BOCHNER TECHNIQUE

Throughout this section we will fix a connected Riemannian manifold (M, g). (Sometimes we will drop the assumption that M is connected - but we will state this clearly). We also let n be the dimension of M.

0.1. **Summary.**

0.2. **Killing Fields.** If X is a vector field on a manifold M, we will write ϕ_t^X for the time t flow of X. This flow may not always be well defined. As a result, it is a function from a subset S of M to M where S is the set of points p where the time t flow of p along X is well defined. Such a set S is open. A point $p \in S$ is called a point where *the flow of X is well defined*. We call S the *domain* of ϕ_t^X.

Definition 0.1. A vector field X on a Riemannian manifold (M, g) is called a *Killing field* if its local flows preserve the metric. In other words, if $U \subset X$ is any open subset so that the flow $U_t := \phi_t^X(U)$ is well defined for $|t| < \epsilon$ then $(\phi_t^X)^*(g|_{U_t}) = g|_U$.

Proposition 0.2. *A vector field X on a Riemannian manifold (M, g) is a Killing field if and only if $L_X g = 0$.*

Proof. Let $F^t := \phi_t^X$ be the flow of X. Recall: $L_X g$ is a symmetric $(0, 2)$-tensor on M (I.e. a fiberwise linear map $L_X : TM \otimes TM \longrightarrow \mathbb{R}$ satisfying $L_X(v, w) = L_X(w, v)$) defined by

$$L_X g(v, w) := \frac{d}{dt} g(DF^t(v), DF^t(w))|_t.$$

Note that this is how one defines the Lie derivative of any tensor field. We have:

$$\frac{d}{dt} g(DF^t(v), DF^t(w))|_{t=t_0} = \frac{d}{dt} g(DF^{t-t_0} DF^{t_0}(v), DF^{t-t_0} DF^{t_0}(v)(w))|_{t=t_0}$$

$$\frac{d}{dt} g(DF^s DF^{t_0}(v), DF^s DF^{t_0}(v)(w))|_{s=0}$$

$$= L_X(g)(DF^{t_0}(v), DF^{t_0}(w)).$$

Hence $L_X(g) = 0$ if and only if $\frac{d}{dt} g(DF^t(v), DF^t(w)) = \frac{d}{dt}(F^t)^* g$ is constant. I.e. if and only if the flow of X preserves g. □

Proposition 0.3. *(Skew symmetry property) X is a Killing field if and only if the map*

$$TM \longrightarrow TM, \quad v \longrightarrow \nabla_v X$$

is a skew symmetric $(1, 1)$-tensor. In other words, $(0, 2)$-tensor

$$(v, w) \longrightarrow g(\nabla_v X, w)$$

is anti-symmetric, and hence a 2-form. This 2-form is also exact, and equal to $\frac{1}{2} d\theta_X$ where θ_X is the 1-form given by

$$\theta_X : TM \longrightarrow \mathbb{R}, \quad \theta_X(Y) = g(X, Y).$$

Proof. Recall that a $(1, 1)$ tensor $f : TM \longrightarrow TM$ is *Skew symmetric* if $g(f(v), w) = -g(f(w), v)$ for each $v, w \in T_x M$, $x \in M$.

Recall that $\nabla_v X$ is a vector defined uniquely as follows: Let θ_X be the unique 1-form defined by

$$\theta_X(Y) = g(X, Y), \quad \forall\ Y \in TM.$$

Then $\nabla_v X$ is defined uniquely by the formula:

$$g(\nabla_v X, Y) = \frac{1}{2}\left(L_X(g)(v,Y) + d\theta_X(v,Y)\right).$$

(This is the unique affine connection on TM preserving g which is torsion free). Since $L_X(g)$ is symmetric, we have that the above formula splits the $(0,2)$ tensor

$$T_X : TM^{\otimes 2} \longrightarrow \mathbb{R}, \quad T_X(v,Y) \longrightarrow g(\nabla_v X, Y)$$

uniquely into its symmetric and anti-symmetric components. Hence it is anti-symmetric if and only if its symmetric component $L_X(g)$ vanishes. □

Proposition 0.4. *Let $p \in M$ and let X be a Killing field. Then X is uniquely determined by $X|_p$ and $\nabla X|_p$.*

Proof. Suppose we have another Killing field Y satisfying $Y|_p = X|_p$ and $\nabla Y|_p = \nabla X|_p$. Then $X - Y$ is a Killing field satisfying $X - Y|_p = 0$ and $\nabla X - Y|_p = 0$. Therefore, from now on, we can assume $X|_p = 0$ and $\nabla X|_p = 0$ and we wish to show that the set of points for which X vanishes is both open and closed in M. The set of points for which X vanishes is certainly closed since it is the zero set of a section of the bundle TM. Therefore we need to show that $X^{-1}(0)$ is open.

Let F^t be the flow of X. Since $p \in X^{-1}(0)$, this flow fixes p and hence we get a linear map

$$D^t := DF^t|_p : T_pM \longrightarrow T_pM.$$

We will now show that D^t is the identity map for each t. Now since $DF^t = DF^{t-t_0}DF^{t_0}$ for each $t_0 \in \mathbb{R}$, we have

$$\frac{d}{dt}D^t|_{t=t_0} = \left(\frac{d}{dt}D^t|_{t=0}\right)D^{t_0}.$$

Hence it is sufficient for us to show that

$$\frac{d}{dt}D^t|_{t=0} = 0.$$

Let Y be a vector field defined in a neighborhood of p and define $v := Y|_p$. Since $X|_p = 0$ and $\nabla_X|_p = 0$, we get

$$L_XY|_p = [X,Y]|_p = \nabla_Y X|_p - \nabla_X Y|_p = 0$$

and so

$$\frac{d}{dt}D^tY|_{t=0} = L_XY|_p = 0.$$

Hence D^t is the identity map for each t.

This implies that $F^t = 0$ in a small neighborhood of p. Why? Let $U \subset T_pM$ be a small open neighborhood of 0 so that the exponential map

$$\exp : U \longrightarrow M$$

is an embedding. Now since F^t is an isometry for each t, we have that $F^t(\exp(v)) = \exp D^tv = \exp v$ for each $v \in U$ sufficiently small. Hence F^t is the identity map near v. □

Theorem 0.5. *(Killing submanifold theorem)*

Let $(X_i)_{i\in I}$ be a collection of Killing fields. Then $N := \cap_{i\in I} X_i^{-1}(0)$ is a disjoint union of totally geodesic submanifolds, each of even codimension.

Let $p \in T_pN$ and let $T_p^\perp N := \{v \in T_pM \ : \ g(v,w) = 0 \ \forall \ w \in T_pN\}$ be the corresponding orthogonal subspace. Then the map

$$\rho_i := T_pM \longrightarrow T_p^\perp N, \quad \rho_i(v) := \nabla_v X_i$$

is well defined and $T_pN = \cap_{i\in I}\ker(\rho_i)$.

Also, there exists constants $\alpha_1, \cdots, \alpha_l$ so that $N = X^{-1}(0)$ where $X = \sum_{i=1}^l \alpha_i X_i$.

Recall that a submanifold $S \subset M$ is *totally geodesic* if every geodesic tangent to S is contained in S. Before we prove this theorem, we have the following lemma (proven in the previous course?):

Lemma 0.6. *Let Q be a set of isometries of (M,g). Let*

$$\mathrm{Fix}(Q) := \{x \in M \ : \ f(x) = x \ \ x \in Q\}$$

be the fixed points set of Q. Then $\mathrm{Fix}(Q)$ is a disjoint union ot totally geodesic submanifolds. Also for each $p \in \mathrm{Fix}(Q)$,

$$T_p\mathrm{Fix}(Q) = \{v \in T_pM \ : \ DF(v) = v \quad \forall \ f \in Q\}.$$

Proof. The key idea is to use the exponential map $\exp : U \longrightarrow M$, $U \subset TM$ of the metric to construct charts on $\mathrm{Fix}(Q)$. We will choose U to be a small neighborhood of the zero section of TM so that $\exp|_{U|_x}$ is a smooth embedding. Recall that geodesics $\gamma : I \longrightarrow M$ where $I \subset \mathbb{R}$ is a connected interval containing 0 are uniquely determined by $\gamma(0)$ and $\gamma'(0) = \frac{d}{dt}\gamma(t)|_{t=0}$. Let

$$T := \{v \in TM \ : \ Df(v) = v, \ f \in Q\}.$$

This will be the tangent space of our fixed point set. Let $x \in \mathrm{Fix}(Q)$ and let $T|_x := T \cap TM|_x$. Then $\exp|_{T_x\cap U}$ is a smooth embedding. It is contained in $\mathrm{Fix}(Q)$ for the following reason: Each $f \in Q$ sends geodesics to geodesics. If γ is a geodesic satisfying $\gamma'(0) \in T|_x$ then $f \circ \gamma$ has the same property. Hence $f \circ \gamma = \gamma$ for each such geodesic. Therefore $f \circ \exp|_{T_x\cap U} = \exp|_{T_x\cap U}$ for each $f \in Q$. Hence $\exp|_{T_x\cap U} \subset \mathrm{Fix}(Q)$. Also $\exp|_{T_x\cap U}$ is a neighborhood of x in $\mathrm{Fix}(Q)$ for the following reason: If $x' \in \mathrm{Fix}(Q)$ is sufficiently close to x then there is a unique shortest geodesic γ connecting x with x'. Now $f \circ \gamma$ is another geodesic connecting x with x' for each f in Q. This geodesic has the same length since f is an isometry and so $f \circ \gamma = \gamma$. Hence x' is in the image of $\exp|_{T|_x\cap U}$ if x' is sufficiently close to x.

