Geometric Applications

MAT 569

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Geometric Applications.

We will be using the following proposition, proven in Section 6 of Peterson,

Proposition: Suppose that $|\sec(M,g)| \leq K$ and $\mathrm{inj}(M,g) \geq i_0$. Then on $B_{i_0}(p)$,

$$\max\{|D\exp_p|,\ |D\exp_p^{-1}|\} \leq \exp(f(n,K,i_0))$$

where f only depends on n, K, i_0 and $f(n,K,0) \to 0$ as $K \to 0$.

Hence, we have the following theorem:

Theorem: For every $Q > 0$, there exists $r > 0$ depending only on ι_0 and K so that for any complete Riemannian manifold (M, g) satisfying $|\sec(M, g)| \leq K$ and $\mathrm{inj}(M, g) \geq i_0$, we have $\|(M, g)\|_{C^0, r} \leq Q$. Furthermore, if (M_i, g_i, p_i) satisfy $\mathrm{inj} M_i \geq i_0$ and $|\sec(M_i, g_i)| \leq K_i \longrightarrow 0$, then a subsequence converges in the pointed Gromov-Hausdorff topology to a flat Riemannian manifold satisfying $\mathrm{inj} \geq i_0$.

We would like a stronger version of this theorem since C^0 bounds from C^2 bounds on the Riemannian metric seem a bit strong.

Instead of using exponential charts, we will use distance functions to construct coordinate charts with appropriate bounds.

Lemma: Suppose $\text{inj}(M, g) \geq i_0$, $|\sec(M, g)| \leq K$. Let $p \in M$ and consider the distance function $d_p(x) := d(x, p)$ where d is the distance metric on (M, g). Then d_p is smooth on $B_{i_0}(p) - p$ and its Hessian is bounded in absolute value on $B_{i_0}(p) - B_{i_0/2}(p)$ by $F(n, K, i_0)$.

Proof idea: The smoothness property just follows from the injectivity radius assumption. The derivative bounds come from the following formula from Chapter 6 of Peterson: In polar coordinates,

$$\sqrt{K}\cot(\sqrt{K}r)g_r \leq \text{Hess} d_p \leq \sqrt{K}\coth(\sqrt{K}r)g_r$$

(here $g_r = r^2 ds_{n-1}^2$ where s_{n-1} is the round metric on the unit sphere). . This gives us our result. □

We will now describe the coordinate system near $p \in M$ that we will use. We let d be the distance metric on M. Also suppose $|\sec(M,g)| \leq K$ and $\operatorname{inj}(M,g) \geq i_0$. Let $e_1, \cdots, e_n \in T_pM$ be an orthonormal basis. Let $\gamma_i(t)$ be the unique geodesic satisfying $\gamma_i(0) = p$, $\dot{\gamma}_i(0) = e_i$ for each i. Define

$$d_i : M \longrightarrow [0, \infty), \quad d_i(x) := d(x, \gamma_i(i_0 \cdot (4\sqrt{K})^{-1})).$$

Define

$$\phi(x) := (d_1(x), \cdots, d_n(x)).$$

The previous lemma tells us that the Hessian of ϕ has a uniform bound when restricted to $B_{i_0 \cdot (8\sqrt{K})^{-1}}(p)$. Our potential chart will be ϕ^{-1}, but we have to show that ϕ restricted to a smaller ball is smoothly invertible first. The proof of the next theorem will address this issue.

Theorem: (The Convergence Theorem of Riemannian Geometry). Given $K, i_0 > 0$, there exists $Q, r > 0$ so that for each (M, g) satisfying

$$|\operatorname{sec}(M, g)| \leq K, \quad \operatorname{inj}(M, g) \geq i_0,$$

we have

$$\|(M, g)\|_{C^{1}, r} \leq Q.$$

Hence this class of manifolds is compact in the pointed C^{α} topology for all $\alpha < 1$.

