Harmonic Norms and Ricci Curvature

MAT 569

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The Harmonic Norm

Let $A \subset M$ be a subset. The *harmonic norm*, denoted by

$$\|A \subset \|^{\text{harm}}_{C^{m,\alpha},r}$$

is defined in the same way as $\|A \subset \|^{\text{harm}}_{C^{m,\alpha},r}$ but we replace condition (n3) with the condition that ϕ_s is harmonic with respect to the Riemannian metric.

Let us spell out this definition in detail as a reminder.

Definition:Let A be a subset of M. The *Harmonic $C^{m,\alpha}$-norm on the scale r* of A, denoted by

$$\|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}$$

is $\leq Q$,if we can find charts:

$$\phi_s : U_s \subset M \longrightarrow B_r(0) \subset \mathbb{R}^n,\ s \in I$$

so that

(h1) For each $p \in A$, there exists $s \in I$ so that $B_{\frac{1}{10}e^{-Q}r}(p) \subset U_s$.

(h2) $|D\phi_s|, |D\phi_s^{-1}| \leq e^Q$.

(h3) ϕ_s is harmonic with respect to the Riemannian metric. Equivalently, in local coordinates from this chart:

$$\frac{1}{\sqrt{g_{st}}}\partial_i \left(\sqrt{g_{st}} \cdot g^{ij}\right) = 0.$$

(h4) $r^{|j|+\alpha}\|D^j((\phi_s^{-1})^* g)\|_\alpha \leq Q$ for all multi-indices j satisfying $0 \leq |j| \leq m$.

Note that the above definition only requires ϕ_s to be $C^{m+1,\alpha}$ and g to be $C^{m,\alpha}$.

Definition: Any chart satisfying (h1)-(h4) is called a (h1)-(h4)-chart.

Since (n2) + (n4) $\implies$ (n3), we have that each (h1)-(h4)-chart is a (n1)-(n4)-chart. Hence we get the following lemma:

Lemma: Let $0 < r_1 < r_2$, $n, m \in \mathbb{N}$, $\alpha \in (0, 1]$. We have

$$\|A \subset (M, g)\|_{C^{m,\alpha}, r} \leq \|A \subset (M, g)\|^{\text{harm}}_{C^{m,\alpha}, r}$$

Therefore we get the following corollary of the fundamental theorem of convergence theory:

Corollary: Let $Q > 0$, $n \geq 2$, $m \geq 0$, $\alpha \in (0, 1]$ and $r > 0$. Consider the class of complete, pointed Riemannian n-manifolds (M, g, p) satisfying

$$\|A \subset (M, g)\|^{\text{harm}}_{C^{m,\alpha},r} \leq Q.$$

This class is closed in the pointed $C^{m,\alpha}$-topology and compact in the pointed $C^{m,\beta}$-topology for all $\beta < \alpha$.

Proof: The only thing to check here is that harmonic charts converge to harmonic charts. This is fairly straightforward.

Proposition (M. Anderson, 1990) Suppose $A \subset M$ is precompact. Then

1. $\|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r} = \|A \subset (M,\lambda^2 g)\|^{\text{harm}}_{C^{m,\alpha},\lambda r}$ for each $r > 0$.
2. The function $r \longrightarrow \|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}$ is continuous. Also if $m \geq 1$ then this function tends to 0 as $r \to 0$.
3. If $(M_i, g_i, p_i) \stackrel{C^{m,\alpha}}{\longrightarrow} (M_\infty, g_\infty, p_\infty)$, then for every precompact domain $A_\infty \subset M_\infty$, there exists precompact domains $A_i \subset M_i$ for each $i \in \mathbb{N}$ so that

   $$\|A_i \subset (M_i, g_i)\|^{\text{harm}}_{C^{m,\alpha},r} \longrightarrow \|A_\infty \subset (M_\infty, g_\infty)\|^{\text{harm}}_{C^{m,\alpha},r}, \ \forall\ r > 0.$$

   If M_i is closed for each $i \in \mathbb{N} \cup \{\infty\}$ then we can assume $A_i = M_i$ for each $i \in \mathbb{N} \cup \{\infty\}$.
4. $\|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r} = \sup_{p \in A} \|\{p\} \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}$.