Hence $\exp|_{T|_x\cap U} : T|_x \cap U \longrightarrow \mathrm{Fix}(Q)$ is a chart for $\mathrm{Fix}(Q)$ for each $x \in \mathrm{Fix}(Q)$. Hence $\mathrm{Fix}(Q)$ is a smooth totally geodesic submanifold of M. □

Proof of Theorem 0.5. Let F_i^t be the time t flow of X_i for each $i \in I$. Then by the previous lemma, the fixed point set

$$S := \mathrm{Fix}(F_i^t, t \in \mathbb{R}) = \{p \in M \ : \ F_i^t(p) = p, \ \forall \ t \in \mathbb{R}, \ i \in I\}.$$

is a disjoint union of totally geodesic submanifolds whose tangent space is

$$TS = \{v \in TM \ : \ DF_i^t(v) = v, \quad \forall \ t \in \mathbb{R}, \ i \in I\}. \tag{0.1}$$

Let v be any vector field on M and let $p \in S$. Then $[v, X_i]|_p = \nabla_{X_i|_p} v - \nabla_{v|_p} X_i$ and so

$$\nabla_{v|_p} X_i = -\nabla_{X_i|_p} v - [v, X_i]|_p = [X_i, v]|_p = \frac{d}{dt}(F_i^t v|_p)|_{t=0}.$$

since $p \in S \subset X_i^{-1}(0)$. Note gradients are always Lie brackets when the vector field X vanishes - I.e. the metric does not matter here.

Therefore by Equation (0.1), $\nabla_{v|_p} X_i = 0$ for each $i \in I$ if and only if $v|_p \in TS$. Also if $w \in T_pS$ and since $F_i^t w = w$ we get

$$g(\nabla_{v|_p} X_i, w) = g(\frac{d}{dt}(F_i^t v|_p)|_{t=0}, w)$$

$$= g(\frac{d}{dt}(F_i^t v|_p)|_{t=0}, w) + g(v, \frac{d}{dt}(F_i^t w)|_{t=0}) = \frac{d}{dt} g(F_i^t v, F_i^t w)|_{t=0} = 0.$$

Hence

$$\nabla_v X_i \in T_p S^\perp$$

for each $v \in T_pM$. Hence we have well define skew symmetric maps:

$$\rho_i : T_pM \longrightarrow T_p^\perp S, \quad \rho_i(v) := \nabla_v X_i, \ i \in I$$

(Proposition (Skew symmetry property)). Also since $v \in T_pM$ satisfies

$$v \in T_pS \quad \text{if and only if } \nabla_v X_i = 0 \quad \forall \ i \in I,$$

we have that

$$T_pS = \cap_{i \in I} \ker(\rho_i).$$

Now by a linear algebra exercise, we can find $i_1, \cdots, i_l$ in I and constants $\alpha_1, \cdots, \alpha_l$ so that

$$\cap_{i \in I} \ker(\rho_i) = \ker(\rho)$$

where $\rho = \sum_{j=1}^l \alpha_j \rho_{i_j}$. Let $X := \sum_{j=1}^l \alpha_j X_{i_j}$. Then $\nabla X|_{T_p^\perp S} = \rho$. Hence we have a well defined map:

$$\nabla X : T_p^\perp S \longrightarrow T_p^\perp S, \quad v \longrightarrow \nabla_v X.$$

Since the kernel ρ is T_pS, we get that this map is invertible. Since this map is also skew symmetric, we get that the dimension of $TS^\perp$ is even (since non-degenerate skew symmetric forms only exist on even dimensional vector spaces). Hence the codimension of S is even dimensional. $\square$

Definition 0.7. We define $\mathfrak{iso}(M, g) \subset \mathrm{Vect}(X)$ to be the subspace of Killing fields of (M, g). We define $\mathrm{Iso}(M, g) \subset \mathrm{Diff}(M)$ to be the subgroup of isometries of M.

Theorem 0.8. *$\mathfrak{iso}(M, g)$ is a Lie subalgebra of $\mathrm{Vect}(M)$ (with the usual Lie bracket) whose dimension is $\leq \frac{n(n+1)}{2}$.*

Proof. Now $L_{[X,Y]} = [L_X, L_Y]$ and so if $L_X g = 0$ and $L_Y g = 0$ then $L_{[X,Y]} g = 0$. Hence $\mathfrak{iso}(M, g)$ is a Lie subalgebra of $\mathrm{Vect}(M)$. Now let $p \in M$. We have shown earlier that there is an injective map

$$\mathfrak{iso}(M, g) \longrightarrow T_pM \times \wedge^2 T_p^* M, \quad X \longrightarrow (X|_p, d\theta_X|_p)$$

is an injective homomorphism where $\theta_X(-) = g(\nabla_v X, -)$. This gives us our dimension restriction since the dimension of $T_pM \times \wedge^2 T_p^* M$ is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$. $\square$

Theorem 0.9. *Suppose that (M, g) is complete (I.e. the geodesic flow is well defined for all time). $\mathrm{Iso}(M, g)$ is naturally a Lie group (with respect to the compact open topology) whose Lie algebra at the identity is $\mathfrak{iso}(M, g)$.*

Proof. We will not prove that $\mathrm{Iso}(M,g)$ is a Lie group (This is the Myers-Steenrod theorem). To show that $\mathfrak{iso}(M,g)$ is the Lie algebra of $\mathrm{Iso}(M,g)$ it is sufficient to show that the 1-parameter subgroup generated by $X \in \mathfrak{iso}(M,g)$ lies in $\mathrm{Iso}(M,g)$. But this was proven above. $\square$

Before we go to an example, let us recall some facts about curvature.

Definition 0.10. The *curvature tensor* is a $(1,3)$ tensor given by

$$R : TM^{\otimes 3} \longrightarrow TM, \quad R(X,Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

This is sometimes written as a $(0,4)$ tensor:

$$R : TM^{\otimes 4} \longrightarrow \mathbb{R}, \quad R(X,Y,Z,W) := g(R(X,Y)Z,W).$$

Now R is skew symmetric in the first two and last two entries, and so we can also view it as fiberwise linear map:

$$R : \wedge^2 TM \otimes \wedge^2 TM \longrightarrow \mathbb{R}, \quad (X \wedge Y, Z \wedge W) \longrightarrow R(X,Y,Z,W).$$

For any two vectors $v, w \in T_pM$, define $\mathrm{Area}^{\square}(v,w) := g(v,v)g(w,w) - g(v,w)^2$. This is the area of the parallelogram spanned by v, w. We define the *sectional curvature* of (M,g) to be the $(0,2)$ tensor

$$\sec : TM^{\otimes 2} := \frac{R(w \wedge v, v \wedge w)}{\mathrm{Area}^{\square}(v,w)}.$$

This only depends on $\mathrm{Span}(v,w)$ and hence if 2-plane(M) is the subspace of 2-planes in TM then we have

$$\sec : \text{2-plane}(M) \longrightarrow \mathbb{R}, \quad \sec(\mathrm{Span}(v,w)) := \sec(v,w).$$

Recall that we have natural simply connected spaces S^n_k of constant sectional curvature k for each $k \in \mathbb{R}$.

(1) If $k > 0$ then S^n_k is the round sphere of curvature $k = r^{-2}$ where r is the radius of this sphere.
(2) If $k = 0$ then S^n_k is just $\mathbb{R}^n$ with the flat metric.
(3) If $k < 0$ then S^n_k can be viewed as the upper half plane

$$M = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \ : \ x^n > 0\}$$

with the metric

$$g = \left(\frac{1}{x^n}\right)^2 ((dx^1)^2 + \cdots + ((dx^n)^2).$$

A more uniform view of S^n_k would be as follows: Consider the $(0,2)$-tensor

$$g := (dx^0)^2 + k\left((dx^1)^2 + \cdots + (dx^n)^2\right)$$

on $\mathbb{R}^{n+1}$. If we think of this as a 'metric' on $\mathbb{R}^{n+1}$, then we can look at radius k sphere

$$S^n_k = \left\{(x^0, \cdots, x^n) \in \mathbb{R}^{n+1} \ : \ (x^0)^2 + k((x^1)^2 + \cdots + (x^n)^2) = k^2\right\}$$

with the induced metric $g|_{S^n_k}$. These spaces are isometric to the ones above.

Example 0.11. We have $\dim(\mathfrak{iso}(S^n_k)) = \frac{n(n+1)}{2}$ (from the previous course, or just via linear algebra). This is the maximal dimension and hence the bound above on the dimension of $\mathfrak{iso}(S^n_k)$ is sharp.

Example 0.12. In the previous course, we showed that every complete Riemannian manifold of constant sectional curvature has universal cover equal to a product $S := \prod_{i=1}^m S_k^n$. Hence such a space is equal to S/Γ where $\Gamma \subset \mathrm{Iso}(S)$ acts freely and discontinuously on S. Hence $G := \mathrm{Iso}(S/\Gamma)$ is the subgroup of $\mathrm{Iso}(S)$ consisting of those isometries that commute with each element of Γ. If $\dim(G)$ is maximal, then G is a Lie group containing the connected component of $\mathrm{Iso}(S)$ containing the identity. Therefore the only elements of $\mathrm{Iso}(S)$ commuting with all elements of G are $\pm I$ and hence $G \subset \pm I$. Now $-I$ only acts freely on the sphere (I.e. when $k > 0$). Hence the only space of constant sectional curvature so that $\dim(\mathrm{Iso})$ is maximal and with non-trivial fundamental group is $\mathbb{R}P^n$.

In fact, we have the following proposition:

Proposition 0.13. *If (M, g) is complete and $\dim(\mathrm{Iso}(M, g))$ is maximal (I.e. $\dim(\mathrm{Iso}(M, g)) = \frac{n(n+1)}{2}$). Then (M, g) has constant sectional curvature.*

Before we prove this proposition, we need a new (and important definition).

Definition 0.14. The *Frame bundle* of (M, g) is the principal $O(n)$-bundle FM constructed as follows: As a set, FM is the collection of points $(p, e_1, \cdots, e_n)$ where $p \in M$ and $e_1, \cdots, e_n \in T_pM$ is an orthonormal basis with respect to g. The bundle maps is the map,

$$\pi : FM \to M, \quad (p, e_1, \cdots, e_n) \to p.$$

The local trivializations defining this bundle are constructed as follows: Let $x_1, \cdots, x_n$ be a chart $U \subset M$. By the Gram-Schmidt process, we can find smooth sections $s_1, \cdots, e_n$ of $TM|_U$ so that $s_1(p), \cdots, s_n(p)$ is an orthonormal basis for each $p \in U$. Then we have a natural map

$$\tau + U : \pi^{-1}(U) \longrightarrow U \times O(n), \quad \tau(p, e_1, \cdots, e_n) = (p, A_p)$$

where $A_p \in O(n)$ is the unique orthogonal matrix sending $e_1, \cdots, e_n$ to $s_1(p), \cdots, s_n(p)$. These trivializations τ_U over all charts U in M give us the collection of trivialization defining our frame bundle FM.

