Homework

(1) (Page 229 Exercise 2). Let $\mathfrak{t} \subset \mathfrak{iso}(M, g)$ be an abelian subalgebra corresponding to the torus subgroup $T^k \subset \mathrm{Iso}(M, g)$. Let $\mathfrak{p} \subset \mathfrak{t}$ be the set of Killing fields whose flow generates a circle action. Show that $\mathfrak{p}$ is a vector space over $\mathbb{Q}$ of dimension k.

(2) (Page 229 Exercise 4). Given two killing fields X, Y, develop a formula for $\Delta g(X, Y)$. Use this to give a formula for the Ricci curvature in a frame consisting of Killing fields.

(3) (Page 230 Exercise 11). Let (M, g) be an n-dimensional Riemannian manifold which is isometric to $(\mathbb{R}^n, g_{\mathrm{std}})$ outside a compact set. Suppose also that $\mathrm{Ric}(M, g) \geq 0$. Show that $(M, g) = (\mathbb{R}^n, g_{\mathrm{std}})$.
Hint: See Corollary 19 in Chapter 6 of Peterson.

(4) Show that $d_{\mathrm{GH}}(X, Y) \geq \frac{1}{2} |\mathrm{diam}(X) - \mathrm{diam}(Y)|$. Therefore, compute the Gromov-Hausdorff distance between $X = S_r$ and $Y = S_R$ for $r, R > 0$ where S_x is the sphere of radius x in $\mathbb{R}^n$ for each $x > 0$, with chord metric (I.e. the distance between two points is realized by a straight line in $\mathbb{R}^n$).

(5) (Peterson Page 331 Exercise 7). Let $\mathcal{C}$ be a class of closed Riemannian manifolds which is compact in the $C^{m,\alpha}$ topology. Show that there exists a function $f(r)$ satisfying $f(r) \to 0$ as $r \to 0$ so that $\|(M, g)\|_{C^{m,\alpha}, r} \leq f(r)$ for each (M, g) in $\mathcal{C}$.

Definition: The *conjugate radius* at a point $p \in M$ is defined to be the supremum over all $r > 0$ so that the exponential map

$$\exp : \{v \in T_pM \ : \ |v| < r\} \longrightarrow M$$

is an immersion. The *conjugate radius* $\mathrm{conj.rad}(M, g)$ of (M, g) is the supremum over all $p \in M$ of the conjugate radius of p.

(6) (Peterson Page 332, Exercise 11). Let $r_0, \Lambda, v > 0$. Consider the class of complete pointed Riemannian n-manifolds (M, g, p) satisfying

$$\mathrm{conj.rad}(M, g) \geq r_0$$
$$|\mathrm{Ric}(M, g)| \leq \Lambda$$
$$\mathrm{Vol} B_1(q) \geq v, \ \forall \ q \in M.$$

Using the techniques from Cheeger's lemma (Lemma 51 from Peterson), show that the injectivity radius is bounded below. Conclude that the class of such manifolds is precompact in the $C^{1,\alpha}$-topology.