Proof: 1. and the first part of 2. are proven in exactly the same way as we did for $\|\cdot\|_{C^{m,\alpha},r}$.

Now we need to show that if $m \geq 1$ then the function

$$r \longrightarrow \|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}$$

converges to 0 as $r \to 0$.

This is done by constructing appropriate harmonic coordinate charts near each $p \in M$. We construct such charts in the same way we did earlier. By using the exponential map, we start with a smooth chart (with coordinates y) so that $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$ at p. On a small ball $B_r(p)$, around p, we solve the Dirichlet problem:

$$\Delta x^k = 0, \quad x^k|_{\partial B_r(0)} = y^k|_{\partial B_r(0)}.$$

One then uses elliptic estimates to get appropriate bounds (h1)-(h4).

Proof of 3. We wish to show: If $(M_i, g_i, p_i) \xrightarrow{C^{m,\alpha}} (M_\infty, g_\infty, p_\infty)$, then for every precompact domain $A_\infty \subset M_\infty$, there exists precompact domains $A_i \subset M_i$ for each $i \in \mathbb{N}$ so that

$$\|A_i \subset (M_i, g_i)\|^{\text{harm}}_{C^{m,\alpha},r} \longrightarrow \|A_\infty \subset (M_\infty, g_\infty)\|^{\text{harm}}_{C^{m,\alpha},r}, \ \forall \ r > 0.$$

First of all we need to construct A_i. This is done in the same way as before. Let is recall these steps: By definition, there is a domain $\Omega \supset A$ so that for all large i we have smooth embeddings $F_i : \Omega \longrightarrow M_i$ satisfying $F_i^* g_i \longrightarrow g_\infty$ in $C^{m,\alpha}$ on Ω. We define $A_i := F_i(A_\infty)$.

Proof of 3. continued.

Let $Q > \|A_\infty \subset (M_\infty, g)\|_{C^{m,\alpha},r}$. Because the norm is continuous with respect to r, there exists $\epsilon > 0$ so that

$$\|A_\infty \subset (M_\infty, g_\infty)\|_{C^{m,\alpha},r+\epsilon} < Q$$

Now choose charts $\phi_s : U_s \subset M_\infty \longrightarrow B_{r+\epsilon}(0) \subset \mathbb{R}^n$ covering A_∞ satisfying (h1)-(h4) with r replaced by $r+\epsilon$. Define

$$U_{is} := F_i(\phi_s^{-1}(B_{r+\epsilon/2}(0))).$$

Define

$$\phi'_{is} := \phi_s \circ F_i^{-1} : U_{is} \subset M_i \longrightarrow B_r(0) \subset \mathbb{R}^n.$$

These charts are NOT harmonic. We have to solve a Dirichlet problem and use elliptic estimates to perturb ϕ'_{is} so that they become harmonic with the right properties. Here we have shrunk $r+\epsilon$ to $r+\epsilon/2$ so that ∂U_{is} has a smooth boundary.

Proof of 3. continued. We now solve the Dirichlet problem:

$$\phi_{is} : U_{is} \longrightarrow \mathbb{R}^n,$$

$$\Delta_{g_i}\phi_{is} = 0,$$

$$\phi_{is} = \phi'_{is} \text{ along } \partial U_{is}.$$

We now need to show that $\phi_{is}, s \in \mathbb{N}$ is a (h1)-(h4)-atlas for i sufficiently large. To do this we need to compare

$$\phi_{is} \circ F_i \circ \phi_{\infty s}^{-1}$$

with the identity map I. Note that these maps agree on $\partial B_{r+\epsilon/2}(0)$. Define $g_{(is)} := (\phi_{is}^{-1})^* g_i$ for each $i \in \mathbb{N} \cup \{\infty\}$. In local coordinates:

$$\Delta_{g_{(is)}} = g_{(is)}^{kl}\partial_k\partial_l + \frac{1}{\sqrt{\det g_{(is)}}}\partial_k\left(\sqrt{\det g_{(is)}} \cdot g_{(is)}^{kl}\right)\partial_l.$$

Proof of 3. continued.