Proof of Proposition 0.13. Any $F \in \mathrm{Iso}(M, g)$ gives us an induced map

$$\widetilde{F} : FM \longrightarrow FM, \quad \widetilde{F}(p, DF(e_1), \cdots, DF(e_n).$$

Such a map does not have any fixed points since isometries are uniquely determined by their action on a tangent plane. Hence the Lie group $\mathrm{Iso}(M, g)$ acts freely on FM. This group acts smoothly on FM as well, and so each orbit is a submanifold of dimension $\frac{n(n+1)}{2}$. Since the dimension of FM is also $\frac{n(n+1)}{2}$, this implies that this orbit contains a connected component of FM.

If M is orientable then FM has two connected components (corresponding to both orientations of the frames) and if M is non-orientable then it has only one.

In either case, one can show that $\mathrm{Iso}(M, g)$ sends each 2-plane in TM to any other 2-plane. Since each 2-plane is mapped to any other 2-plane via an isometry, we get that (M, g) must have constant sectional curvature. □

Killing Fields in negative Ricci Curvatre. Let us recall facts about Ricci curvature first. Recall that the Ricci curvature should be thought of as the 'Laplacian' of the metric g. Recall the following definition:

Definition 0.15. Let $f : M \longrightarrow \mathbb{R}$. Define ∇f to be the vector field uniquely defined by

$$g(\nabla f, w) = df(w), \quad \forall\ w \in TM.$$

We define the *Hessian* of f to be the symmetric $(0,2)$-tensor:

$$\mathrm{Hess}(f) := \frac{1}{2} L_{\nabla f} g$$

or, equivalently, the $(1,1)$ tensor:

$$S : TM \longrightarrow TM, \quad S(X) := \nabla_X \nabla f.$$

The *Laplacian of* f is defined to be the function

$$\Delta f := \mathrm{tr}(S) = \sum_{i=1}^{n} g(\nabla_{e_i} \nabla f, e_i) = \frac{1}{2} \sum_{i=1}^{n} (L_{\nabla f}) g(e_i, e_i).$$

Definition 0.16. The *Ricci curvature* is the symmetric $(0,2)$ tensor equal to the trace of R with respect to the first and fourth entries. In other words, if $e_1, \cdots, e_n \in T_pM$ is an orthonormal basis then

$$\mathrm{Ric}(v, w) = \mathrm{tr}(x \to R(x, v)w) = \sum_{i=1}^{n} g(R(e_i, v)w, e_i)$$

$$= \sum_{i=1}^{n} g(R(v, e_i)e_i, v) = \sum_{i=1}^{n} g(R(e_i, w)v, e_i).$$

If I took the trace with respect to, say, the first and second entries, I would get 0 because R is anti-symmetric in the first two entries. As a result, Ricci curvature is the only reasonable way to take the trace of R.

Definition 0.17. Equivalently, the *Ricci curvature* is a $(1,1)$ tensor given by

$$\mathrm{Ric}(v) := \sum_{i=1}^{n} R(v, e_i)e_i.$$

These two definitions are equivalent after identifying the $(0,2)$ tensor with the $1-1$ tensor in the usual way (I.e. by converting the second parameter from a vector to a covector using the metric g).

Definition 0.18. We write $\mathrm{Ric} \geq k$ if all the eigenvalues of the $(1,1)$-tensor Ric are $\geq k$. Or equivalently, if

$$\mathrm{Ric}(v, v) \geq kg(v, v),\ \forall\ v \in TM.$$

If $\mathrm{Ric}(v) = kv$ for some v, or equivalently $\mathrm{Ric}(v, w) = kg(v, w)$ then we say that (M, g) is an *Einstein manifold* with *Einstein constant* k.

Example 0.19. S^n_k is an Einstein manifold with Einstein constant $(n-1)k$.

Definition 0.20. For a $(1,1)$-tensor T, its *norm* $|T|$ at $p \in M$ is defined by

$$|T|^2 := \mathrm{tr}(T \circ T^*) = \sum_{i=1}^{n} g(T(e_i), T(e_i))$$

where $e_1, \cdots, e_n$ any orthonormal basis at p and T^* is the adjoint of T (I.e. the unique $(1,1)$ tensor satisfying $g(v, Tw) = g(Tvw)$ for each v, w).

In other words, if you think T as a matrix with respect to an orthogonal basis, then its norm is the sum of the squares of all the entries of this matrix.

Proposition 0.21. *(Killing Field derivative identities) Let X be a Killing field on (M, g) and consider the function*

$$f : M \longrightarrow \mathbb{R}, \quad f = \frac{1}{2}|X|^2.$$

Then

(1) $\nabla f = -\nabla_X X$.
(2) $\mathrm{Hess} f(V, V) = g(\nabla_V X, \nabla_V X) - R(V, X, X, V)$.
(3) $\Delta f = |\nabla X|^2 - \mathrm{Ric}(X, X)$.

Proof. **Proof of (1)**:

$$df(V) = \nabla_V f = \frac{1}{2}\nabla_V(g(X, X)) = \frac{1}{2}\left(g(\nabla_V X, X) + g(X, \nabla_V X)\right) = g(\nabla_V X, X) = -g(\nabla_X X, V)$$

The last equality follows from the Skew symmetry property of the tensor:

$$(v, w) \longrightarrow g(\nabla_v X, w).$$

(See Proposition 0.2).

Since the above equality is true for each $V \in TM$, we get $\nabla f = -\nabla_X X$.

Proof of (2): We will repeatedly use the fact that $V \longrightarrow \nabla_V X$ is a skew symmetric $(1, 1)$-tensor.

$$\begin{aligned}
\mathrm{Hess}(f(V, V)) &= g(\nabla_V(\nabla f), V) = g(\nabla_V(-\nabla_X X), V) \\
&= -g(R(V, X)X, V) - g(\nabla_X \nabla_V X, V) - g(\nabla_{[V,X]} X, Y) \\
&= -R(V, X, X, V) - g(\nabla_X \nabla_V X, V) + g(\nabla_{\nabla_X V} X, V) - g(\nabla_{\nabla_V X} X, V) \\
&= -R(V, X, X, V) - g(\nabla_X \nabla_V X, V) - g(\nabla_V X, \nabla_X V) + g(\nabla_V X, \nabla_V X) \\
&= -R(V, X, X, V) - D_X(g(\nabla_V X, V)) + g(\nabla_V X, \nabla_V X) \\
&= -R(V, X, X, V) + g(\nabla_V X, \nabla_V X)
\end{aligned}$$

since $g(\nabla_V X, V) = 0$ by skew symmetry.

Proof of (3):

$$\begin{aligned}
\Delta f &= \sum_{i=1}^{n} \mathrm{Hess} f(E_i, E_i) \\
&\overset{(2)}{=} \sum_{i=1}^{n} g(\nabla_{E_i} X, \nabla_{E_i} X) - \sum_{i=1}^{n} R(E_i, X, X, E_i) \\
&= \sum_{i=1}^{n} g(\nabla_{E_i} X, \nabla_{E_i} X) - \mathrm{Ric}(X, X) \\
&= |\nabla X|^2 - \mathrm{Ric}(X, X).
\end{aligned}$$

□

Definition 0.22. Let $\mathrm{Tens}^{r,s}$ be the vector space (over $\mathbb{R}$) of (r,s) tensor fields on M. We have

$$\mathrm{Tens}^{\bullet,\bullet} := \oplus_{r,s\geq 0}\mathrm{Tens}^{r,s}$$

is an algebra with multiplication given by tensor product. This algebra is $\mathbb{Z}\times\mathbb{Z}$ graded. For any vector field V, we define the vector space map

$$\nabla_V : \mathrm{Tens}^{r,s} \longrightarrow \mathrm{Tens}^{r,s}$$

to be the unique vector space map satisfying:

(1) $\nabla V f = df(V)$ for each $f \in C^\infty(M) = \mathrm{Tens}^{0,0}$,
(2) $\nabla_V \nu = i_V d\nu$ for each $\nu \in \Omega^1(M) = \mathrm{Tens}^{0,1}$,
(3) $\nabla_V X$ on $\mathrm{Vect}(M) = \mathrm{Tens}^{1,0}$ coincides with the covariant derivative as usual
(4) Also, ∇_V satisfies the Leibniz rule

$$\nabla_V(S\otimes T) = (\nabla_V S)\otimes T + S\otimes(\nabla_V T)$$

for each $S \in \mathrm{Tens}^{r,s}(M)$, $T \in \mathrm{Tens}^{r',s'}(M)$.

If we have a $(0,r)$ tensor or a $(1,r)$ tensor S, then

$$\nabla_V S(Y_1,\cdots,Y_r) = \nabla_V((S(Y_1,\cdots,Y_r))) - \sum_{i=1}^r S(Y_1,\cdots,\nabla_V Y_i,\cdots,Y_r)$$

for each r-tuple of vector fields.

Recall that a tensor S is *parallel* if $\nabla S = 0$.

