Hölder Spaces and Schauder Estimates

MAT 569

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- **Definition:** A *bounded domain* $\Omega \subset \mathbb{R}^n$ is a connected open subset which is bounded.
- We define $C^0(\Omega, \mathbb{R}^k)$ to be the normed vector space of bounded continuous functions $u : \Omega \longrightarrow \mathbb{R}^k$ with norm

$$\|u\|_{C^0} = \sup_{x \in \Omega} |u(x)|.$$

- **Lemma**: $(C^0(\Omega, \mathbb{R}^k), \|\cdot\|_{C^0})$ is a Banach space.

- We can also define the Banach space:

$$(C^m(\Omega, \mathbb{R}^k), \|\cdot\|_k)$$

consisting of C^m functions $u : \Omega \longrightarrow \mathbb{R}$, whose derivatives up to order m are bounded and with norm

$$\|u\|_{C^m} := \sum_{|i| \le m} \sup_{\Omega} |\partial^i u|.$$

- The problem with this space is that $C^m(\Omega, \mathbb{R}^k)$ is not a closed subspace of $C^{m-1}(\Omega, \mathbb{R}^k)$.
- For example $f(x) = |x|$ is in the closure of

$$C^1([-1,1], \mathbb{R}) \subset C^0([-1,1], \mathbb{R}).$$

- For each $\alpha \in (0,1]$, define the C^α-*pseudonorm* of $u : \Omega \longrightarrow \mathbb{R}^k$ to be:

$$\|u\|_\alpha := \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

- For $\alpha = 1$ this is the best Lipschiz constant of u.
- We define the *Hölder space* $C^{m,\alpha}(\Omega, \mathbb{R}^k)$ as the space of functions in $C^m(\Omega, \mathbb{R}^k)$ whose derivatives up to order m have finite C^α-pseudonorm. We have the following norm:

$$\|u\|_{C^{m,\alpha}} = \|u\|_{C^m} + \sum_{|i|=m} \|\partial^i u\|_\alpha$$

  on this space.
- Sometimes we write $\|u\|_{C^{m,\alpha},\Omega}$ if we need to highlight the domain.

Lemma: $(C^{m,\alpha}(\Omega, \mathbb{R}^k), \| \cdot \|_{C^{m,\alpha}})$ is a Banach space. Also for $\beta < \alpha$, the inclusion map $C^{m,\alpha}(\Omega, \mathbb{R}^k) \hookrightarrow C^{m,\beta}(\Omega, \mathbb{R}^k)$ is compact. I.e. closed bounded sets map to compact sets.

Proof The general case follows from the $m = 0$ case (*Exercise*). Let $(u_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in $C^\alpha(\Omega, \mathbb{R}^k)$. Since it is Cauchy in C^0, $u_i \to u$ for some $u \in C^0(\Omega, \mathbb{R}^k)$. For each $x \neq y$, we have

$$\frac{|u_i(x) - u_i(y)|}{|x - y|^\alpha} \longrightarrow \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Since the left hand side is uniformly bounded, $u \in C^\alpha(\Omega, \mathbb{R}^k)$.
We now need to show $u_i \xrightarrow{C^\alpha} u$. Let $\epsilon > 0$. Choose $N > 0$ so that

$$\frac{|u_i(x) - u_j(x) - (u_i(y) - u_j(y))|}{|x - y|^\alpha} \leq \epsilon, \quad \forall\ i, j \geq N,\ x, y \in \Omega.$$

Letting $j \to \infty$, we get: Choose $N > 0$ so that

$$\frac{|u_i(x) - u(x) - (u_i(y) - u(y))|}{|x - y|^\alpha} \leq \epsilon, \quad \forall\ i \geq N,\ x, y \in \Omega$$

and so $u_i \xrightarrow{C^\alpha} u$.