Since $m \geq 1$, we get a C^α bound on the term:

$$\frac{1}{\sqrt{\det g_{(is)}}}\partial_k\left(\sqrt{\det g_{(is)}}\cdot g^{kl}_{(is)}\right).$$

Hence elliptic estimates give us:

$$\begin{aligned} &\|I - \phi_{is}\circ F_i \circ \phi^{-1}_{\infty s}\|_{C^{m+1,\alpha}} \\ \leq\ & C\|\Delta_{g_{(is)}}(I - \phi_{is}\circ F_i\circ\phi^{-1}_{\infty s})\|_{C^{m-1,\alpha}} \\ =\ & C\|\Delta_{g_{(is)}} I\|_{C^{m-1,\alpha}}. \end{aligned}$$

Now

$$\|\Delta_{g_{(is)}} I\|_{C^{m-1,\alpha}} = \left\|\frac{1}{\sqrt{\det g_{(is)}}}\partial_k\left(\sqrt{\det g_{(is)}}\cdot g^{kl}_{(is)}\right)\right\|_{C^{m-1,\alpha}}$$

$$\longrightarrow \left\|\frac{1}{\sqrt{\det g_{(\infty s)}}}\partial_k\left(\sqrt{\det g_{(is)}}\cdot g^{kl}_{(is)}\right)\right\|_{C^{m-1,\alpha}} = \|\Delta_{g_{(\infty s)}} I\|_{C^{m-1,\alpha}} = 0.$$

Proof of 3. continued. Hence ϕ_{is} become (h1) – (h4)-coordinates for large i. Hence

$$\|A_i \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r} < Q$$

for large i. (some details of the proof might be missing, so we refer to Peterson).

Proof of 4. We wish to show:
$\|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r} = \sup_{p\in A} \|\{p\} \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}$. Since there is no transition function condition (n3), we have the following property:

$$\|A \cup B \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}$$

$$= \max\left\{\|A \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}, \|B \subset (M,g)\|^{\text{harm}}_{C^{m,\alpha},r}\right\}.$$

Hence the norm is realized locally and we get our result. □

The good thing about harmonic coordinates is that the metric seems to be controlled by the Ricci curvature.

Lemma (Anderson 1990) Suppose that $|\mathrm{Ric}(M,g)| \leq \Lambda$. Let $r_1 < r_2$, $K \geq \|A \subset (M,g)\|^{\mathrm{harm}}_{C^1, r_2}$, and let $\alpha \in (0,1)$. Then there is a constant $C = C(n, \alpha, K, r_1, r_2, \Lambda)$ so that

$$\|A \subset (M,g)\|^{\mathrm{harm}}_{C^{1,\alpha}, r_1} \leq C.$$

Moreover, if g is an Einstein metric, I.e. $\mathrm{Ric} = kg$, then for each $m \in \mathbb{Z}$, there is a constant $C = C(n, \alpha, K, r_1, r_2, \Lambda, m)$ so that

$$\|A \subset (M,g)\|^{\mathrm{harm}}_{C^{m+1,\alpha}, r_1} \leq C.$$

Proof: We need to bound g_{ij} in a fixed harmonic coordinate chart. To do this we will use the formula:

$$\Delta g_{ij} = -2\mathrm{Ric}ij - 2Q(g, \partial g) \tag{1}$$

combined with elliptic estimates. First, recall that $\Delta g_{ij} = g^{ij}\partial_i\partial_j$. Hence we can use the elliptic estimate:

$$\|g_{ij}\|_{C^{1,\alpha},B(0,r_1)} \leq C\left(\|\Delta g_{ij}\|_{C^0,B(0,r_2)} + \|g_{ij}\|_{C^\alpha,B(0,r_2)}\right)$$

for some $C = C(n, \alpha, K, r_1, r_2)$. Also, by Equation (1) and our assumption $|\mathrm{Ric}| \leq \Lambda$, we get

$$\|\Delta g_{ij}\|_{C^0,B(0,r_2)} \leq 2\Lambda\|g_{ij}\|_{C^0,B(0,r_2)} + \widehat{C}\|g_{ij}\|_{C^1,B(0,r_2)}.$$