Proposition 0.23. *(Bochner 1946) Suppose (M,g) is compact, oriented and has $\mathrm{Ric}\leq 0$. Then every Killing field is parallel. Furthermore, if $\mathrm{Ric}<0$ then there are no non-trivial Killing fields.*

Proof. Let X be a Killing field and let $f := \frac{1}{2}|X|^2$. Then

$$0 = \int_M \Delta f d\mathrm{vol} = \int_M |\nabla X|^2 - \mathrm{Ric}(X,X) d\mathrm{vol} \geq \int_M |\nabla X|^2 d\mathrm{vol} \geq 0.$$

Hence $|\nabla X| = 0$ and so X must be parallel. The above inequalities then tell us

$$\int_M \mathrm{Ric}(X,X)d\mathrm{vol} = 0.$$

Since we are assuming $\mathrm{Ric}\leq 0$, this implies $\mathrm{Ric}(X,X)=0$. Hence, if $\mathrm{Ric}<0$ then $X=0$. $\square$

Corollary 0.24. *If (M,g) is as above, then $\dim(\mathfrak{iso}(M,g)) = \dim(\mathrm{Iso}(M,g)) \leq M$ and $\mathrm{Iso}(M,g)$ is finite if $\mathrm{Ric}(M,g)<0$ (in fact, we just need $\mathrm{Ric}(v,v)<0$ for each nontrivial vector v).*

Proof. Recall that the map

$$\text{Killing Fields} \to T_pM \times \mathrm{Hom}(T_pM, T_pM), \quad X \longrightarrow (X|_p, v \to \nabla_v X)$$

is injective. Now since $\nabla X = 0$, we get that the map

$$\text{Killing Fields} \to T_pM, \quad X \longrightarrow X|_p$$

is injective which implies that $\dim(\mathfrak{iso}(M,g)) \leq \dim(M)$. Also if $\mathrm{Ric}<0$ then $X|_p = 0$ as well ensuring that the dimension of $\mathrm{Iso}(M,g)$ is zero. Since M is compact, we get that $\mathrm{Iso}(M,g)$ is finite (since it acts freely and properly discontinuously on the compact manifold FM of frames). $\square$

Corollary 0.25. *If (M, g) is as above and $p := \dim(\mathfrak{iso}(M, g))$, we have that the universal cover $\widetilde{M}$ is a product*

$$\widetilde{M} = \mathbb{R}^p \times N$$

with the product metric. The metric on $\mathbb{R}^p$ is the usual Euclidean metric. The Killing fields here are just given by linear vector fields on $\mathbb{R}^p$.

Proof. We have p linearly independent Killing fields $X_1, \cdots, X_p$ which we can assume are orthonormal. Also since $\widetilde{M}$ is simply connected and since $\nabla X_i = 0$, are equal to $\nabla f_1, \cdots, \nabla f_p$ for some smooth functions $f_i : M \longrightarrow \mathbb{R}$, $i = 1, \cdots p$. The function $f := (f_1, \cdots, f_p) : \widetilde{M} \to \mathbb{R}^p$ is no critical points (since $X_1, \cdots, X_p$ are linearly independent). Hence $N = f^{-1}(0)$ is a smooth manifold. Since M is compact, the vector fields $X_1, \cdots, X_p$ on $\widetilde{M}$ are also integrable. Let F_i^t be the time t flow of X_i. Hence we have a diffeomorphism:

$$\mathbb{R}^p \times N \longrightarrow \widetilde{M}, \quad (t_1, \cdots, t_p, n) \longrightarrow F_1^{t_1} \circ \cdots \circ F_p^{t_p}(n).$$

Also the Lie bracket $[X_i, X_j] = \nabla_{X_j} X_i - \nabla_{X_i} X_j = 0$ vanishes since these vector fields are parallel. This implies that the flows F_i^t, $i = 1, \cdots, p$ commute with each other. This implies that the metric splits as a product (flat metric) $\times\, g_N$ on $\mathbb{R}^p \times N$ under the diffeomorphism above.

□

Killing Fields in positive Ricci Curvatre. Here we will see how these Bochner techniques can be used to constrain the topology of manifolds with positive Ricci curvature.

Recall that the *Euler characteristic* of a manifold is defined to be:

$$\chi(M) := \sum_{i=1}^{n} (-1)^i \dim(H^i(M; \mathbb{R})).$$

Also recall that we have the following theorem:

Theorem 0.26. *If the Euler characteristic of M is non-zero then every vector field on M vanishes at some point.*

We have the following conjecture:

Hopf conjecture: Every compact even dimensional manifold with positive sectional curvature is positive Euler characteristic.

We will explain partial results in this direction. In dimension 2 this conjecture is true since the 2-sphere is the only 2-manifold with positive sectional curvature and its Euler characteristic is 2.

Proposition 0.27. *The Hopf conjecture is true in dimension* 4.

Proof. We showed earlier that if M has positive sectional curvature then it has finite fundamental group (if it is orientable then $\pi_1 = 0$ by a theorem of Synge 1936 and the non-orientable case follows by a covering trick. The key point of the proof here is to start with a shortest non-contractible geodesic and show that the second variation of the energy is negative. This shows that there are even shorter ones giving a contradiction).

Therefore $H_1(M;\mathbb{R}) = 0$. By Poincaré duality, we get $H^3(M;\mathbb{R}) = 0$ as well. Hence all the odd dimensional cohomology vanishes and so the Euler characteristic is at least $1 = \dim(H^0(M;\mathbb{R}))$. $\square$

Theorem 0.28. *(Berger 1965) If M is even dimensional, compact and has positive sectional curvature then every Killing field has a zero.*

Proof. Let X be a Killing field and let $f = \frac{1}{2}|X|^2$. Since M is compact, f has a minimum $p \in M$. Since it is a minimum, $\mathrm{Hess} f|_p \geq 0$. Also by Proposition 0.21 (Killing Field derivative identities), $\mathrm{Hess} f(V,V) = g(\nabla_V X, \nabla_V X) - R(V,X,X,V)$. By assumption

$$g(R(V,X)X,V) > 0$$

if X, V are linearly independent. Consider the skew symmetric map (Proposition 0.2)

$$\eta : T_pM \longrightarrow T_pM, \quad v \longrightarrow \nabla_v X.$$

Now $\nabla_X X|_p = -\nabla f|_p = 0$ by Proposition 0.21 (Killing Field derivative identities) since f has a minimum at p.

Now suppose, for a contradiction, that X is nowhere zero. Then η has non-trivial Kernel since $\nabla_X X|_p = 0$. Since the dimension of T_pM is even, we get that the kernel is even dimensional and hence has dimension at least 2. Therefore, let $v \in \ker(\eta)$ be linearly independent from X. Then

$$\mathrm{Hess} f(v,v) = g(\nabla_v X, \nabla_v X) - R(v,X,X,v) = -R(v,X,X,v) < 0$$

since $\sec < 0$. This contradicts the fact that $\mathrm{Hess} f|_p \geq 0$. $\square$

The above result can be used to get some more results hinting at the Hopf conjectures. For instance we know that the zero sets of Killing fields are totally geodesic submanifolds. Therefore, one might hope there is some kind of induction on dimension procedure.

Theorem 0.29. *(Conner 1957) Let X be a Killing field on a compact Riemannian manifold. Let $N_i \subset M$ be the connected components of the totally geodesic submanifold $X^{-1}(0)$. Then*

(1) $\chi(M) = \sum_i \chi(N_i)$
(2) $\sum_p \dim(H^{2p}(M;\mathbb{R})) \geq \sum_i \sum_p \dim(H^{2p}(N_i;\mathbb{R}))$
(3) $\sum_p \dim(H^{2p+1}(M;\mathbb{R})) \geq \sum_i \sum_p \dim(H^{2p+1}(N_i;\mathbb{R}))$

We won't prove this theorem.

Corollary 0.30. *Suppose M is a compact 6-manifold with positive sectional curvature. Suppose that M admits a non-zero Killing field X. Then $\chi(M) > 0$*

Proof. We know that $\chi^{-1}(0)$ is non-empty and each connected component of this submanifold has even codimension. This each connected component is 0,2 or 4 dimensional. Since these are totally geodesic submanifolds, they all have positive sectional curvature. They all have trivial odd Betti numbers (by the proof of Proposition 0.27). Hence $\chi(M) > 0$ by the previous theorem. $\square$

Sometimes we have stronger constraints by having positive sectional curvature.

Theorem 0.31. *(Hsiang-Kleiner, 1989) If M^4 is compact orientable 4-manifold with positive sectional curvature which admits a Killing field, then $\chi(M) \leq 3$. This implies that M is homeomorphic to S^4 or $\mathbb{C}P^2$ (using work of Freedman which says that the homeomorphism type of a four manifold only depends on the intersection form and the Kirby-Siebenmann invariant - this invariant vanishes for PL manifolds and hence smooth ones).*

We won't prove this theorem. But we will prove theorems which look similar in spirit. Usually it is easier to prove things if we have more symmetries.

Definition 0.32. The *rank* of a compact Lie group is the dimension of the largest abelian subalgebra of its associated Lie algebra. The *symmetry rank* of (M, g) is the rank of $\mathrm{Iso}(M, g)$. Let $\mathfrak{h}(M, g) \subset \mathfrak{iso}(M, g)$ be any abelian subalgebra whose rank is the symmetry rank of (M, g) and define

$$\mathcal{Z}(\mathfrak{h}(M, g)) := \{N \ : \ N \text{ is a connected component of} \\ \cap_{i=1}^{m} X_i^{-1}(0) \text{ for some } X_1, \cdots, X_m \in \mathfrak{h}(M, g)\}.$$

This set has a natural partial order given by inclusion.

Proposition 0.33. *(1) All $X \in \mathfrak{h}(M, g)$ are tangent to the submanifold $N \in \mathcal{Z}(\mathfrak{h}(M, g))$.*

(2) If $N, N' \in \mathcal{Z}(\mathfrak{h}(M, g))$ then each connected component of $N \cap N'$ is contained in $\mathcal{Z}(\mathfrak{h}(M, g))$.

(3) $N \in \mathcal{Z}(\mathfrak{h}(M, g))$ is maximal if and only if the restriction of $\mathfrak{h}(M, g)$ to N has dimension $\dim(\mathfrak{h}(M, g)) - 1$.

(4) If $N \in \mathcal{Z}(\mathfrak{h}(M, g))$ then there are only finitely many maximal sets $N_1, \cdots, N_l \in \mathcal{Z}(\mathfrak{h}(M, g))$ containing N and they satisfy

$$N = N_1 \cap \cdots \cap N_l.$$

Proof. (1) Suppose N is a connected component of the zero set of $X \in \mathfrak{h}(M, g)$. Let $Y \in \mathfrak{h}(M, g)$. Then

$$0 = (L_Y g)(X, X) = d(|X|^2)(Y) - 2g(L_Y X, X) = d(|X|^2)(Y)$$

by the Leibniz rule and the fact that $[X, Y] = 0$. Hence the flow of Y preserves the level sets of $|X|^2$ and hence is tangent to N.

