct maps

$$F_{il} : \ \Omega_{\infty l} := \cup_{s=1}^{l} U_{\infty s} \longrightarrow \Omega_{il} := \cup_{s=1}^{l} U_{is}$$

that get closer to f_{is} as $i \to \infty$ in the $C^{m,\beta}$ sense for each $s = 1, \cdots, l$ (as in the Definition above). We will construct F_{il} by induction on l. Choose a $C^{m+1,\beta}$ partition of unity $(\lambda_s)_{s\in\mathbb{N}}$ subordinate to $(U_{\infty s})_{s\in\mathbb{N}}$.

For $l = 1$, we define $F_{i1} := f_{i1}$. Now suppose that we have constructed F_{il}. If $U_{\infty(l+1)} \cap \Omega_{\infty l} = \emptyset$, we define

$$F_{i(l+1)}(x) := \begin{cases} F_{il}(x) & \text{if } x \in \Omega_{\infty l} \\ f_{i(l+1)} & \text{if } x \in U_{\infty(l+1)}. \end{cases}$$

Proof of Step 4. continued. Now suppose $U_{\infty(l+1)} \cap \Omega_{\infty l} \neq \emptyset$, we do the following: Define

$$\lambda_{\leq l} := \sum_{s=1}^{l} \lambda_s, \quad \lambda_{>l} := \sum_{s=l+1}^{\infty} \lambda_s.$$

Define $F_{i(l+1)} : \Omega_{\infty(l+1)} \longrightarrow \Omega_{i(l+1)}$,

$$F_{i(l+1)}(x) :=$$

$$\phi_{i(l+1)}^{-1} \circ \left(\lambda_{>l}(x) \cdot \phi_{i(l+1)} \circ f_{i(l+1)}(x) + \lambda_{\leq l}(x) \cdot \phi_{i(l+1)} \circ F_{il}\right).$$

The claim is that $F_{i(l+1)}$ gets closer to f_{is} in the $C^{m,\beta}$ sense as $i \to \infty$ for each $s \in \mathbb{N}$. We will not give the details here, but refer to Peterson (page 315). □

Corollary: The norm $\|A \subset (M,g)\|_{C^{m,\alpha},r}$ for compact A is always realized by (n1)-(n4)-charts

$$\phi_s : U_s \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n, \; s \in \mathbb{N}$$

with Q replaced by $\|A \subset (M,g)\|_{C^{m,\alpha},r}$.

Proof: Choose (n1)-(n4)-charts

$$\phi_s^Q : U_s^Q \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n, \; s \in \mathbb{N}$$

for each $Q > \|A \subset (M,g)\|_{C^{m,\alpha},r}$. By the proof of the fundamental theorem, these charts have a limit as $Q \to \|A \subset (M,g)\|_{C^{m,\alpha},r}$. □

Corollary: If $\|(M,g)\|_{C^{m,\alpha},r} = 0$ for some $r > 0$ then M is a flat manifold. If $\|(M,g)\|_{C^{m,\alpha},r} = 0$ for all $r > 0$, then $(M,g) = (\mathbb{R}^n, g_{\text{std}})$.

Proof: The proof even works when $m = \alpha = 0$. By **Claim 0**, part a (page 11), M can be covered by charts $\phi : U \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n$ satisfying:

a. $d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \leq e^Q |x_1 - x_2|$,

b. $d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \geq \min(e^{-Q}|x_1 - x_2|, e^{-Q}(2r - |x_1| - |x_2|))$

where d is the distance metric on M for each $Q > 0$. Now let $Q \to 0$ and use Arzela-Ascoli on ϕ^{-1}. This gives us maps $\phi^{-1} : B_r(0) \longrightarrow M$:

a. $d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \leq |x_1 - x_2|$,

b. $d(\phi^{-1}(x_1), \phi^{-1}(x_2)) \geq \min(|x_1 - x_2|, (2r - |x_1| - |x_2|))$.

Hence ϕ^{-1} is an isometry onto its image (at least near 0) and hence ϕ is a well defined flat chart. Hence M is locally flat.

Alternative Norms

Properties (n1)-(n4)can be replaced by the following properties: We have charts

$$\phi_s : U_s \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n,\ s \in I$$

so that

(n1') The Lebesgue number of $(U_s)_{s\in I}$ is $f_1(n, Q, r)$. Recall that a cover has Lebesgue number λ if any ball of radius λ sits inside an element of this cover.

(n2') $|D\phi_s|, |D\phi_s^{-1}| \leq f_2(n, Q)$.

(n3') $\|\phi_s^{-1} \circ \phi_t\|_{C^{m+1,\alpha}} \leq f_3(n, Q, r)$.

(n4') $r^{|j|+\alpha}\|D^j((\phi_s^{-1})^* g)\|_\alpha \leq f_4(n, Q)$ for all multi-indices j satisfying $0 \leq |j| \leq m$

where f_i are continuous functions, $f_1(n, \infty, r) = 0$ and $f_2(n, 0) = 1$.

For the fundamental theorem of convergence, we have assumed $\alpha > 0$. However, if $m = \alpha = 0$, then $\mathcal{M}^0(n, Q, r)$ (the class of complete, pointed Riemannian n-manifolds satisfying $\|(M, g)\|_{C^0, r} \leq Q$),is only precompact in the pointed Gromov-Hausdorff topology. Also the characterization of flatness still holds.