Convergence

MAT 569

April 9, 2020

Gromov-Hausdorff Distance

- **Definition**. Let (X, d) be a metric space and let $A, B \subset X$ be subsets. The *distance* between A and B is defined to be:

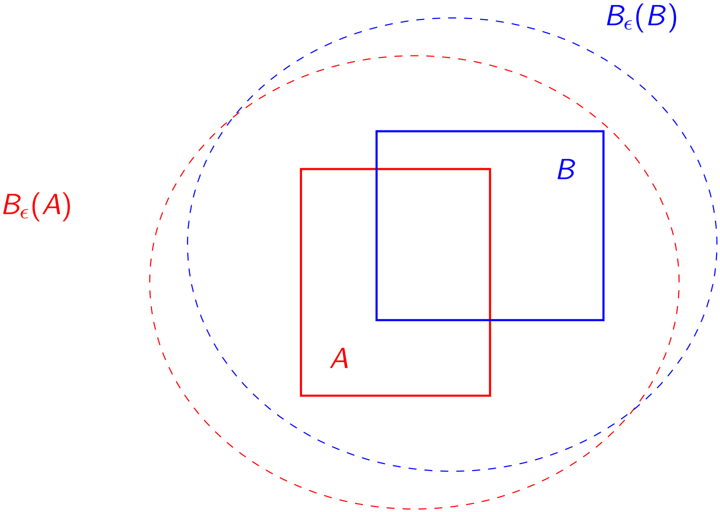
$$d(A, B) := \inf\{d(a, b) \ : \ a \in A, \ b \in B\}.$$

- **Definition**. For $\epsilon > 0$, the ϵ-*neighborhood* of A is defined to be

$$B_\epsilon(A) := \{x \in X \ : \ d(x, A) < \epsilon\}.$$

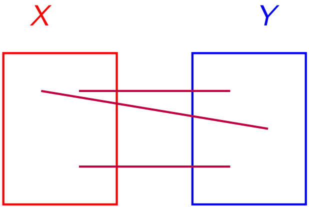
- **Definition**. The *Hausdorff distance* between A and B is defined to be:

$$d_H(A, B) := \inf\{\epsilon > 0 \ : \ A \subset B_\epsilon(B), \ B \subset B_\epsilon(A)\}.$$


$B_\epsilon(B)$
B
$B_\epsilon(A)$
A

- **Lemma**: *(Exercise)*. Suppose A, B are compact. Then the Hausdorff distance between A and B is 0 if and only if $A = B$.
- The Hausdorff distance can be infinite. E.g. Consider two different lines in $\mathbb{R}^2$.
- However, it is finite if A and B are compact (Exercise).

- We will be mainly interested in compact metric spaces. We will also sometimes be interested in *proper metric spaces*: I.e. balls are relatively compact.
- **Definition**. Let (X, d_X), (Y, d_Y) be metric spaces. An *admissible metric* on the disjoint union $X \sqcup Y$, is a metric d satisfying $d|_X = d_X$, $d|_Y = d_Y$. In other words, a metric on $X \sqcup Y$ in which X and Y are isometric subspaces.



- **Definition**. Let X and Y be metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined to be:

$$d_{\mathrm{GH}}(X,Y) := \inf \left\{ d_H(X,Y) \ : \ \begin{array}{c} \text{admissible metrics} \\ d \text{ on } X \sqcup Y \end{array} \right\}.$$

- **Lemma**. $d_{\mathrm{GH}}(X,X) = 0$. *Proof:* Let $\epsilon > 0$ Consider the topological space $[0,\epsilon] \times X$ with the product metric. Consider the subspace $\{0,\epsilon\} \times X = X \sqcup X$. The Gromov-Hausdorff distance between these two copies of X is $\epsilon > 0$. $\square$

Example:. Let A, B be finite sets equipped with the *discrete metric* (I.e. the unique metric satisfying $d(x, y) = 1$ if $x \neq y$). Then $d_{\mathrm{GH}}(A, B)$ is equal to 1 if and only if $A \neq B$.

In other words, the Gromov-Hausdorff metric on the the set of (metric isomorphism classes) of finite sets with the discrete metric is the discrete metric.

Why? Consider a metric d on $A \sqcup B$ and suppose $|A| < |B|$.Suppose every point $a \in A$ is within distance ϵ of a point $b_a \in B$ for some $\epsilon > 0$.Now let $b \in B - \cup_{a \in A}\{b_a\}$.Then by the triangle inequality, $d_H(A, b) \geq 1 - \epsilon$.