Hence by combining the previous two displayed equations:

$$\|g_{ij}\|_{C^{1,\alpha,B(0,r_1)}} \leq C(2\Lambda + \widehat{C} + 1)\|g_{ij}\|_{C^1,B(0,r_2)}.$$

Hence we get a bound on $\|A \subset (M, g)\|^{\mathrm{harm}}_{C^{1,\alpha},r_1}$ from $\|A \subset (M, g)\|^{\mathrm{harm}}_{C^1,r_2}$.

Proof continued: Now consider the case where g is an Einstein metric. We will use a boostrap method.

Since $\mathrm{Ric} = kg$, we get $C^{1,\alpha}$ bounds on Ric from the $C^{1,\alpha}$ bounds on g demonstrated above. Hence by Equation (1), we get a C^{α} bound of Δg_{ij} from $\|g_{ij}\|_{C^{1,\alpha}}$. Hence by elliptic estimates:

$$\|g_{ij}\|_{C^{2,\alpha},B(0,r_1)} \leq C\left(\|\Delta g_{ij}\|_{C^{\alpha},B(0,r_2)} + \|g_{ij}\|_{C^{\alpha},B(0,r_2)}\right)$$

for some $C = C(n, \alpha, K, r_1, r_2)$. This is

$$\leq C \cdot C' \cdot \|g_{ij}\|_{C^{1,\alpha},B_{r_2}(0)}.$$

Hence Δg_{ij} is bounded in $C^{2,\alpha}$ and using the same argument, we get $C^{3,\alpha}$ bounds for g_{ij}. We continue by induction until we get a $C^{m+1,\alpha}$ bound of g_{ij}. $\square$

By combining the above lemma with the fundamental theorem of convergence, we get:

Corollary. Let $n \geq 2$, $Q, r, \Lambda \in (0, \infty)$. Then the class of pointed Riemannian n-manifolds (M, g, p) satisfying

$$\|(M, g)\|^{\text{harm}}_{C^1, r} \leq Q,$$

$$|\text{Ric}| \leq \Lambda$$

is precompact in the pointed $C^{1,\alpha}$-topology for any $\alpha \in (0, 1)$.

We now have stronger convergence theorems (when using harmonic norm).

Theorem (Anderson). Let $n \geq 2$, $\alpha \in (0,1)$ and $\Lambda, i_0, Q > 0$. Then there exists $r = r(n, \alpha, \Lambda, i_0, Q)$ so that for any complete Riemannian manifold (M, g) satisfying

$$|\mathrm{Ric}| \leq \Lambda,$$

$$\mathrm{inj} \geq i_0,$$

we have

$$\|(M,g)\|^{\mathrm{harm}}_{C^{1,\alpha},r} \leq Q.$$

Proof. We argue by contradiction. Therefore, suppose there exists $Q > 0$ and a sequence of complete Riemannian Riemannian manifolds (M_i, g_i),$i \in \mathbb{N}$ so that

$$|\mathrm{Ric}(M_i, g_i)| \leq \Lambda,$$

$$\mathrm{inj}(M_i, g_i) \geq i_0,$$

$$\|(M, g)\|^{\mathrm{harm}}_{C^{1,\alpha}, i^{-1}} > Q.$$

Since the harmonic norm is a continuous function of scale, we can find $r_i \in (0, i^{-1})$ so that

$$\|(M, g)\|^{\mathrm{harm}}_{C^{1,\alpha}, r_i} = Q.$$

Now rescale the manifolds: $\overline{g}_i = r_i^{-1} g_i$. Then $(M_i, \overline{g}_i)$ satisfies:

$$|\mathrm{Ric}(M_i, \overline{g}_i)| \leq r_i \Lambda,$$

$$\mathrm{inj}(M_i, g_i) \geq r_i^{-1} i_0, \quad \|(M, g)\|^{\mathrm{harm}}_{C^{1,\alpha}, 1} = Q.$$