Proof continued.. We now need to show: for $\beta < \alpha$, the inclusion map $C^{m,\alpha}(\Omega, \mathbb{R}^k) \hookrightarrow C^{m,\beta}(\Omega, \mathbb{R}^k)$ is compact. Let $(u_i)_{i\in\mathbb{N}}$ be a bounded sequence in $C^\alpha(\Omega, \mathbb{R}^k)$. It is equicontinuous in $C^0(\Omega, \mathbb{R}^k)$, and so $u_i \xrightarrow{C^0} u \in C^0(\Omega, \mathbb{R}^k)$ after passing to a subsequence by Arezela-Ascoli. Let $v_i := u_i - u$. Then for each $x, y \in \Omega$,

$$\frac{|v_i(x) - v_i(y)|}{|x-y|^\beta} = \left(\frac{|v_i(x) - v_i(y)|}{|x-y|^\alpha}\right)^{\beta/\alpha} \cdot |v_i(x) - v_i(y)|^{1-\beta/\alpha}$$

and so

$$\|v_i\|_\beta \leq (\|v_i\|_\alpha)^{\beta/\alpha} \cdot (2\|v_i\|_{C^0})^{1-\beta/\alpha}.$$

This converges to 0 as $i \to \infty$ since $\beta < \alpha \leq 1$, $u_i \xrightarrow{C^0} u$ and $(\|v_i\|_\alpha)_{i\in\mathbb{N}}$ is bounded. Hence $u_i \xrightarrow{C^\beta} u$.

Elliptic Estimates

- Consider a second order linear differential operator on functions $\Omega \longrightarrow \mathbb{R}$. This is an operator of the form:

$$L(u) = a^{ij}\partial_i\partial_j u + b^i\partial_i u$$

  where $(a^{ij})_{i,j=1}^n$, $(b^i)_{i=1}^n$ are C^α functions $\Omega \longrightarrow \mathbb{R}$ satisfying $a^{ij} = a^{ji}$.
- L is *elliptic* if (a^{ij}) is positive definite.
- Let us fix $\lambda > 0$ so that all eigenvalues of (a^{ij}) are in $[-\lambda, \lambda]$ and so that

$$\|a^{ij}\|_\alpha \leq \lambda^{-1}, \quad \|b^j\|_\alpha \leq \lambda^{-1}.$$

- **Theorem**: Let $\Omega \subset \mathbb{R}^n$ be an open domain of diameter $\leq D$ and $K \subset \Omega$ a subdomain satisfying $d(K, \partial\Omega) > \delta$ for some $\delta > 0$. Let $\alpha \in (0,1)$. Then there is a constant $C = C(n, \alpha, \lambda, \delta, D)$ satisfying

$$\|u\|_{C^{2,\alpha},K} \leq C\left(\|Lu\|_{C^{\alpha},\Omega} + \|u\|_{C^{\alpha},\Omega}\right)$$

$$\|u\|_{C^{1,\alpha},K} \leq C\left(\|Lu\|_{C^{0},\Omega} + \|u\|_{C^{\alpha},\Omega}\right)$$

  Furthermore, if Ω has a smooth boundary and $u = \phi$ on $\partial\Omega$, then there exists $C = C(n, \alpha, \lambda, D)$ so that:

$$\|u\|_{C^{2,\alpha},\Omega} \leq C\left(\|Lu\|_{C^{\alpha},\Omega} + \|\phi\|_{C^{2,\alpha},\partial\Omega}\right).$$

- We won't prove this theorem. However, we will give some ideas of the proof.

- The first step is to prove it in the case $L = \Delta$.
- The second idea is that a^{ij} "look" constant near each point in Ω and hence we we can use the previous step when b^i and f are zero.
- Finally one can rewrite the equation as:

$$Lu = a^{ij}\partial_i\partial_j u + b^i\partial_i u = \partial_i(a^{ij}\partial_j u).$$

Then we have integration by parts:

$$\int_\Omega (\partial_i(a^{ij}\partial_j u))h = -\int_\Omega a^{ij}\partial_j u \partial_i h$$

if $h|_{\partial\Omega} = 0$. One can then use the previous step.