Example (Exercise). If X is a metric space and $Y = \{y\}$ is a single point then

$$d_{\mathrm{GH}}(X, Y) = \mathrm{rad}(X)$$

where $\mathrm{rad}(X)$ is the radius of the smallest ball covering X

$$\mathrm{rad}(X) = \inf_{x \in X} \sup_{y \in Y} d(X, y).$$

Example: (Exercise) $d_{\mathrm{GH}}([0,1], \mathbb{R}/\mathbb{Z}) \geq 1/4$.
Why? Suppose that the distance is $l < 1/4$. Then for each $x \in [0,1]$ there exists $y_x \in \mathbb{R}/\mathbb{Z}$ whose distance is at most $l' < 1/4$. Then $d(y_0, y_1) \leq 1/2$ and so by the triangle inequality $d(0,1) < 1$ in $[0,1]$ giving us a contradiction.

Lemma: $d_{\mathrm{GH}}(X,Y)$ is equal to the infimum I over all metric spaces Z of $d_H(X,Y)$ where X and Y are isometric subsets of Z.

Proof: $d_{\mathrm{GH}}(X,Y) \geq I$ since we can choose (Z,d) to be $X \sqcup Y$. Conversely consider some metric space (Z,d) and consider $d_H(X,Y)$. Let $\epsilon > 0$ and consider the space $[0,\epsilon] \times Z$. The induced metric on

$$X \sqcup Y = \{0\} \times X \sqcup \{\epsilon\} \times Y \subset [0,\epsilon] \times Z$$

is admissible.Hence $d_{\mathrm{GH}}(X,Y) \leq I + \epsilon$ for each $\epsilon > 0$. □

- **Definition:** The *diameter* of a metric space X is

$$\operatorname{diam}(X) := \sup_{x,y \in X} d(x,y).$$

- **Lemma:** Let X, Y be metric spaces. Then

$$d_{\mathrm{GH}}(X,Y) \le D := \frac{1}{2}\min(\operatorname{diam}(X), \operatorname{diam}(Y)).$$

Proof: We need to construct an appropriate admissible metric on $X \sqcup Y$. Define the distance between $x \in X$ and $y \in Y$ to be $D/2$. This gives us an admissible metric on $X \sqcup Y$ and hence our inequality. $\square$

Definition: We let $\mathcal{M}$ be the collection of isometry classes of compact metric spaces.

Note that $\mathcal{M}$ is a set. This is because compact metric spaces have cardinality at most $\mathbb{R}$

(Why? any such space is determined by a countable dense subset and the collection of all metrics on a countable metric space at worst a countable number of copies of $\mathbb{R}$).

Goal: Show that $(\mathcal{M}, d_{\mathsf{GH}})$ is a metric space.

Lemma: (*Exercise*). We have

1. $d_{\mathrm{GH}}(X, Y) \geq 0$,
2. $d_{\mathrm{GH}}(X, Y) = d_{\mathrm{GH}}(Y, X)$,
3. $d_{\mathrm{GH}}(X, Z) \leq d_{\mathrm{GH}}(X, Y) + d_{\mathrm{GH}}(Y, Z)$.

As a result, d_{GH} is almost a metric. The hard part is to show that $d_{\mathrm{GH}}(X, Y) = 0$ implies that X is isometric to Y.
We need some definitions, lemmas and a theorem first before we prove this.

- We want a notion of "approximate" isometry.
- **Definition:** Let (X, d_X), (Y, d_Y) be metric spaces and let $\epsilon \geq 0$. A map $f : X \longrightarrow Y$ is an ϵ-*isometry* if

$$|d_Y(f(x), f(x')) - d_X(x, x')| < \epsilon, \quad \forall\ x, x' \in X$$

and for each $y \in Y$, there exists $x \in X$ so that

$$d_Y(y, f(x)) < \epsilon.$$

- In other words, it f almost preserves the metric, and it is almost surjective.

Lemma: Let (X, d_X), (Y, d_Y) be metric spaces satisfying $d_{\mathrm{GH}}(X, Y) < \epsilon$ for some $\epsilon > 0$. Then there is a 2ϵ-isometry from X to Y.