Proof: Let $p \in M$ and let d_i be the distance coordinate system described above and $\phi = (d_1, \cdots, d_n)$ as above. Let $g_{ij} := g(\nabla d_i, \nabla d_j)$. First note that $g_{ij} = \text{id}$ at p. Therefore our bound on the Hessian of the distance functions tell us there exists $Q, \epsilon > 0$ depending only on ι_0, K so that $|D\phi| \leq e^Q$ on $B_\epsilon(p)$ and $|D\phi^{-1}| \leq e^Q$ on $B_\epsilon(0)$. Hence by the implicit function theorem, there exists $\widehat{\epsilon} > 0$ only depending on Q, n so that $\phi^{-1} : B_{\widehat{\epsilon}}(p) \longrightarrow \mathbb{R}^n$ is a smooth chart satisfying (n2). Conditions (n3),(n4) come from the Hessian estimates. Condition (n1) is true since our charts are centered at every point $p \in M$. $\square$

Corollary: (Cheeger 1967) Given $n > 1$ and $k > 0$, the class of Riemannian $2n$-manifolds with $k \leq \sec \leq 1$ is compact in the C^α-topology. Hence there are only finitely many diffeomorphism types of such manifolds.

Proof: This follows directly from the following two propositions:

Theorem: (Klingenberg's Estimate for injectivity radius, 1959). Suppose (M, g) is an even dimensional manifold satisfying $0 < \sec < 1$. Then $\mathrm{inj} > \pi/2$.

Theorem: (Hopf-Rinow, 1931, Myers 1932). Suppose (M, g) is complete and satisfies $\sec > k > 0$. Then M is compact and satisfies $\mathrm{diam}(M, g) \leq \pi/\sqrt{k} = \mathrm{diam}(S^n_k)$.
(Here the diameter bound is used because we are talking about convergence, instead of pointed convergence).

Now we wish to give conditions ensuring that the injectivity radius is bounded below.

Lemma (Cheeger 1967). Let $n \geq 2$, $v, K > 0$ and a compact n-manifold (M, g) satisfying:

$$|\sec(M, g)| \leq K,$$

$$\mathrm{vol} B_1(p) \geq v, \quad \forall\, p \in M.$$

Then $\mathrm{inj}(M, g) \geq i_0$ where i_0 only depends on n, K, v.

Proof: Suppose (for a contradiction), there exists (M_i, g_i) satisfying the assumptions stated in this lemma with $\mathrm{inj}(M_i, g_i) \to 0$ as $i \to \infty$. Choose $p_i \in M_i$ so that $\mathrm{inj}_{p_i} = \mathrm{inj}(M_i, g_i)$. Now consider the sequence of pointed manifolds: $(M_i, \overline{g}_i, p_i)$, $i \in \mathbb{N}$ where $\overline{g}_i = (\mathrm{inj}(M_i, g_i))^{-2} g_i$. Then

$$\mathrm{inj}(M_i, \overline{g}_i) = 1, \quad |\sec(M_i, \overline{g}_i)| \leq (\mathrm{inj}(M_i, g_i))^{-2} K = K_i \to 0.$$

Proof continued. Now by the convergence theorem for Riemannian geometry stated above together with the first theorem stated on these slides we get that $(M_i, \overline{g}_i, p_i)$ converges in the pointed C^α topology to a flat manifold $(M_\infty, \overline{g}_\infty, p_\infty)$ for each $\alpha < 1$.

Claim 1: $\mathrm{inj}_{p_\infty}(M_\infty, \overline{g}_\infty) \leq 1$.

Proof of **Claim 1**: We need the following Theorem by Klingenberg:

Theorem: Suppose $|\sec(M, g)| \leq K$. Then

$$\mathrm{inj}_p(M, g) \geq$$

$$\min\left\{\frac{\pi}{\sqrt{K}}, \frac{1}{2} \cdot (\text{length of shortest geodesic loop based at } p)\right\}.$$

Now since $\sqrt{K_i} \to \infty$, the above theorem tells us that there is a geodesic loop of length 2 based at p_i in $(M_i, \overline{g}_i)$. These geodesic loops must converge to a geodesic loop on $(M_\infty, \overline{g}_\infty)$ and hence $\mathrm{inj}_{p_\infty}(M_\infty, \overline{g}_\infty) \leq 1$. QED for **Claim 1**.

Proof continued. **Claim 2**: $(M_\infty, \overline{g}_\infty) = (\mathbb{R}^n, g_{\text{std}})$. This contradicts **Claim 1**, and so we are done if we can prove **Claim 2**.