- **Corollary**: Suppose, in addition,

$$\|a^{ij}\|_{C^{m,\alpha}} \leq \lambda^{-1}, \quad \|b^j\|_{C^{m,\alpha}} \leq \lambda^{-1},$$

then there is a constant $C = C(n, m, \alpha, \lambda, \delta, D)$ so that

$$\|u\|_{C^{m+2,\alpha},K} \leq C\left(\|Lu\|_{C^{m,\alpha},\Omega} + \|u\|_{C^{\alpha},\Omega}\right)$$

and on a domain with smooth boundary (as above):

$$\|u\|_{C^{m+2,\alpha},\Omega} \leq C\left(\|Lu\|_{C^{m,\alpha},\Omega} + \|\phi\|_{C^{m+2,\alpha},\partial\Omega}\right).$$

- **Theorem**: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Then the Dirichlet problem:

$$Lu = f, \quad u|_{\partial\Omega} = \phi$$

  has a unique solution $u \in C^{2,\alpha}(\Omega)$ if $f \in C^{\alpha}(\Omega)$ and $\phi \in C^{2,\alpha}(\partial\Omega)$.
- The uniqueness is straightforward from the estimates above (*Exercise*). However, existence is harder.

Harmonic Coordinates

- **Definition:** A *Harmonic corrdinate system* is a coordinate system $(x^1, \cdots, x^n)$ where x^i is a Harmonic function with respect to the Laplacian on (M, g).
- **Theorem:** For any point $p \in M$, there is a harmonic coordinate system $U \longrightarrow \mathbb{R}^n$ satisfying $U \ni p$.
- *Proof:*. Start with a coordinate system $y = (y^1, \cdots, y^n)$ centered at our given point and let $g_{ij} = g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$ be our metric. We need to find a coordinate transformation $y \to x$ so that

$$\Delta x^k = \frac{1}{\sqrt{\det g_{ij}}} \partial_i \left(\sqrt{\det g_{ij}} \cdot g^{ij} \cdot \partial_j x^k \right) = 0.$$

To do this, let $B_\epsilon(0)$ be a small ball in the chart y and solve the Diriclet problem:

$$\Delta x^k = 0, \quad x^k|_{\partial B_\epsilon(0)} = y^k|_{\partial B_\epsilon(0)}.$$

Proof continued: We need to show that x^k are coordinates for $\epsilon > 0$ small enough. We will use the Schauder estimates above to do this. We have:

$$\|x - y\|_{C^{2,\alpha},B_\epsilon(0)} \leq$$

$$C\left(\|\Delta(x-y)\|_{C^\alpha,B_\epsilon(0)} + \left\|(x-y)|_{\partial B_\epsilon(0)}\right\|_{C^{2,\alpha},\partial B_\epsilon(0)}\right)$$

$$= C\|\Delta y\|_{C^\alpha,B_\epsilon(0)}.$$

If we can show

$$\|\Delta y\|_{C^\alpha,B_\epsilon(0)} \to 0, \text{ as } \epsilon \to 0$$

then this would be enough to show that x is a coordinate system near 0. We only need to do this when $\alpha < 1$. To do this, we will assume that $y^k = \exp \circ z^k$ for each k where $z^1, \cdots, z^k$ are orthogonal linear coordinates on T_pM. In this coordinate system, Δ is equal to the flat Laplacian at 0. Hence $\Delta y = 0$ at 0 and so the equation above holds for $\alpha < 1$ since y is smooth. □

- **Lemma**: Let $x : U \longrightarrow \mathbb{R}^n$ be a harmonic coordinate system on (M, g). Then for each smooth function u on U,
  1. $\Delta u = g^{ij}\partial_i\partial_j u$.
  2. There is a universal analytic expression $Q(g, \partial g)$ that is polynomial in g, quadratic in ∂g and with a denominator depending on $\sqrt{\det g_{ij}}$ so that:

$$\frac{1}{2}\Delta g_{ij} + Q(g, \partial g) = -\mathrm{Ric}_{ij}.$$

  $\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j)$.