Proof: Let d be an admissible metric on $X \sqcup Y$ so that $d_H(X, Y) < \epsilon$. For each $x \in X$, define $f(x) \in Y$ to be any point satisfying $d(x, f(x)) < \epsilon$. Then

$$d_Y(f(x), f(x')) \leq d(x, f(x)) + d(x, x') + d(x', f(x'))$$

$$\leq d(x, x') + 2\epsilon.$$

Also, $d_Y(f(x), f(x')) \geq -d(x, f(x)) + d(x, x') - d(x', f(x'))$

$$\geq d(x, x') - 2\epsilon.$$

Hence

$$|d_Y(f(x), f(x')) - d_X(x, x')| < 2\epsilon, \quad \forall\ x, x' \in X.$$

I.e. f almost respects the metric.

Proof continued.
We now need to show that f is almost surjective. Let $y \in Y$. Choose $x \in X$ so that $d(x, y) < \epsilon$. Then

$$d(f(x), y) \leq d(x, f(x)) + d(x, y)$$

$$< 2\epsilon.$$

□

Lemma: Conversely suppose that there is an ϵ-isometry f between metric spaces X and Y. Then $d_{\mathrm{GH}}(X, Y) \leq \epsilon$.

Proof: Let $\delta > 0$ be small. We let d be an admissible metric on $X \sqcup Y$ so that for $x \in X$ and $y \in Y$,

$$d(x, y) = \inf_{x' \in X} (d(x, x') + d(f(x'), y)) + \delta.$$

This is a metric (*Exercise*).

Let $y \in Y$. Choose $x \in X$ so that $d(f(x), y) < \epsilon$. Then $d(x, y) < \epsilon + \delta$.Hence $B_{\epsilon+\delta}(Y)$ contains X.Now let $x \in X$.Then $d(f(x), x) \leq \delta$. Hence $B_{\epsilon+\delta}(X)$ contains Y.Hence $d_{\mathrm{GH}}(X, Y) \leq \epsilon$.

- **Definition:** Let (X, d_X),(Y, d_Y) be topological spaces. Let $C^0(X, Y)$ be the space of continuous maps $X \longrightarrow Y$ equipped with the *compact open topology*. This is the topology generated by sets

$$V(K, U) := \{f \in C^0(X, Y) \ : \ f(K) \subset I\}$$

for each compact subset $K \subset X$ and open subset $U \subset Y$.
- **Definition:** A subset $F \subset C^0(X, Y)$ is *equicontinuous* if for each $x \in X$ and each $\epsilon > 0$ there exists a neighborhood U_x so that $d_Y(f(x), f(y)) < \epsilon$ for each $y \in U_x$ and each $f \in F$.
- **Theorem:** *(Arzela-Ascoli)*
Let X be a compact Hausdorff space and let Y be a compact metric space. Then every equicontinuous subset $F \subset C^0(X, Y)$ is relatively compact (I.e. has compact closure).

Lemma: Let X, Y be compact metric spaces and let $(\epsilon_i)_{i\in\mathbb{N}}$ be positive constants converging to 0. Let $f_i : X \longrightarrow Y$ be a ϵ_i-isometry for each $i \in \mathbb{N}$. Then f_i converges in $C^0(X, Y)$ to an isometry.

Proof: The maps $(f_i)_{i\in\mathbb{N}}$ are equicontinuous. Hence $f_i \to f_\infty$ with respect to the compact open topology by Arzela-Ascoli. Hence f_i converges pointwise to f_∞ (*Exercise*). Hence

$$d_Y(f_\infty(x), f_\infty(x')) \leq$$

$$d_Y(f_i(x), f_\infty(x)) + d(f_i(x), f_i(x')) + d_Y(f_i(x'), f_\infty(x'))$$
$$\to d_X(x, x').$$

A similar argument tells us that

$$d_Y(f_\infty(x), f_\infty(x')) \geq d_X(x, x').$$

□

Proof continued.

Hence f_∞ is an isometric embedding from X to Y. We now need to show that f_∞ is surjective.

Let $y \in Y$.Then, there exists $x_i \in X$ so that $d_Y(f_i(x_i), y) < \epsilon_i$ for each $i \in \mathbb{N}$. Now $x_i \to x_\infty$ after passing to a subsequence since X is compact. Hence $f_i(x_i) \to f_\infty(x_\infty)$. Since $f_i(x_i) \to y$, we then get $f_\infty(x_\infty) = y$and so f_∞ is surjective.

□

Proposition: Suppose $X, Y \in \mathcal{M}$ satisfies $d_{\mathrm{GH}}(X, Y) = 0$ then X is isometric to Y.