(2) Let $N, N' \in \mathcal{Z}(\mathfrak{h}(M, g))$. Then, by the Killing field theorem, we can find $X, X' \in \mathfrak{h}(M, g)$ so that $N = X^{-1}(0)$ and $N' = (X')^{-1}(0)$. Let $p \in N \cap N'$. Choose α, α' so that $Y := \alpha X + \alpha' X'$ satisfies

$$\ker \nabla Y|_p = \ker \nabla X|_p \cap \ker \nabla X'|_p$$

where

$$\nabla Y : T_p M \longrightarrow T_p M, \quad v \longrightarrow \nabla_v Y$$

and where ∇X and $\nabla X'$ are defined likewise. Let Q be the connected component of $Y^{-1}(0)$ containing p. Then $T_p Q = T_p N \cap T_p N'$ (by the Killing field theorem). Similarly $T_q Q = T_q N \cap T_q N'$ for each $q \in Q$. Hence N and N' intersect cleanly along Q (I.e. the dimension of the intersections of their tangent spaces is constant along Q). Therefore by the constant rank theorem, this implies that Q is a connected component of $N \cap N'$ and hence $Q \in \mathcal{Z}(\mathfrak{h}(M, g))$.

(3) Let $N \in \mathcal{Z}(\mathfrak{h}(M, g))$. Suppose that the restriction of $\mathfrak{h}(M, g)$ to N has dimension d not equal to $\dim(\mathfrak{h}(M, g)) - 1$. Since there exists $X \in \mathfrak{h}(M, g)$ satisfying $N \subset X^{-1}(0)$, we get that $d < \dim(\mathfrak{h}(M, g)) - 1$. This implies that there are two linearly independent vector fields X and Y in $\mathfrak{h}(M, g)$ satisfying $N \subset X^{-1}(0) \cap Y^{-1}(0)$. Let $p \in N$ abd $V := T_p^{\perp} N$. By Theorem 0.5 we get that V is even dimensional and we also have two skew symmetric linear maps

$$\nabla X|_p, \nabla Y|_p : V \longrightarrow V.$$

Since $[X, Y] = 0$, these two skew symmetric linear maps commute (Why???? - exercise: Let F^t be the flow of X then $\nabla_v X = -\nabla_X v - [v, X] = [X, v] = \frac{d}{dt}(F^t v)|_{t=0}$

- Note gradients are always Lie brackets when the vector field X vanishes - I.e. the metric does not matter here. We get a similar formula if G^t is the flow of Y. So our result follows since F^t and G^t commute). Since V is even dimensional, this implies that V decomposes as a direct sum

$$V := E_1 \oplus \cdots \oplus E_l$$

of 2-dimensional subspaces so that ∇X and ∇Y preserve these spaces. The space of skew symmetric maps on a 2-dimensional space is 1-dimensional. Hence, there exists $\alpha, \beta \in \mathbb{R}$ so that $\alpha \nabla X + \beta \nabla Y|_{E_1} = 0$. Since Killing fields are uniquely determined by their value and derivative at p, this implies that the zero set of $\alpha X + \beta Y$ contains a manifold strictly larger than N. Hence N cannot be maximal. Hence the dimension of all maximal elements of $\mathcal{Z}(\mathfrak{h}(M,g))$ are greater than or equal to $\dim(\mathfrak{h}(M,g)) - 1$.

Now suppose that $N \in \mathcal{Z}(\mathfrak{h}(M,g))$ is not maximal. Then there exists $N' \in \mathcal{Z}(\mathfrak{h}(M,g))$ so that N is a proper submanifold of N'. Hence there exists $X \in \mathfrak{h}(M,g)$ so that $N' \subset X^{-1}(0)$. Also since N is a proper submanifold, there exists $Y \in \mathfrak{h}(M,g)$ so that $N \subset Y^{-1}(0)$ and $Y|_{N'} \neq 0$. Therefore $N \subset X^{-1}(0) \cap Y^{-1}(0)$ and X, Y are linearly independent. Hence the dimension of $\mathfrak{h}(M,g)$ restricted to N is smaller than $\dim(\mathfrak{h}(M,g)) - 1$.

(4) Let $N \in \mathcal{Z}(\mathfrak{h}(M,g))$. Let $p \in N$ and let

$$\mathfrak{h}_0 := \{X \in \mathfrak{h}(M,g) \ : \ N \subset X^{-1}(0)\}.$$

By the Killing field theorem ∇X is a skew symmetric linear transformation on $V = T_pN^\perp$ for each $X \in \mathfrak{h}_0$. Since these transformations all commute (since the vector fields in $\mathfrak{h}_0$ commute and since each $X \in \mathfrak{h}_0$ vanishes at p and so $\nabla_v X|_p = [X, v]$), we get a decomposition

$$V = E_1 \oplus \cdots E_l$$

into two dimensional subspaces as above, where ∇X preserves these subspaces for each $X \in \mathfrak{h}_0$. The kernel $\nabla X|_p$, for each $X \in \mathfrak{h}_0$, is equal to

$$E_{i_1} \oplus \cdots E_{i_k}$$

for some $i_1, \cdots, i_k$. Now suppose N' contains N. Then the above argument implies that T_pN' is equal to T_pN plus a finite number of subspaces E_i. Also N' is uniquely determined by T_pN' (why?, because it is a totally geodesic submanifold and hence a neighborhood of p in N' is determined by T_pN and hence all of N' is determined by T_pN by an open closed argument). Hence there are only finitely many $N' \in \mathcal{Z}(\mathfrak{h}(M,g))$ containing N.

Let $N_1, \cdots, N_k$ be the maximal elements of $\mathcal{Z}(\mathfrak{h}(M,g))$ containing N. We will now show that $N = \cap_{i=1}^l N_i$. Let $X_1, \cdots, X_l \in \mathfrak{h}_0$ be the unique elements (up to scalar multiplication) satisfying $N_i = X_i^{-1}(0)$ (see (3)). Put an equivalence relation $\sim$ on $\{1, \cdots, l\}$. We say that $i \sim j$ if, for each $X, X' \in \mathfrak{h}_0$, $\nabla X|_{E_i \oplus E_j}$ is proportional to $\nabla X'|_{E_i \oplus E_j}$. Let $I_1, \cdots, I_s \subset \{1, \cdots, l\}$ be the corresponding equivalence classes and define

$$E'_i := \oplus_{j \in I_i} E_i, \quad i = 1 \cdots, s.$$

Let $u_i : V \longrightarrow V$ be an element of $\mathfrak{h}_0$ whose restriction to E_j is trivial for each $j \neq i$ and which is non-zero on E_i. Then $u_1, \cdots, u_s$ is a basis for $\mathfrak{h}_0$. One can then show that for each i, $u_i = \nabla X_j|_p$ for some j. Hence $\cap \ker \nabla X_j|_p = 0$. This implies $T_pN = \cap_{i=1}^l T_pN_i$ and hence (via a similar argument to (2)), $N = \cap_i N_i$.

$\square$

Recall that a result of Berger 1955 (above) says that every Killing field on an even dimensional manifold has a zero. For odd dimensional manifolds this is not true: Consider the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ with the induced metric coming from the Euclidean metric. The flow

$$F^t : S^{2n-1} \longrightarrow S^{2n-1}, \quad F^t(z) = e^{it} z$$

is the flow of a Killing field which does not vanish anywhere. In this case, the dimension of $\mathfrak{h}(M, g)$ is 1 (we will see this later). What happens if $\dim(\mathfrak{h}(M, g)) \geq 2$? Here is a result when $\dim(\mathfrak{h}(M, g))$ is large.

Theorem 0.34. *(Grove-Searle 1994)*
Let M be a compact n-manifold with positive sectional curvature and symmetry rank k (I.e. $\dim(\mathfrak{iso}(M, g)) = k$). If $k \geq n/2$ then M is either diffeomorphic to a sphere, complex projective space or a cyclic quotient $S^n/\mathbb{Z}_q$ where $\mathbb{Z}_q$ acts freely by isometries.

Proof. We will first show that there is a maximal element of $\mathcal{Z}(\mathfrak{h}(M, g))$ of codimension 2. For each $B \in \mathcal{Z}(\mathfrak{h}(M, g))$, let $\mathfrak{h}_B \subset \mathfrak{h}(M, g)$ be the subspace of elements tangent to B.

Suppose (for a contradiction) there is a maximal element $N \in \mathcal{Z}(\mathfrak{h}(M, g))$ of codimension > 2 (this could be the empty set!). Then

$$\dim(\mathfrak{h}_N) = \dim(\mathfrak{h}(M, g)) - 1 \geq n/2 > \dim(N)/2 + 1$$

(well, actually it cannot be empty by the formula above!). Now choose a maximal element N_1 of $\mathcal{Z}(\mathfrak{h}(N, g|_N))$. Since N_1 is of even codimension inside N, we get

$$\dim(\mathfrak{h}_{N_1}) > \dim(N_1)/2 + 1.$$

Keep choosing maximal elements $N_2 \subset N_1$, $N_3 \subset N_2$ etc until we reach a 1 or 2 dimensional submanifold N_l so that $\mathfrak{h}(N_l) \geq 2$. If N_l has dimension 1 then $\dim(\mathfrak{h}(N_1)) \leq 1$ since N_l must be a circle, giving us a contradiction. If $\dim(N_l) = 2$ then we can choose linearly independent Killing fields X, Y on $(N_l, g|_{N_l})$. Since N_l is even dimensional, we get that $X^{-1}(0)$ is a non-empty zero dimensional submanifold. Since $[X, Y] = 0$, we get that the flow of Y preserves $|X|^2$ and hence the flow of Y preserves $X^{-1}(0)$. Hence $X^{-1}(0) = Y^{-1}(0)$. Choose $p \in X^{-1}(0)$. Since $\nabla X|_p$ and $\nabla Y|_p$ are skew symmetric operators on a two dimensional subspace $T_p N_l$, we get that they are proportional. Since Killing fields on N_l are determined by their value and derivative at p, this implies X and Y are proportional, giving us a contradiction.

Hence, (M, g) admits a Killing field X so that $X^{-1}(0)$ contains a submanifold of codimension 2. Our result now follows from the following theorem: □

Theorem 0.35. *(Grove-Searle 1994). Suppose (M, g) is a closed n-manifold with positive sectional curvature admitting a Killing field X so that $X^{-1}(0)$ has a connected component of codimension 2. Then M is diffeomorphic to a sphere, or complex projective space or a quotient of the round sphere by a finite group acting freely by isometries.*

We will prove a weaker version of this theorem instead. Before we do that, we give an application of the Grove-Searle result above:

Theorem 0.36. *(Putnam-Searle 2002) Suppose M^{2n} is a closed $2n$-manifold of positive sectional curvature and symmetry rank $k \geq \frac{2n-4}{4}$. Then $\chi(M) > 0$.*

Proof. We will induct on dimension (in a similar manner as above). However it is better to have a slightly stronger induction hypothesis: We will show that (M, g) and all $N \in \mathcal{Z}(\mathfrak{h}(M, g))$ have positive Euler characteristic. For the base case, we know we know that the above statement is true when $2n = 2, 4$.