Proof continued. By M. Anderson's 1990 proposition part 4., we can choose $p_i \in M_i$ so that:

$$\|p_i \in (M, g)\|^{\text{harm}}_{C^{1,\alpha},1} \in [Q/2, Q].$$

Now by the previous lemma (Anderson 1990), the $C^{1,\gamma}$-norm (with respect to some r) is bounded for each $\gamma \in (0,1)$. In particular this is true for each $\gamma \in (\alpha, 1)$. Hence by the fundamental theorem of convergence,

$$(M_i, \overline{g}_i, p_i) \xrightarrow{C^{1,\alpha}} (M_\infty, \overline{g}_\infty, p_\infty)$$

for some pointed Riemannian manifold $(M_\infty, \overline{g}_\infty, p_\infty)$ of class $C^{1,\gamma}$. Since the $C^{1,\alpha}$ norm is continuous with respect to the pointed $C^{1,\alpha}$-topology, we also have:

$$\|p_\infty \in (M_\infty, \overline{g}_\infty)\|^{\text{harm}}_{C^{1,\alpha},1} \in [Q/2, Q].$$

Claim: $(M_\infty, \overline{g}_\infty) = (\mathbb{R}^n, g_{\text{std}})$.

Note that if we can prove this claim then we are done, because this contradicts the following fact:

$$\|p_\infty \in (M_\infty, \overline{g}_\infty)\|^{\text{harm}}_{C^{1,\alpha},1} \in [Q/2, Q].$$

Proof of **Claim**: The manifolds in the sequence $(M_i, \overline{g}_i)$, $i \in \mathbb{N}$ are covered by harmonic coordinate charts that converge to harmonic charts of $(M_\infty, \overline{g}_\infty)$. In such charts, the metric satisfies:

$$\frac{1}{2}\Delta \overline{g}_{kl} + Q(g, \partial g) = -\text{Ric}_{kl}.$$

Also, $|\text{Ric}_{kl}| \leq r_i \Lambda$. Since $r_i \to 0$ as $i \to \infty$, we get

$$\frac{1}{2}\Delta \overline{g}_{kl} + Q(g, \partial g) = 0$$

in the limiting harmonic coordinate chart of $(M_\infty, \overline{g}_\infty)$.

Proof of **Claim** *continued.*

Hence $(M_\infty, \overline{g}_\infty)$ is a weak solution to the Einstein equation $\mathrm{Ric} = 0$. The manifold itself admits as smooth coordinate chart, since it is C^1 (we won't prove this.) Hence it can be covered by smooth harmonic coordinate charts and so by the last lemma of the Schader estimate slides, the metric must also be smooth since it satisfies the Einstein equation.

Also $\mathrm{inj}(\overline{M}_i, \overline{g}_i) \to \infty$ as $i \to \infty$, and any geodesic in $(M_\infty, \overline{g}_\infty)$ is a limit of geodesics of $(M_i, \overline{g}_i)$, $i \in \mathbb{N}$ (in the C^0_{loc} sense). One can then use this fact to show $\mathrm{inj}(M_\infty, \overline{g}_\infty) = \infty$. Then claim now follows from the Cheegar-Gromoll splitting theorem (proven Chapter 9 of Peterson).

QED for **Claim** and QED for the proof of our proposition.

We get the following immediate corollary of the theorem above combined with the fundamental theorem of compactness.

Corollary: (Anderson 1990). Let $n \geq 2$, $\alpha \in (0,1)$, $\Lambda, D, i \in (0,\infty)$. The class of closed Riemannian n-manifolds satisfying

$$|\mathrm{Ric}| \leq \Lambda$$

$$\mathrm{diam} \leq D$$

$$\mathrm{inj} \geq i$$

is precompact in the $C^{1,\alpha}$-topology.. Hence there are only finitely many diffeomorphism types of such manifolds.

The proof of the theorem above relied on a characterization of $(\mathbb{R}^n, g_{\text{std}})$ via the splitting theorem. Instead, one can also use volume comparison results instead.