Proof:
Since $d_{\mathrm{GH}}(X, Y) < 1/i$ for each $i \in \mathbb{N}$, there is a $1/i$-isometry $f_i : X \longrightarrow Y$ for each $i \in \mathbb{N}$.
By the previous lemma, f_i converges to an isometry $X \to Y$. □

- **Definition:** Let X, Y be metric spaces. Define

$$d_{\mathfrak{iso}}(X,Y) := \{\epsilon > 0 \ : \ \exists \ \epsilon\text{-isometries } X \to Y \text{ and } Y \to X\}.$$

- **Proposition** (*Exercise*). $(\mathcal{M}, d_{\mathfrak{iso}})$ is a metric space.
- **Corollary**: $(\mathcal{M}, d_{\mathrm{GH}})$ is a metric space.

Proof:
We only need to show:

$$d_{\mathrm{GH}}(X,Y) = 0 \implies X \text{ is isometric to } Y.$$

This follows from:

$$d_{\mathfrak{iso}}(X,Y) \leq d_{\mathrm{GH}}(X,Y) \leq 2d_{\mathfrak{iso}}(X,Y).$$

□

Example: (Berger spheres).
Consider the unit sphere $S^3 \subset \mathbb{C}^2$. Let $U(1)$ act diagonally on S^3 I.e. we have an action

$$\Phi : U(1) \times S^3 \longrightarrow S^3, \quad (e^{i\vartheta}, x) := e^{i\vartheta} x.$$

Now we 'stretch' the metric in the direction of this action.We did this earlier with Berger-Cheeger perturbations.Let g_{S^3}, $g_{U(1)}$ be the standard metrics on S^3 and $U(1)$. Now consider the product metric $g_{S^3} + \lambda g_{U(1)}$ on $S^3 \times U(1)$, $\lambda > 0$.This descends a metric g_λ on:

$$S^3 = S^3 \times_{U(1)} S^3.$$

As $\lambda \to 0$, we have that S^3 Gromov-Hausdorff converges to $S^2 = S^3/U(1)$.

Lemma: For each compact metric space (X, d_X) and each $\epsilon > 0$, there is a finite metric space (Y, d_Y) so $d_{\mathrm{GH}}(X, Y) < \epsilon$. In other words, every compact metric space can be approximated by a finite one.

Proof: Choose a countable dense subset $A \subset X$. Then $B_\epsilon(a)$, $a \in A$ is a cover and hence has a finite subcover $B_\epsilon(a_i)$, $i = 1, \cdots, k$. The Hausdorff distance between X and $\{a_1, \cdots, a_k\}$ is at most ϵ. By an earlier Lemma, this gives us our result.

Proposition: $(\mathcal{M}, d_{\mathsf{GH}})$ is separable.

Recall that a metric space is *separable* if it contains a countable dense open subset.

Proof: By the previous lemma, finite metric spaces form a dense subset $F \subset \mathcal{M}$. Moreover, finite metric spaces whose associated metrics take rational values gives us a dense subset of F. Hence $(\mathcal{M}, d_{\mathsf{GH}})$ is separable.

Theorem: $(\mathcal{M}, d_{\mathrm{GH}})$ is complete.

Proof: It is sufficient to show that $(\mathcal{M}, d_{\mathfrak{iso}})$ is complete. Let $(X_i, d_i)_{i\in\mathbb{N}}$ be Cauchy. After passing to a subsequence, we can assume that there is a $1/2^i$ isometry:

$$f_i : X_i \longrightarrow X_{i+1}$$

for each i. We let X_∞ be the set of equivalence classes of sequences $(x_i)_{i\in\mathbb{N}}$, where

- $x_i \in X_i$ for each i.
- $d_{i+1}(f_i(x_i), x_{i+1}) \leq 2^{i+1}$ for all sufficiently large i.
- $(x_i)_{i\in\mathbb{N}}$ is *equivalent* to $(x_i')_{i\in\mathbb{N}}$ if $d_i(x_i, x_i') \leq 1/2^{i-1}$ for all sufficiently large i.

We define the metric d_∞ on X_∞ to be

$$d_\infty((x_i)_{i\in\mathbb{N}}, (x_i')_{i\in\mathbb{N}}) = \limsup_{i\in\mathbb{N}} d_i(x_i, x_i').$$

Exercise: Prove that this is a metric.

Proof continued.
Claim: (X_∞, d_∞) is compact.