Now suppose that it is true for each even dimensional manifold of dimension $< 2n$. Let $N \in \mathcal{Z}(\mathfrak{h}(M,g))$ and let $N' \in \mathcal{Z}(\mathfrak{h}(M,g))$ be a maximal component. If the codimension of N' is ≥ 4 then $\chi(N) > 0$ by the induction hypothesis (combined with the fact that $\mathfrak{h}(M,g)$ restricted to N' has rank one less). If the codimension of N is 2 then the Grove-Searle result tells us that N is either a finite quotient of a sphere or complex projective space. Then the odd dimensional homology groups (over $\mathbb{Q}$) vanish. Now we can use part (3) of Conners theorem (proven earlier). Let $X \in \mathfrak{h}(M,g)$ be such that N is a connected component of $X^{-1}(0)$ and let N_i be the connected components of $X^{-1}(0)$. Then

$$0 = \sum_p \dim(H^{2p+1}(M;\mathbb{R})) \geq \sum_i \sum_p \dim(H^{2p+1}(N_i;\mathbb{R})).$$

Hence the odd dimensional homology groups of N vanish and so $\chi(N) > 0$. □

We now wish to prove a simplified version of the Grove-Searle result stated above. This is enough for the proposition above:

Proposition 0.37. *(Weak Grove-Searl theorem) Suppose (M,g) is a closed n-manifold with positive sectional curvature admitting a Killing field X so that $X^{-1}(0)$ has a connected component of codimension 2. Then the Betti numbers b_i of M satisfy:*

$$b_{2p+1} = 0, \quad \forall\ 2p+1 < n$$
$$b_{2p} = b_{2p+2} \quad \forall 2p+2 \leq n.$$

Before we prove this proposition, we need the following lemma:

Proposition 0.38. *(Connectedness Principle with symmetries Wilking 2003).*

Suppose (M,g) is a closed Riemannian manifold of positive sectional curvature. If $N \in \mathcal{Z}(\mathfrak{h}(M,g))$ has codimension k, then it is $(n-2k+2)$-connected.

Before we prove this lemma, we recall some facts about path spaces and the energy functional which will be useful. For each $p,q \in M$, we define $\Omega_{p,q}M$ to be the space of smooth paths $\gamma : [0,1] \longrightarrow M$ satisfying $\gamma(0) = p$ and $\gamma(1) = q$ equipped with the C^∞ topology. In other words, a set $K \subset \Omega_{p,q}M$ is closed if and only if for every sequence of maps $\gamma_i : [0,1] \longrightarrow M$ in $\Omega_{p,q}$, $i \in \mathbb{N} \cup \{\infty\}$ so that $\gamma_i(x) \to \gamma_\infty(x)$ and so that the derivatives of γ_i at x converge to the derivatives of γ_∞ at x for each $x \in [0,1]$ and where $\gamma_i \in K$, we have $\gamma_\infty \in K$ too.

Now consider the function:

$$E : \Omega_{p,q}(\gamma) \longrightarrow \mathbb{R}, \quad E(\gamma) := \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt.$$

This is called the *energy functional*. We should pretend that this is a "smooth function" on an "infinite dimensional manifold". How do we compute its derivatives? Let us compute its first derivative. A *smooth curve* in $\Omega_{p,q}M$ is a family $(\gamma_s)_{s\in I}$ of elements in $\Omega_{p,q}$ where $I \subset \mathbb{R}$ is an interval so that the map

$$\gamma_\bullet : I \times [0,1] \longrightarrow M, \quad \gamma_\bullet(s,x) := \gamma_s(x)$$

is smooth. Let us suppose $I = (-\epsilon, \epsilon)$. Then

$$\frac{d}{dt}\gamma_s|_{t=0}$$

is a smooth section of $\gamma_0^* TM$ whose value at $x \in [0,1]$ is

$$D\gamma_\bullet(\frac{\partial}{\partial_s})|_{(0,x)}.$$

As a result, a *tangent vector* at $\gamma \in \Omega_{p,q}$ is a smooth section v of γ^*TM which is zero at $\partial[0,1] = \{0,1\}$ and a curve tangent to this tangent vector is a smooth curve $(\gamma_s)_{t\in I}$ so that s $\gamma_{s_0} = \gamma$ and $\frac{d}{dt}\gamma_s|_{s=s_0} = v$. We define $T_\gamma\Omega_{p,q}M = C^\infty(\gamma^*TM)$.

What is dE? This is a linear function

$$T_\gamma\Omega_{p,q}M \longrightarrow \mathbb{R}.$$

It is defined as follows: let $(\gamma_s)_{t\in(-\epsilon,\epsilon)}$ be a smooth curve in $\Omega_{p,q}M$ tangent to v at $s = 0$. Then

$$dE(v) := \frac{d}{ds}E(\gamma(s))|_{s=0}.$$

This does not depend on the choice of γ. We can compute this formula explicitly:

$$\begin{aligned}\frac{dE(\gamma_s)}{ds} &= \int_0^1 \frac{1}{2}\frac{\partial}{\partial s}g\left(\frac{\partial}{\partial t}\gamma_s(t), \frac{\partial}{\partial t}\gamma_s(t)\right)dt\\ &= \int_0^1 g\left(\frac{\partial^2}{\partial s\partial t}\gamma_s(t), \frac{\partial}{\partial t}\gamma_s(t)\right)dt\end{aligned}$$

(to understand this, one can choose an appropriate trivialization of γ^*TM so that g is constant and equal to the Euclidean metric in this trivialization). Then the above expression is

$$\begin{aligned}&= \int_0^1 g\left(\frac{\partial^2}{\partial s\partial t}\gamma_s(t), \frac{\partial}{\partial t}\gamma_s(t)\right)dt = \int_0^1 g\left(\frac{\partial^2}{\partial t\partial s}\gamma_s(t), \frac{\partial}{\partial t}\gamma_s(t)\right)dt\\ &= \int_0^1 \frac{\partial}{\partial t}g\left(\frac{\partial}{\partial s}\gamma_s(t), \frac{\partial}{\partial t}\gamma_s(t)\right) - g\left(\frac{\partial}{\partial s}\gamma_s(t), \frac{\partial^2}{\partial^2 t}\gamma_s(t)\right)dt\\ &= v(1) - v(0) - \int_0^1 g(\ddot\gamma(t), v(t))dt.\\ &= -\int_0^1 g(\ddot\gamma(t), v(t))dt.\end{aligned}$$

Note a more invariant expression of $\ddot\gamma$ is $\nabla_{\dot\gamma}\dot\gamma$. Now, if we choose $v = \ddot\gamma(t)$, then

$$dE(v) = -\int_0^1 |\ddot\gamma(t)|^2 dt.$$

Hence $dE|_\gamma = 0$ if and only if $\ddot\gamma = 0$. In other words, if and only if γ is a geodesic.

Recall that a *Morse function* is a smooth function

$$f : M \longrightarrow \mathbb{R}$$

whose Hessian at each critical point is a non-degenerate quadratic form. The *index* of this critical point p is the dimension of the negative Eigenspace of $\mathrm{Hess}_p(f)$. Morse functions give us explicit descriptions of M as a CW complex. Each critical point of index k corresponds to attaching a k cell. Instead of proving this I will just draw a picture to demonstrate. draw picture. The corresponding cell of a critical point p is the *descending manifold* of p:

$$W^u(p) := \{x \in M \ : \ \lim_{t\to\infty}\phi_t^{\nabla f}(x) = p\}.$$

Now we can also formally define $\mathrm{Hess}E$ as well. It turns out that for generic metric g, E is a "Morse function" which means that the dimension of the kernel of the Hessian is zero. I.e. $\mathrm{Hess}_pE(v,v) = 0$ implies that $v = 0$. Also the dimension of the negative Eigenspace is finite (although the positive Eigenspace is infinite dimensional). Using Morse theoretic ideas, one can show that $\Omega_{p,q}M$ is homotopic to a CW complex and each critical point of E of index k corresponds to a k cell. As a result, it is important for us to compute the Hessian of E at each critical point.

Theorem 0.39. *(Synges second variation formula 1926). Let $\overline{\gamma} : (-\epsilon, \epsilon) \times [0,1] \longrightarrow M$ represent a smooth path $(\gamma_s)_{s \in (-\epsilon,\epsilon)}$ so that γ_0 is a geodesic. Then*

$$\frac{d^2E(\gamma_s)}{ds}|_{s=0} = \int_0^1 \left| \frac{\partial^2 \overline{\gamma}}{\partial t \partial s} \right|^2 - \int_0^1 g\left(R\left(\frac{\partial \overline{\gamma}}{\partial s}, \frac{\partial \overline{\gamma}}{\partial t} \right) \frac{\partial \overline{\gamma}}{\partial t}, \frac{\partial \overline{\gamma}}{\partial s} \right) + g\left(\frac{\partial^2 \overline{\gamma}}{\partial s^2}, \frac{\partial \overline{\gamma}}{\partial t} \right) \Big|_0^1$$

For any $v = \frac{d}{ds}\gamma_s \in T_{\gamma_0}\Omega_{p,q}M$ as above, we define $d^2E(v) := \frac{d^2E(\gamma_s)}{ds}|_{s=0}$. As a result d^2E is a quadratic form on $T_{\gamma_0}\Omega_{p,q}M$. Note that if γ_s is a path in $\Omega_{p,q}$ then the last term does not matter in Synges formula.

We won't prove this theorem.

Definition 0.40. The *index* of γ_0 as described above is the dimension of the subspace spanned by the vectors

$$\left\{ v \in T_{\gamma_0}\Omega_{p,q} \ : \ d^2E(v) < 0 \right\} \subset T_{\gamma_0}\Omega_{p,q}.$$

Example 0.41. If M has non-positive sectional curvature then Snyges formula tells us that all critical points of E have index 0. This implies that the $\Omega_{p,q}M$ is homotopic to a zero dimensional CW complex. It also tells us that there is exactly one geodesic in each connected component of $\Omega_{p,q}M$.