- *Proof of 1.* $0 = \Delta x^k = \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ij} \cdot \partial_j x^k\right)$

$$= g^{ij}\partial_i\partial_j x^k + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ij}\right) \cdot \partial_j x^k$$

$$= 0 + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ij}\right) \cdot \delta_j^k$$

$$\frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ik}\right).$$

Proof of 1. continued. From the previous slide we have:

$$0 = \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ik}\right).$$

Hence

$$\begin{aligned}\Delta u &= \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ij} \cdot \partial_j u\right)\\ &= g^{ij}\partial_i\partial_j u + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ij}\right) \cdot \partial_j u\\ &= g^{ij}\partial_i\partial_j u.\end{aligned}$$

Proof of 2.
Since x^k is harmonic for each k, we can use the Bochner formula:

$$\frac{1}{2}\Delta g(\nabla x^k, \nabla x^k) = \Delta\left(\frac{1}{2}|x^k|^2\right) = |\mathrm{Hess}(x^k)|^2 + \mathrm{Ric}(\nabla x^k, \nabla x^k).$$

Polarizing this quadratic expression gives:

$$\frac{1}{2}\Delta g(\nabla x^i, \nabla x^j) - g(\mathrm{Hess}x^i, \mathrm{Hess}x^k) = \mathrm{Ric}(\nabla x^i, \nabla x^j).$$

Proof of 2. continued.

Now $\nabla x^k = g^{ij}\partial_j x^k \partial_i = g^{ik}\partial_i$.Hence $g(\nabla x^i, \nabla x^j) = g^{ij}$. and so:

$$\frac{1}{2}\Delta g^{ij} - g(\mathrm{Hess}x^i, \mathrm{Hess}x^k) = \mathrm{Ric}(\nabla x^i, \nabla x^j)$$

Hence

$$\frac{1}{2}\Delta g^{ij} - g(\mathrm{Hess}x^i, \mathrm{Hess}x^k) = g^{ik} \cdot \mathrm{Ric}(\partial_i, \partial_l) \cdot g^{lj}.$$

Since $g_{ik} \cdot g^{kj} = \delta_i^j$, we get

$$0 = \Delta(g_{ik} \cdot g^{kj}) =$$

$$(\Delta g_{ik})g^{kj} + 2\sum_k g(\nabla g_{ik}, \nabla g^{kl}) + g_{ik}\Delta g^{kj}$$

$$(\Delta g_{ik})g^{kj} + 2(\nabla g_{ik}) \cdot (\nabla g^{kl}) + g_{ik}\Delta g^{kj}$$

(here we we can assume that g_{ij} is the standard metric at the origin.)

Proof of 2. continued.

By combining the previous two equations, we get

$$\Delta g_{ij} = -2(\nabla g_{ij})(\nabla g^{kl})g_{lj} - g_{ik}(\Delta g^{kl})g_{lj}$$

$$= -2(\nabla g_{ij})(\nabla g^{kl})g_{lj} - 2(\nabla g_{ik})g(\mathrm{Hess}x^k, \mathrm{Hess}x^l)g_{lj}$$

$$-2(\nabla g_{ik})(\nabla g^{ks})\mathrm{Ric}(\partial_s, \partial_t)g^{tl}g_{lj}$$

$$= -2(\nabla g_{ij})(\nabla g^{kl})g_{lj} - 2(\nabla g_{ik})g(\mathrm{Hess}x^k, \mathrm{Hess}x^l)g_{lj} - 2\mathrm{Ric}(\partial_i, \partial_j)$$

$$= -2Q_{ij}(g, \partial g) - 2\mathrm{Ric}_{ij}.$$

Hence

$$\frac{1}{2}\Delta g_{ij} + Q_{ij}(g, \partial g) = -\mathrm{Ric}_{ij}.$$

□

Let us consider this formula when we have an Einstein metric:

$$\mathrm{Ric}_{ij} = (n-1)kg_{ij}.$$

In this case:

$$\frac{1}{2}g_{ij} = -(n-1)kg_{ij} - Q(g, \partial g).$$

Now if g is only $C^{1,\alpha}$, the formula above makes sense (I.e. using weak derivatives).So if g is $C^{1,\alpha}$, then the LHS is C^{α}, and so g is in fact $C^{2,\alpha}$.Repeating this argument tells us that g is $C^{k,\alpha}$ for each k and so g is smooth. **Conclusion:**

Corollary: Suppose g is a $C^{1,\alpha}$ metric satisfying Einstein's equations and we have a smooth harmonic coordinate system $(x^k)_{k=1,\cdots,n}$, then g is also a smooth metric.