Proof of claim: Let $(z_i)_{i\in\mathbb{N}}$ be a sequence in X_∞. Then $z_i = (x_{ij})_{j\in\mathbb{N}}$, $x_{ij} \in X_j$. Since X_j is compact for each j, we have, by a diagonal argument, $x_{ij} \to x_{i\infty}$ after passing to a subsequence for each j. After passing to a further subsequence, we can assume that $f_i(x_{ij})$ converges to $f_i(x_{i\infty})$ as $j \to \infty$ for each i. Now $z_\infty := (x_{i\infty})_{i\in\mathbb{N}}$ is in X_∞ and z_i converges to z_∞ as $i \to \infty$ (*Exercise*).

□

Claim: $(X_i, d_i) \xrightarrow{d_{\mathfrak{iso}}} (X_\infty, d_\infty)$.

Proof of Claim: Let $i \in \mathbb{N}$. Construct a map $X_i \to X_\infty$ by sending $x_i \in X_i$ to a sequence $(y_j)_{j\in\mathbb{N}}$ where y_j is any point you like for $j < i$, and $y_j = f_{j-1} \circ \cdots \circ f_i$. This is a $1/2^i$-isometry for each i. □

Hence $X_i \xrightarrow{d_{\mathrm{GH}}} X_\infty$ and so $(\mathcal{M}, d_{\mathrm{GH}})$ is complete.

Pointed Convergence.

- Sometimes we wish to deal with "Gromov-Hausdorff" convergence for non-compact metric spaces.
- To do this, we should think of such a space as a union of compact metric spaces, and then look at the convergence of all of these individual spaces.
- Let us now gives some more precise definitions.

- **Definition:** A *pointed metric space* is a triple (X, d, x) where (X, d) is a metric space and $x \in X$ is a *basepoint*. Sometimes we just write (X, x) if we do not wish to describe the metric.
- Two such spaces are *based isometric* if there is an isometry sending one basepoint to the other.
- **Definition** Let (X, d_X, x), (Y, d_Y, y) be compact pointed metric spaces. The *Gromov-Hausdorff distance* between them is:

$$d_{\mathrm{GH}}((X, d_X, x), (Y, d_Y, y)) :=$$

$$\inf \{ d_H(X, Y) + d(x, y) \ : \ d \text{ is an admissible metric on } X \sqcup Y \} .$$

- **Definition** A *proper pointed metric space* is a pointed metric space (X, d, x) so that $B_R(x)$ is precompact for each $R > 0$.
- We define $\mathcal{M}_*$ to be the set of based isometry classes of pointed metric spaces.
- **Definition**: $B_R(x)$ (resp. $\overline{B}_R(x)$) denotes the open (resp. closed) ball of radius R about x.
- **Definition**: A sequence of pointed metric spaces (X_i, d_i, x_i), $i \in \mathbb{N}$ *Gromov-Hausdorff converges* to $(X_\infty, d_\infty, x_\infty)$ if for each $R > 0$, there exists a sequence $R_i \longrightarrow R$ so that the compact pointed metric spaces $(\overline{B}_{R_i}(x_i), d_i, x_i)$, $i \in \mathbb{N}$ Gromov-Hausdorff converge to $(\overline{B}_R(x_\infty), d_\infty, x_\infty)$.

You might think that all you need is that $(\overline{B}_R(x_i), d_i, x_i)$, $i \in \mathbb{N}$ Gromov-Hausdorff converge to $(\overline{B}_R(x_\infty), d_\infty, x_\infty)$ for each $R > 0$ in the definition above (as stated in the Wikpedia definition). However the following example shows that this is not good.

Example: Let $X_i = \{0, 1 + 1/i\}$ with the metric induced from $\mathbb{R}$ and let 0 be the basepoint for each $i \in \mathbb{N} \cup \{\infty\}$. Then $X_i \xrightarrow{d_{\mathrm{GH}}} X_\infty$. However, $B_1(0) = \{0\} \subset X_i$ does not converge in d_{GH} to $B_1(0) = \{0, 1\} \subset X_\infty$.

In other words, such a modified definition is too strong.

Exercise. Let (X_i, d_i, x_i), $i \in \mathbb{N} \cup \{\infty\}$ be a sequence of compact pointed metric spaces satisfying

$$d_{\mathrm{GH}}((X_i, d_i, x_i), (X_\infty, d_\infty, x_\infty)) \to 0$$

then (X_i, d_i, x_i),$i \in \mathbb{N}$ Gromov-Hausdorff converges to $(X_\infty, d_\infty, x_\infty)$.