Now we can generalize the discussion above as follows.

Definition 0.42. Let $N \subset M$ be a submanifold. We define $\Omega_N M$ to be the space of smooth map $\gamma : [0,1] \longrightarrow M$ satisfying $\gamma(0), \gamma(1) \in N$. We define $E : \Omega_N M$ as above. The critical points of E are now geodesics $\gamma : [0,1] \longrightarrow M$ in $\Omega_N M$ so that $\dot{\gamma}(0) \perp TN$ and $\dot{\gamma}(1) \perp TN$. We have exactly the same variation formula for d^2E as above.

Metric "stretched" by a group action - Berger-Cheeger perturbations.

Suppose that we have a Lie group G of isometries acting on (M, g). Now equip G with a left invariant metric $(,)$. Then we get a metric $g + \lambda(,)$ on $M \times G$ which is invariant under the natural $G \times G$ action. Hence it descends to a metric g_λ on

$$M \times_G G$$

which is the quotient by the diagonal action. Now an inclusion

$$M \overset{x \to (x,e)}{\longrightarrow} M \times G \twoheadrightarrow M \times_G G$$

gives us a natural diffeomorphism $M \cong M \times_G G$. Hence we get a family of metrics g_λ, $\lambda \geq 0$ on M. What do these metrics look like? First of all these metrics are all preserved by G. What else? The tangent space T_pM is decomposed into $V_p \oplus H_p$ where V_p is tangent to the group action and W_p is perpendicular. One can show that the metrics are unchanged when restricted to H_p but the get bigger and bigger when restricted to V_p. In particular the size of the Killing fields $|X|_{g_\lambda}$ goes to zero as $\lambda \to \infty$. Also the sectional curvatures restricted to H_p can only increase (see page 88 in Peterson's book for an explicit calculation). Finally if γ is a geodesic that is tangent to V_p at each point p along γ then it remains a geodesic for g_λ for each λ (see). We will use these facts in the proof below.

We now wish to prove the following proposition:

Proposition 0.43. *(Connectedness Principle with symmetries Wilking 2003).*

Suppose (M, g) is a closed Riemannian manifold of positive sectional curvature. If $N \in \mathcal{Z}(\mathfrak{h}(M, g))$ has codimension k, then it is $(n - 2k + 2)$-connected.

Recall that $\mathfrak{h}(M, g) \subset \mathfrak{iso}(M, g)$ is any abelian subalgebra whose rank is the symmetry rank of (M, g) and

$$\mathcal{Z}(\mathfrak{h}(M, g)) := \{N \ : \ N \text{ is a connected component of}\}$$
$$\cap_{i=1}^{m} X_i^{-1}(0) \text{ for some } X_1, \cdots, X_m \in \mathfrak{h}(M, g)\}.$$

Proof of proposition above. The key idea here is to show that $\Omega_N M$ is highly connected. We have a natural inclusion map $\iota : N \hookrightarrow \Omega_N M$ via constant loops and a natural projection map $p : \Omega_N M \longrightarrow N$. Since $\iota \circ p$ is the identity map, this implies by functoriality of homotopy groups that the natural map

$$\pi_k N \longrightarrow \pi_k \Omega_N M$$

is injective since the composition:

$$\pi_k(N) \longrightarrow \pi_k(\Omega_N M) \longrightarrow \pi_k(N)$$

is the identity map. Therefore to prove this proposition it is sufficient to show that $\Omega_N M$ is $(n - 2k + 2)$-connected. By the discussion above, it is therefore sufficient for us to show that all geodesics starting and ending at N and which are orthogonal to N at their endpoints have Morse index $\geq n - 2k + 2$. Therefore it is sufficient for us to construct $n - 2k + 2$ linearly independent vector fields along each such geodesic where the second derivative of E along this vector field is negative.

Let us now compute the index of the critical points of the energy functional

$$E : \Omega_N M \longrightarrow \mathbb{R}, \quad E(\gamma) := \int_0^1 |\dot{\gamma(t)}|^2 dt.$$

Let γ be a critical point of E. Then γ is a geodesic starting and ending at N and which is perpendicular to N at its endpoints.

We will now construct vector fields along γ in the negative Eigenspaces of d^2E. In fact, this will only be true once we replace g with the "stretched metric" g_λ constructed above. Consider the space $\mathcal{S}_0$ of sections $F \in C^\infty(\gamma^* TM)$ satisfying:

$$F(0) \in T_{\gamma(0)} N$$
$$\dot{F} = -\frac{g(F, \nabla_{\dot{\gamma}} X)}{|X|^2} X.$$

We will show our negative Eigenspace contains:

$$\mathcal{S} := \{F \in \mathcal{S}_0 \ : \ F(1) \in T_{\gamma(0)} N\} \subset \mathcal{S}_0$$

and that the dimension of $\mathcal{S}$ is at least $n - 2k + 2$.

We will first compute the dimension of $\mathcal{S}$. Consider the evaluation map

$$\text{ev} : \mathcal{S}_0 \longrightarrow T_{\gamma(1)} M, \quad \text{ev}(F) := F(1).$$

We will first show that $F(1)$ is orthogonal to $\dot{\gamma}$ and $\nabla_{\dot{\gamma}} X$ for each $F \in \mathcal{S}_0$. Since these two vectors $\dot{\gamma}$ and $\nabla_{\dot{\gamma}} X$ span a two dimensional space (since ∇X is anti-symmetric and non-degenerate on $TN^\perp$ and $\dot{\gamma} \in TN^\perp$), we get that the codimension of the image of ev is ≥ 2. This implies that the codimension of $\mathcal{S}$ inside $\mathcal{S}_0$ is at most $k - 2$. Since the dimension of $\mathcal{S}$ is $\dim(N) = n - k$. Hence the dimension of $\mathcal{S}$ is at least $n - k - k + 2 = n - 2k + 2$.

Let us now show $g(F(1), \nabla_{\dot\gamma} X) = 0$. To do this, we will first compute:

$$\frac{d}{dt} g(F, X) = g(\dot F, X) + g(E, \nabla_{\dot\gamma} X)$$

$$= g(-\frac{g(F, \nabla_{\dot\gamma} X)}{|X|^2}, X) + g(E, \nabla_{\dot\gamma} X) = 0.$$

Also $X|_{\gamma(0)} = 0$ which implies $g(F, X) = 0$ everywhere. Hence by the previous two equations:

$$g(F(1), \nabla_{\dot\gamma} X) = 0$$

since $X|_{\gamma(1)} = 0$.

Now we will show that $g(F(1), \dot\gamma) = 0$. Again we compute

$$\frac{d}{dt} g(F, \dot\gamma) = g(\dot F, \dot\gamma) = g(-\frac{g(F, \nabla_{\dot\gamma} X)}{|X|^2}, \dot\gamma) = -\frac{g(F, \nabla_{\dot\gamma})}{|X|^2} g(X, \dot\gamma).$$

Now

$$\frac{d}{dt} g(X, \dot\gamma) = g(\nabla_{\dot\gamma} X, \dot\gamma) = 0$$

by skew symmetry and $X|_{\gamma(0)} = 0$. Hence $g(X, \dot\gamma) = 0$ which implies $\frac{d}{dt} g(F, \dot\gamma) = 0$ by the equation above. Since $\dot\gamma|_{\gamma(0)} \in TN^\perp$ and $F(0) \in TN$, we get $g(F, \dot\gamma)|_{\gamma(0)} = 0$ and so $g(F, \dot\gamma) = 0$ by the previous two equations.

Hence we have shown that the dimension of $\mathcal{S}$ is at least $n - 2k + 2$. We now need to show that the negative Eigenspace of the energy functional

$$E_\lambda : \Omega_N M \longrightarrow \mathbb{R}, \quad E_\lambda(\gamma) := \int_0^1 |\dot\gamma|^2_{g_\lambda}$$

converges to $\mathcal{S}$ as $\lambda \to \infty$ (the rate of convergence does not depend on γ either! - this is important!). Since γ is a geodesic for g_λ for each λ, this will give us our result.

By Synges second variation formula with $\overline{\gamma}$ satisfying $\frac{d}{ds}\overline{\gamma} = \dot F$:

$$\left.\frac{d^2 E_\lambda}{ds^2}\right|_{s=0} = \int_0^1 \left|\dot{F(t)}\right|^2_{g_\lambda} dt - \int_0^1 g_\lambda(R(\dot F, \dot\gamma)\dot\gamma, F) dt$$

$$= \int_0^1 \left| -\frac{g(F, \nabla_{\dot\gamma} X)}{|X|^2_g} X \right|^2_{g_\lambda} - \int_0^1 \sec_{g_\lambda}(F, \dot\gamma) \mathrm{Area}^{\square}_{g_\lambda}(F, \dot\gamma) dt$$

$$\leq \int_0^1 \left| -\frac{g(F, \nabla_{\dot\gamma} X)}{|X|^2_g} X \right|^2_{g_\lambda} - \int_0^1 \sec_g(F, \dot\gamma) |F|^2_g dt$$

(this is because sectional curvature increases as λ increases and because $|\dot\gamma|_{g_\lambda}$ shrinks in comparison to $\dot F$ which is defined using an ODE with shrinking derivatives.)

$$\leq \int_0^1 \frac{g(F, \nabla_{\dot\gamma} X)^2}{|X|^4_g} |X|^2_{g_\lambda} - \int_0^1 \sec_g(F, \dot\gamma)|F|^2_g dt \longrightarrow -\int_0^1 \sec_g(F, \dot\gamma)|F|^2_g dt$$

as $\lambda \to \infty$. Hence the dimension of the negative Eigenspace of E_λ is at least $2 - 2k + 2$ for λ sufficiently large. This proves our proposition.