Proof. Since $\text{vol}B_1(p_i) \geq v$ inside (M_i, g_i) for each $i \in \mathbb{N}$, there exists $v' = v'(n, K, v)$ so that $\text{vol}B_r(p_i) \geq v' r^n$ for each $r \leq 1$. Hence $\text{vol}B_r(p_i) \geq v' r^n$ for each $r \leq (\text{inj}(M_i, g_i))^{-1}$ inside the rescaled manifold $(M_i, \overline{g}_i)$. Since $(M_i, \overline{g}_i, p_i)$ converges in the pointed C^α topology to a flat manifold $(M_\infty, \overline{g}_\infty, p_\infty)$, we then get $\text{vol}B_r(p_\infty) \geq v' r^n$ for all r.

Hence $(M_\infty, \overline{g}_\infty) = (\mathbb{R}^n, g_{\text{std}})$. Why? If it wasn't, then there would be a covering $\widehat{M} \longrightarrow M_\infty$ equal to a cylinder $\mathbb{R}^n/\mathbb{Z}$ where $\mathbb{Z}$ acts via translations. However, in this case:

$$\lim_{r \to \infty} \frac{\text{vol}B_r(p_\infty)}{r^n} = 0$$

giving us a contradiction. □

Corollary (Cheeger 1967) Let $n \geq 2$, $\Lambda, D, v > 0$. The class of closed Riemannian n-manifolds with

$$|\sec| \leq \Lambda$$

$$\text{diam} \leq D$$

$$\text{vol} \geq v$$

is precompact in the C^α topology for any $\alpha \in (0,1)$. Hence there are only finitely many diffeomorphism types of such manifolds.

By modifying the proof of the convergence theorem in Riemannian geometry, we have the following theorem:

Theorem (S-h Zhu) Given $K, i_0 > 0$, there exists $Q, r > 0$ so that for each (M, g) satisfying

$$\sec(M, g) \geq -K^2, \quad \mathrm{inj}(M, g) \geq i_0,$$

we have

$$\|(M, g)\|_{C^1, r} \leq Q.$$

Hence this class of manifolds is compact in the pointed C^α topology for all $\alpha < 1$.

Proof idea: It suffices to get upper and lower Hessian bounds for the distance function $d_p(x) = d(x, p)$. As before, we get an upper bound

$$\mathrm{Hess} d(x) \leq K \cdot \coth(k \cdot d_p(x)) g_r.$$

Proof continued

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We now need a lower bound. Now suppose $x_0 \in B_{i_0}(p)$. Let γ be the unique unit speed geodesic minimizing the distance between p and x_0 and define $y_0 := \gamma(i_0)$. Define

$$f_{x_0} : B_{i_0}(p) \longrightarrow [0, \infty), \quad f_{x_0}(x) := i_0 - d(x, y_0).$$

Then $f_{x_0}(x_0) = d_p(x_0)$ and $f_{x_0} - d_p \leq 0$ by the triangle inequality. Hence $\text{Hess} d_p \geq \text{Hess} f_{x_0}$ at x_0. Also from chapter 6 in Peterson:

$$\text{Hess} f_{x_0} \geq -K \cdot \coth(d(x_0, y_0) \cdot K) g_r = -K \cdot \coth(K(i_0 - r(x_0))) g_r.$$

This gives us the appropriate two sided bounds (details missing). □

The following example shows that the bounds by Zhu are sharp.

Let

$$f_\epsilon : [0,\infty) \longrightarrow [0,\infty)$$

be a concave function so that:

$$f_\epsilon(r) = \begin{cases} r & \text{if } 0 \le r \le 1-\epsilon \\ \frac{3}{4}r & \text{if } 1+\epsilon \le r. \end{cases}$$

Consider the metrics

$$dr^2 + f_\epsilon^2 d\theta^2$$

on $\mathbb{R}^2$. These metrics have sec ≥ 0 and inj ≥ 1. As $\epsilon \to 0$, we get a $C^{1,1}$-manifold with a $C^{0,1}$-metric (M, g). As a result limit spaces can't be expected to be smoother than the above example.