Note: For each $R > 0$, you need to find $R_i \to R$ so that

$$(B_{R_i}(x_i), d_i, x_i) \xrightarrow{d_{\mathrm{GH}}} (B_{R_\infty}(x_\infty), d_\infty, x_\infty).$$

Lemma: $\mathcal{C} \subset \mathcal{M}_*$ is precompact if and only if

$$F_R := \left\{ \overline{B}_R(x) \subset X \ : \ (X, x) \in \mathcal{C} \right\} \subset \mathcal{M}$$

is precompact for each $R > 0$.

Proof. Let $G_R := \left\{ (\overline{B}_R(x), x) \subset X \ : \ (X, x) \in \mathcal{C} \right\} \subset \mathcal{M}_*$. If $\mathcal{C}$ is precompact, then G_R is precompact for each $R > 0$ and so F_R is precompact for each $R > 0$.

Now suppose F_R is precompact for each $R > 0$. Fix $R > 0$. Let $(\overline{B}_R(x_i), x_i) \subset (X_i, x_i)$ be a sequence in F_R. After passing to a subsequence, we have $\overline{B}_R(x_i) \xrightarrow{d_{\mathrm{GH}}} Y_R$ for some $Y_R \in \mathcal{M}$. After passing to another subsequence, there is a $1/2^i$-isometry

$$f_i : \overline{B}_R(x_i) \longrightarrow Y_R$$

for each $i \in \mathbb{N}$. After passing to a subsequence, $f_i(x_i) \to y_R$. Hence $(\overline{B}_R(x_i), x_i) \to (Y_R, y_R)$ in $\mathcal{M}_*$.

Proof continued.
For each $R_1 \leq R$ then $(\overline{B}_{R_1}(x_i), x_i) \to (\overline{B}_{R_1}(y_R), y_R)$ (Exercise). Hence (Y_{R_1}, y_{R_1}) is isometric to $(\overline{B}_{R_1}(y_R), y_R)$ for each $R_1 \leq R$. Hence we can define a pointed metric space: $Y := \cup_{R>0} Y_R$ with basepoint y corresponding to y_R for each $R > 0$. Then $(\overline{B}_R(x_i), x_i)$ converges to $(\overline{B}_R(y), y)$ for each $R > 0$. □

Convergence of maps.

- **Definition** Let $f_k : X_k \longrightarrow Y_k$, $k \in \mathbb{N} \cup \{\infty\}$ be a sequence of maps between compact metric spaces. Suppose

$$X_k \xrightarrow{d_{\mathrm{GH}}} X_\infty, \quad Y_k \xrightarrow{d_{\mathrm{GH}}} Y_\infty.$$

  Then we say that $f_k \xrightarrow{d_{\mathrm{GH}}} f_\infty$ if for each sequence of points $x_i \in X_i$, $i \in \mathbb{N} \cup \{\infty\}$ satisfying $(X_i, x_i) \xrightarrow{d_{\mathrm{GH}}} (X_\infty, x_\infty)$, we have $(Y_i, f(x_i)) \xrightarrow{d_{\mathrm{GH}}} (Y_\infty, f(x_\infty))$.
- Note that we can put appropriate metrics on $\sqcup_{i \in \mathbb{N} \cup \{\infty\}} X_i$ and $\sqcup_{i \in \mathbb{N} \cup \{\infty\}} Y_i$ so that $F = \sqcup_{i \in \mathbb{N} \sqcup \{\infty\}} f_i$ is uniformly continuous if and only if $f_k \xrightarrow{d_{\mathrm{GH}}} f_\infty$.

- **Definition**: A sequence of maps $f_k : X_k \longrightarrow Y_k$, $k \in \mathbb{N}$ is *equicontinuous* if for each $\epsilon > 0$, there is a $\delta > 0$ so that

$$B_\epsilon(f_k(x)) \subset f_k(B_\delta(x)), \quad \forall\, x \in X_k,\ k \in \mathbb{N}.$$

- **Theorem**: Every equicontinuous sequence of maps $f_k : X_k \longrightarrow Y_k$, $k \in \mathbb{N}$ between compact metric spaces has a convergent subsequence.