Theorem: (Exercise 5 in Chapter 9 of Peterson). Every complete pointed Riemannian manifold (M, g, p) satisfying

$$\text{Ric}(M, g) \geq 0$$

$$\lim_{r \to \infty} \frac{\text{vol} B_r(p)}{\omega_n r^n} = 1,$$

where ω_n is the volume of the unit ball in $(\mathbb{R}^n, g_{\text{std}})$, is equal to $(\mathbb{R}^n, g_{\text{std}})$.

Anderson has the following related result.

Lemma: (Anderson 1990). For each $n \geq 2$, there exists a constant $\epsilon = \epsilon(n) > 0$, so that if (M, g, p) is a complete Ricci flat pointed Riemannian manifold satisfying

$$\mathrm{vol} B_r(p) \geq (\omega_n - \epsilon) r^n, \ \forall \ r > 0, \tag{2}$$

then it is equal to $(\mathbb{R}^n, g_{\mathrm{std}})$.

Proof: First of all, we replace ϵ with $\epsilon\omega_n$ so that Equation (2) becomes:

$$\mathrm{vol} B_r(p) \geq (1 - \epsilon)\omega_n r^n, \ \forall \ r > 0.$$

This statement is equivalent to the statement:

$$\lim_{r \to \infty} \frac{\mathrm{vol} B_r(p)}{\omega_n r^n} \geq (1 - \epsilon) \tag{3}$$

by the Bishop-Cheeger-Gromov volume comparison lemma stated at the end of the slides on Gromov-Hausdorff convergence.

Proof continued: Also the above statement is true for some point p iff it is true for all points $p \in M$. Also if we rescale the metric then Equation (3) holds for this new metic iff it holds for the hold metric.

Now suppose (for a contradiction) that the statement of our lemma is false. Then, for each $i \in \mathbb{N}$, we can find pointed Ricci flat manifolds (M_i, g_i) so that

$$\lim_{r\to\infty} \frac{\mathrm{vol} B_r(p_i)}{\omega_n r^n} \geq (1 - i^{-1})$$

$$\|(M_i, g_i)\|^{\mathrm{harm}}_{C^{1,\alpha,r}} \neq 0, \ \forall \ r > 0.$$

(Recall that the harmonic norm vanishes only for flat manifolds by an earlier lemma).

Proof continued.

By scaling the metrics g_i to $\overline{g}_i$ and replacing the basepoints p_i with q_i, we can assume:

$$\lim_{r\to\infty} \frac{\mathrm{vol} B_r(q_i)}{\omega_n r^n} \geq (1 - i^{-1})$$

$$\|(M_i, g_i)\|^{\mathrm{harm}}_{C^{1,\alpha},1} \leq 1.$$

$$\|q_i \in (M_i, g_i)\|^{\mathrm{harm}}_{C^{1,\alpha},1} \in [0.5, 1].$$

From the earlier Theorem by Anderson, a subsequence converges in the $C^{1,\alpha}$-topology to a Ricci flat pointed manifold $(M_\infty, \overline{g}_\infty, q_\infty)$.

Proof continued.

Since Ric $= 0$, we can use one of the earlier lemmas by Anderson to show that these manifolds in fact converge in the $C^{m,\alpha}$-topology after passing to a subsequence. Hence:

$$\lim_{r\to\infty} \frac{\mathrm{vol} B_r(q_\infty)}{\omega_n r^n} \geq 1$$

Such a limit is also ≤ 1 by Bishop-Cheeger-Gromov volume comparison lemma.
Hence by the previous lemma, $(M_\infty, \overline{g}_\infty) = (\mathbb{R}^n, g_{\text{std}})$. But this is impossible since the harmonic norm is continuous, and we assumed:

$$\|q_i \in (M_i, g_i)\|^{\text{harm}}_{C^1,\alpha,1} \in [0.5, 1] \ \forall \ i \in \mathbb{N}.$$

□

Corollary: Let $n \geq 2$, $-\infty < \lambda \leq \Lambda < \infty$, and let $D, i_0 \in (0, \infty)$. Then there exists $\delta = \delta(n, \lambda \cdot i_0^2)$ so that the class of closed Riemannian manifolds satisfying:

$$(n-1)\Lambda \geq \mathrm{Ric} \geq (n-1)\lambda,$$

$$\mathrm{diam} \leq D$$

$$\mathrm{vol} B_{i_0}(p) \geq (1-\delta) v(n, \lambda, i_0),$$

where $v(n, k, r)$ is the volume of the radius r ball in the symmetric space S_k^n, is precompact in the $C^{1,\alpha}$-topology for any $\alpha \in (0, 1)$. Hence there are only finitely many diffeomorphism types of such manifolds.