  Proof: After passing to a subsequence, we can assume that $X_k \xrightarrow{d_{\mathrm{GH}}} X_\infty$ and $Y_k \xrightarrow{d_{\mathrm{GH}}} Y_\infty$. Now choose countable dense subset

$$A_i = \{a_{i1}, a_{i2}, \cdots\} \subset X_i$$

  for each $i \in \mathbb{N}$. By a diagonal argument, we have that the pointed spaces (X_k, a_{ki}), $k \in \mathbb{N}$ converge to $(X_\infty, a_{\infty i})$ for some $a_{\infty i} \in X_\infty$ for each $i \in \mathbb{N}$. Also, after passing to a subsequence, $(X_k, f_k(a_{ki})) \longrightarrow (X_\infty, f_\infty(a_{\infty i}))$ for some $f_\infty(a_{\infty i})$ for each i. Hence $f_k \xrightarrow{d_{\mathrm{GH}}} f_\infty$. □

- We need some conditions ensuring that subsets of $\mathcal{M}$ are relatively compact.
- **Definition:** Let X be a compact metric space. Define

$$\mathrm{Cap}(\epsilon) = \mathrm{Cap}_X(\epsilon) := \text{max number of disjoint } \epsilon/2\text{-balls in } X,$$

$$\mathrm{Cov}(\epsilon) = \mathrm{Cov}_X(\epsilon) := \text{min number of } \epsilon\text{-balls to cover } X.$$

- **Lemma**: $\mathrm{Cov}(\epsilon) \leq \mathrm{Cap}(\epsilon)$.

Proof. Let $B_{\epsilon/2}(x_i)$, $i = 1, \cdots, k$ be disjoint where $k = \mathrm{Cap}(\epsilon)$. Suppose (for a contradiction) $x_{k+1} \in X - \cup_{i=1}^k B_\epsilon(x_i)$. Then $B_{\epsilon/2}(x_i)$, $i = 1, \cdots, k+1$ are disjoint and so $k < \mathrm{Cap}(\epsilon)$. Contradiction. Hence $\cup_{i=1}^k B_\epsilon(x_i) = X$ and so $\mathrm{Cov}(\epsilon) \leq \mathrm{Cap}(\epsilon)$. □

Theorem: (Gromov 1980). Let $\mathcal{C} \subset \mathcal{M}$. The following statements are equivalent:

1. $\mathcal{C}$ is precompact.
2. There exists a function $N_1 : (0, \alpha) \longrightarrow (0, \infty)$ so that

$$\mathrm{Cap}_X(\epsilon) \leq N_1(\epsilon), \ \forall \ \epsilon > 0.$$

3. There exists a function $N_2 : (0, \alpha) \longrightarrow (0, \infty)$ so that

$$\mathrm{Cov}_X(\epsilon) \leq N_2(\epsilon), \ \forall \ \epsilon > 0.$$

The proof of this theorem relies on the following Lemma:

Lemma: (*Exercise*). Let $X, Y \in \mathcal{M}$ satisfy $d_{\mathrm{GH}}(X, Y) < \delta$. Then

$$\mathrm{Cap}_X(\epsilon) \geq \mathrm{Cap}_Y(\epsilon + 4\delta)$$

$$\mathrm{Cov}_X(\epsilon) \geq \mathrm{Cov}_Y(\epsilon + 2\delta)$$

(Similarly with X and Y swapped).

Proof of Gromov's theorem:

1. $\Longrightarrow$ 2. Suppose $\mathcal{C}$ is precompact. Let $\epsilon > 0$. Choose $X_1, \cdots, X_k \in \mathcal{C}$ so that

$$\cup_{i=1}^{k} B_{\epsilon/4}(X_k) \supset \mathcal{C}.$$

Define $N_1(\epsilon) := \max_i \mathrm{Cap}_{X_i}(\epsilon/2)$. Then by the previous lemma,

$$\mathrm{Cap}_X(\epsilon) \le \mathrm{Cap}_{X_i}(\epsilon/2) \le N_1(\epsilon), \ \forall\ X \in \mathcal{C}.$$

2. $\Longrightarrow$ 3. Use $N_1 = N_2$.

3. $\implies$ 1.

It is sufficient for us to show that $\mathcal{C}$ is *totally bounded*. I.e. $\mathrm{Cov}_{\mathcal{C}}(\epsilon) < \infty$ for each $\epsilon > 0$. Let $\epsilon > 0$ and let N_1 be as in 2 and define $N := \lfloor N_1(\epsilon/2) \rfloor$.

Let $X \in \mathcal{C}$. Since $\mathrm{Cov}_X(\epsilon/2) \leq N$, there exists $x_1, \cdots, x_N$ in X satisfying $\cup_{i=1}^{N} B_{\epsilon/2}(x_i) \supset X$. This implies that for each $X \in \mathcal{C}$, there is a metric space A_X with N points so that $d_{\mathrm{GH}}(X, A_X) \leq \epsilon/2$ and satisfying $\mathrm{Cov}_A(\epsilon/2) \leq N$.

Now let $S \subset \mathcal{M}$ be the subset of metric spaces A with N points satisfying $\mathrm{Cov}_A(\epsilon/2) \leq N$. Since $d_H(\mathcal{C}, S) \leq \epsilon/2$, it is now sufficient for us to show that $\mathrm{Cov}_S(\epsilon/2) < \infty$. We leave this finial step as an exercise. $\square$

Corollary: Let $n \geq 2$ be an integer, $k \in \mathbb{R}$ and $D > 0$. Then:

1. The collection of closed Riemannian n-manifolds satisfying $\mathrm{Ric} \geq (n-1)k$ and $\mathrm{diam} \leq D$ is precompact in $\mathcal{M}$.
2. The collection of pointed complete Riemannian n-manifolds satisfying $\mathrm{Ric} \geq (n-1)k$ is precompact in $\mathcal{M}_*$.

We will use the following proposition to prove this corollary (we wont prove this proposition though, but refer to Peterson Ch 9):
Lemma: (Bishop-Cheeger-Gromov): Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$ and let $x \in M$. Then

$$f_x : (0, \infty) \longrightarrow (0, \infty), \;\; f_x(R) := \frac{\mathrm{vol} B_R(x)}{v(n,k,R)}$$

is non-increasing and satisfies $f_x(R) \to 1$ as $R \to 0$ where $v(n,k,R)$ is the volume of a ball of radius R in the constant curvature symmetric space S^n_k.

Proof of the Corollary above: We only need to prove 2. Let $\mathcal{C} \subset \mathcal{M}_*$ be the collection of all Riemannian manifolds from 2. By an earlier lemma, it is suffucient to show that

$$F_R := \left\{ \overline{B}_R(x) \subset X \; : \; (X, x) \in \mathcal{C} \right\} \subset \mathcal{M}$$

is precompact for each $R > 0$. Let $R > 0$ and let $\epsilon > 0$. We need to find $N_1(\epsilon)$ so that $\text{Cap}_{\overline{B}_R(x)}(\epsilon) \leq N_1(\epsilon)$ for each $(M, x) \in \mathcal{C}$. Let $x_1, \cdots, x_k \in M$ be points so that $B_\epsilon(x_i)$ are all disjoint. Let $B_\epsilon(x_i)$ be the ball with the smallest volume. Then, by the lemma above (f_x is non-increasing):

$$k \leq \frac{\text{vol}B_R(x)}{\text{vol}B_\epsilon(x_i)} \leq \frac{\text{vol}B_{2R}(x_i)}{\text{vol}B_\epsilon(x_i)} \leq \frac{v(n, k, 2R)}{v(n, k, \epsilon)}.$$

Hence we choose $N_1(\epsilon)$ to be $\frac{v(n,k,2R)}{v(n,k,\epsilon)}$. □

- **Definition**: The *Mikowski dimension* of a metric space X is

$$\dim X := \limsup_{\epsilon \to 0} \frac{\log \mathrm{Cov}_X(\epsilon)}{-\log(\epsilon)}.$$

- **Fact:** The Mikowski dimension of any Riemannian n-manifold is n.
- The proof of the corollary above tells us that the Riemannian manifolds listed converge (after passing to a subsequence) to a metric space whose Mikowski dimension is $\leq n$.

- The following lemma tells us that Mikowski dimension cannot increase under limits.
- **Lemma**: Suppose $N : (0, \infty) \longrightarrow (0, \infty)$ is a continuous function. Let $\mathcal{C}(N) \subset \mathcal{M}$ be the subset of metric spaces X satisfying $\mathrm{Cov}_X(\epsilon) \leq N(\epsilon)$ for each $\epsilon > 0$. Then $\mathcal{C}(N)$ is compact.

  Proof: We already know that $\mathcal{C}(N)$ is precompact. Suppose $(X_i)_{i \in \mathbb{N}}$ is a sequence in $\mathcal{C}(N)$ converging to X_∞. Then

$$\mathrm{Cov}_X(X_\infty) \leq \mathrm{Cov}_{X_i}(\epsilon - 2d_{\mathrm{GH}}(X, X_i)) \leq N(\epsilon - 2d_{\mathrm{GH}}(X, X_i))$$

$$\to N(\epsilon), \text{ as } i \to \infty$$

  since N is continuous. □