Proof. The proof is similar to the previous theorem with the injectivity radius bound . Suppose (for a contradiction), there exists $Q > 0$ and a sequence of complete pointed Riemannian manifolds (M_i, g_i, p_i),$i \in \mathbb{N}$ so that

$$(n-1)\Lambda \geq \mathrm{Ric}(M_i, g_i) \geq (n-1)\lambda,$$

$$\mathrm{diam}(M_i, g_i) \leq D$$

$$\mathrm{vol}B_{i_0}(p_i) \geq (1-\delta)v(n, \lambda, i_0),$$

$$\|(M_i, g_i)\|^{\mathrm{harm}}_{C^{1,\alpha}, i^{-1}} > Q$$

for some $Q > 0$. Choose $r_i \in (0, i^{-1})$ so that:

$$\|(M_i, g_i)\|^{\mathrm{harm}}_{C^{1,\alpha}, r_i} = Q.$$

Now rescale the manifolds $\overline{g}_i = r_i^{-1} g_i$.

Proof continued. Then,

$$r_i(n-1)\Lambda \geq \mathrm{Ric}(M_i, \overline{g}_i) \geq r_i(n-1)\lambda,$$

$$\mathrm{diam}(M_i, \overline{g}_i) \leq r_i^{-1} D$$

$$\mathrm{vol}_{\overline{g}_i} B_r(p_i) \geq (1-\delta) v(n, \lambda \cdot r_i, i_0 \cdot r_i^{-1/2}),$$

$$\|(M_i, \overline{g}_i)\|^{\mathrm{harm}}_{C^{1,\alpha},1} = Q$$

For large enough i and for $r < i_0 \cdot r_i^{-1}$, we then have:

$$\mathrm{vol}_{\overline{g}_i} B_r(p_i) \geq (1-\delta) v(n, \lambda \cdot r_i, r) \sim (1-\delta)\omega_n r^n.$$

We can adjust our base points p_i so that

$$\|q_i \in (M_i, g_i)\| \| \|^{\mathrm{harm}}_{C^{1,\alpha},1} \in [Q/2, Q].$$

Proof continued

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Now by the fundamental theorem of convergence,

$$(M_i, \overline{g}_i, p_i) \xrightarrow{C^{1,\alpha}} (M_\infty, \overline{g}_\infty, p_\infty)$$

for some pointed Riemannian manifold $(M_\infty, \overline{g}_\infty, p_\infty)$ of class $C^{1,\gamma}$, $\gamma \in (\alpha, 1)$.

Claim: $(M_\infty, \overline{g}_\infty) = (\mathbb{R}^n, g_{\text{std}})$.

If this claim holds, then we get a contradiction since

$$\|p_\infty \in (M_\infty, \overline{g}_\infty)\|^{\text{harm}}_{C^{1,\alpha},1} \in [Q/2, Q].$$

Proof of **Claim***:* The limit space $(M_\infty, \overline{g}_\infty)$ has trivial Ricci curvature and volume bound:

$$\text{vol}_{\overline{g}_\infty} B_r(p_\infty) \geq (1-\delta)\omega_n r^n, \ \forall \ r > 0.$$

By the exercise earlier, this means that $(M_\infty, \overline{g}_\infty) = (\mathbb{R}^n, g_{\text{std}})$.
QED for **Claim**. QED for our corollary.

We now wish to apply some of these compactness results. We will prove some *pinching results*. I.e. statements of the form: if some curvature like quantity is sufficiently close to the curvature of some space, then it is diffeomorphic to such a space.

We will start out with Ricci pinching.

Theorem: Let $n \geq 2$, $i_0, D \in (0, \infty)$ and $\lambda \in \mathbb{R}$. Then there exists $\epsilon = \epsilon(n, \lambda, D, i_0) > 0$ so that any closed Riemannian (M, g) manifold satisfying

$$\text{diam}(M, g) \leq D$$

$$\text{inj}(M, g) \geq i_0$$

$$|\text{Ric}(M, g) - \lambda g| \leq \epsilon$$

is $C^{1,\alpha}$-close to an Einstein metric with Einstein constant λ.

Note that the injectivity radius condition can also be replaced by a volume condition using the previous compactness result.

Proof. Suppose (for a contradiction), there exists a sequence (M_i, g_i), $i \in \mathbb{N}$ satisfying

$$\mathrm{diam}(M_i, g_i) \le D$$

$$\mathrm{inj}(M_i, g_i) \ge i_0$$

$$|\mathrm{Ric}(M_i, g_i) - \lambda g_i| \le 1/i.$$

And g_i is not $C^{1,\alpha}$ close to an Einstein metric with Einstein constant λ. By Anderson's compactness theorem,

$$(M_i, g_i) \xrightarrow{C^{1,\alpha}} (M_\infty, g_\infty).$$

for some (M_∞, g_∞). Now by using harmonic coordinate charts, we have that g_∞ satisfies:

$$\frac{1}{2}\Delta g_\infty + Q(g_\infty, \partial g_\infty) = -\lambda g_\infty$$

in a weak sense. Hence the limiting space is Einstein with Einstein constant λ. Contradiction.

Theorem: $n \geq 2$, $v, D \in (0, \infty)$ and $\lambda \in \mathbb{R}$. Then there exists $\epsilon = \epsilon(n, \lambda, D, i) > 0$ so that any closed Riemannian (M, g) manifold satisfying

$$\mathrm{diam}(M, g) \leq D$$

$$\mathrm{vol}(M, g) \geq v$$

$$|\sec(M, g) - \lambda| \leq \epsilon$$

is $C^{1,\alpha}$-close to a metric of constant curvature.

Proof. Let (M_i, g_i), $i \in \mathbb{N}$ satisfy:

$$\operatorname{diam}(M_i, g_i) \leq D$$

$$\operatorname{vol}(M_i, g_i) \geq v$$

$$|\sec(M_i, g_i) - \lambda| \leq 1/i.$$

Now by a Lemma by Cheeger (page 9 on previous set of slides), the sectional curvature condition tells us that the injectivity radius is bounded below. Hence the previous theorem combined with one of Anderson's compactness results (and comparing Ricci and Sectional curvature) tells us that (after passing to a subsequence),

$$(M_i, g_i) \xrightarrow{C^{1,\alpha}} (M_\infty, g_\infty)$$

where (M_∞, g_∞) is Einstein. We now need to show that (M_∞, g_∞) in fact has constant curvature.

Proof continued.

Let $p_\infty \in M_\infty$, Choose $p_i \in M_i$ so that $p_i \to p_\infty$ (under identification via appropriate $C^{1,\alpha}$ converging harmonic charts). Write $g_i = dr^2 + g_{r,i}$ in polar coordinates for each $i \in \mathbb{N} \cup \{\infty\}$. Then $g_{r,i}$ converge to $g_{r,\infty}$. From chapter 6 of Peterson (Theorem 27 on page 175),

$$\mathrm{sn}^2_{\lambda+\epsilon_i}(r) ds^2_{n-1} \leq g_{r,i} \leq \mathrm{sn}^2_{\lambda-\epsilon_i}(r) ds^2_{n-1}$$

where

- $\epsilon_i \to 0$ as $i \to \infty$,
- sn_κ is a smooth function smoothly depending on κ, (sn_κ is defined on page 12 of Peterson).
- ds^2_{n-1} is the metric on the unit sphere in $\mathbb{R}^n$. Hence we get:

$$\mathrm{sn}^2_\lambda(r) ds^2_{n-1} \leq g_{r,\infty} \leq \mathrm{sn}^2_\lambda(r) ds^2_{n-1}.$$

Hence the limit metric has constant curvature. □