$\square$

Quick summary of Hodge theory. Let us suppose that M is closed. We have a volume form:

$$d\text{vol}(v_1, \cdots, v_n) = \det\left(g(v_i, e_j)\right)$$

where $e_1, \cdots, e_n$ is an oriented orthonormal basis. We also have an inner product:

$$(\omega_1, \omega_2) = \int_M g(\omega_1, \omega_2) d\text{vol}, \quad \omega_1, \omega_2 \in \Omega^k(M)$$

This enables us to define the *Hodge star operator:*

$$\star : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

to be the unique linear map satisfying

$$(\star\omega_1, \omega_2) = \int \omega_1 \wedge \omega_2, \quad \forall \omega_1, \omega_2 \in \Omega^k(M).$$

This gives us an explicit Poincaré duality isomorphism:

$$\star : H^k(M) \longrightarrow H^{n-k}(M).$$

Lemma 0.44.

$$\star^2 : \Omega^k(M) \longrightarrow \Omega^k(M), \quad \star^2 = (-1)^{k(n-k)}$$

Definition 0.45. The *adjoint of the exterior differential* is the unique map

$$\delta : \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$$

satisfying

$$(\delta\omega_1, \omega_2) = (\omega_1, d\omega_2), \quad \forall \omega_1 \in \Omega^{k+1}(M), \quad \omega_2 \in \Omega^k(M).$$

Lemma 0.46.

$$\delta : \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$$

is equal to

$$\delta(\omega) = (-1)^{(n-k)(k+1)} \star d \star .$$

Remark 0.47. The expressions for δ and $\star$ are *local*. As a result they can be defined for non-compact M as well. For instance, in a chart, we can cut of our forms with a bump function and consider them as forms on S^n and use the same arguments.

Definition 0.48. We define the *Hodge Laplacian* or just the *Laplacian* to be the operator

$$\Delta : \Omega^k(M) \longrightarrow \Omega^k(M), \quad \Delta\omega := (d\delta + \delta d)\omega = (d + \delta)^2\omega.$$

A *Harmonic differential form* is an element in the kernel of Δ. We let

$$\mathcal{H}^i = \mathcal{H}^i(M)$$

be the vector space spanned by Harmonic differential forms.

Theorem 0.49. *(Hodge) The harmonic forms give us a basis for de Rham cohomology.*

Bochner technique for forms

Consider a 1-form θ and let X be the g-dual of θ. We are interested in the function:

$$f : M \longrightarrow \mathbb{R}, \quad \frac{1}{2}|\theta|_g^2 = \frac{1}{2}g(X,X) = \frac{1}{2}\theta(X).$$

Proposition 0.50. $\mathrm{div}(X) = -\delta(\theta)$.

Recall that the *divergence* of a vector field X is defined to be

$$\mathrm{tr}(\nabla X) = \sum_{i=1}^{n} g(\nabla_{e_i}X, e_i)$$

where $(e_i)_{i=1}^n$ is an orthonormal frame. Here δ is the adjoint of the exterior differential.

Proof of Proposition above. We will prove this when M is compact and oriented. Then δ is defined by:

$$\int_M g(dh,\omega) = \int_M h\delta\omega$$

for each differential form ω and for each smooth function $h : M \longrightarrow \mathbb{R}$. Therefore, we need to show that

$$\int_M g(dh,\theta)d\mathrm{vol} = -\int_M h\mathrm{div}(X)d\mathrm{vol}$$

for each smooth function h. To prove this,

$$\mathrm{div}(fX) = g(\nabla f, X) + f\mathrm{div}(X) = g(df,\theta) + f\mathrm{div}(X).$$

Integrating:

$$\int_M \mathrm{div}(fX)d\mathrm{vol} = \int_M g(df,\theta)d\mathrm{vol} + \int_M f\mathrm{div}(X)d\mathrm{vol}.$$

Now

$$\mathrm{div}(fX)d\mathrm{vol} = L_{fX}d\mathrm{vol} = di_{fX}d\mathrm{vol}$$

(I won't prove this identity, you can do that) and so the integral on the left hand side is zero by Stokes' theorem.

If M is not compact and oriented, then since our theorem is local, we can cut off θ with a bump function and think of it as a 1-form on a small chart in S^n and then apply the argument above. $\square$

The above proposition says that the Laplacian on function is the same as the old definition:

$$\mathrm{div}\nabla = -\delta d.$$

Proposition 0.51. *Let θ be a 1-form and X the dual vector field. Then ∇X is symmetric if and only if $d\theta = 0$.*

Proof. Recall, by definition

$$g(\nabla_V X, W) = \frac{1}{2}(L_X g)(V,W) + \frac{1}{2}d\theta(V,W).$$

This is the decomposition ∇X into its symmetric and skew-symmetric parts. Hence ∇X is symmetric if and only if $d\theta = 0$. $\square$

Corollary 0.52. *If θ is harmonic then ∇X is symmetric.*

Definition 0.53. For a $(\cdot, r)$ tensor field S, we define $\nabla^2 S = \nabla(\nabla S)$ where ∇S is now a $(\cdot; r+1)$ tensor field. More explicitly,

$$\nabla^2_{X_1, X_2} S(Y_1, \cdots, Y_r) = (\nabla_{X_1}(\nabla S))(X_2, Y_1, \cdots, Y_2)$$
$$= (\nabla_{X_1}(\nabla_{X_2} S))(Y_1, \cdots, Y_r) - (\nabla_{\nabla_{X_1} X_2} S)(Y_1, \cdots, Y_r).$$

We now have the following Bochner identities:

Theorem 0.54. *Let X be a vector field so that ∇X is symmetric (I.e. the dual 1-form θ is closed). Let*

$$f = \frac{1}{2} |X|^2 .$$

Let $p \in M$ and suppose $X = \nabla u$ near p (such a function u exists because θ is closed). Then

(1) $\nabla f = \nabla_X X$.

(2) $\text{Hess} f(V, V) = \text{Hess}^2\ u\ (V, V) + (\nabla_X \text{Hess}\ u)\,(V, V) + R(V, X, X, V)$

$$= g(\nabla_V X, \nabla_V X) + g(\nabla^2_{X,V} X, V) + R(X, V, V, X).$$

(Here $\text{Hess}^2 u$ *means* $S^2(u)$ *where* S *is the associated* $(1,1)$ *tensor*

$$X \longrightarrow \nabla_X \nabla u.$$

I.e. $X \to \nabla_{\nabla_X \nabla u} \nabla u$ *- and then we convert it back to a* $(0,2)$ *tensor, giving us*

$$(X, V) \to g(\nabla_{\nabla_X \nabla u} \nabla u, V).$$

.

(3) $\Delta f = |\text{Hess}\ u|^2 + D_X \Delta u + \text{Ric}(X, X)$

$$= |\nabla X|^2 + D_X \text{div}(X) + \text{Ric}(X, X).$$

Proof. (1) $g(\nabla f, V) = D_V \frac{1}{2} |X|^2 = D_V \frac{1}{2} g(X, X) = g(\nabla_V X, X) = g(\nabla_X X, V)$ for each vector V. (the second to last equality is due to the fact that ∇X is symmetric).

(2) $\text{Hess} u(U, V) = g(\nabla_U \nabla u, V) = g(\nabla_U X, V)$, and so

$$\text{Hess}^2 u(V, V) = g(\nabla_{\nabla_V X} X, V) = g(\nabla_V X, \nabla_V X)$$

since ∇X is symmetric.

$$(\nabla_X \text{Hess} u)(V, V) = \nabla_X g(\nabla_V X, V) - g(\nabla_{\nabla_X V} X, V) - g(\nabla_V X, \nabla_X V)$$
$$= g(\nabla_X \nabla_V X, V) - g(\nabla_{\nabla_X V} X, V) = g(\nabla^2_{X,V} X, V).$$

Hence

$$\text{Hess} f(V, V) = g(\nabla_V \nabla f, V) = g(\nabla_V \nabla_X X, V)$$
$$= g(R(V, X)X, X + \nabla_X \nabla_V X + \nabla_{[V,X]} X, V)$$
$$= R(V, X, X, V) + g(\nabla_X \nabla_V X, V) - g(\nabla_{\nabla_X V} X, V) + g(\nabla_{\nabla_V X} X, V)$$
$$= R(V, X, X, V) + (\nabla_X \text{Hess}\ u)(V, V) + \text{Hess}^2\ u(V, V).$$

(3) Now we take traces of the equation (2) give (3). The trace of $R(X, -, -, X)$ is $\text{Ric}(X, X)$. The trace of $g(\nabla_- X, \nabla_- X)$ is

$$\sum_i g(\nabla_{e_i} X, \nabla_{e_i} X) = |\nabla X|^2.$$

Finally we need to compute the trace of $\nabla_X \text{Hess } u$. Let $p \in M$. If $X|_p \neq 0$ we choose our orthonormal frame $(e_i)_{i=1}^n$ to be invariant under the flow of X. If $X|_p = 0$ then we choose any orthonormal frame. Then

$$\text{tr}(\nabla_X \text{Hess } u) = \sum_{i=1}^n \nabla_X \text{Hess } u(e_i, e_i) =$$

$$\sum_{i=1}^n D_X \text{Hess } u(e_i, e_i) - 2\text{Hess } u(\nabla_X e_i, e_i) =$$

$$\sum_{i=1}^n D_X \text{Hess } u(e_i, e_i) = D_X \sum_{i=1}^n \text{Hess } u(e_i, e_i) = D_X \Delta u.$$

□

Theorem 0.55. *(Bochner 1948) If (M, g) is closed and oriented and has* Ric ≥ 0 *then every harmonic* 1*-form is parallel. In other words, if X is the vector field dual to this* 1*-form θ (I.e. $g(X, -) = \theta(-)$) then $\nabla X = 0$.*

Proof. Let $f = 1/2|X|^2$. The by the proposition above,

$$\Delta f = |\nabla X|^2 + \text{Ric}(X, X).$$

However since $\Delta f d\text{vol} = (\delta df)d\text{vol} = -\text{div}\nabla f d\text{vol} = L_{\nabla f} d\text{vol}$ is exact (see proof of Proposition 0.5 for exactness), we get:

$$0 = \int_M \Delta f d\text{vol} = \int_M |\nabla X|^2 + \text{Ric}(X, X).$$

Now since $|\nabla X|^2$ and $\text{Ric}(X, X)$ are non-negative, this implies $\nabla X = 0$. □

Corollary 0.56. *If, in addition,* Ric > 0 *at some point p of M then all harmonic forms vanish.*

Proof. The above proof tells us that $\text{Ric}(X, X) = 0$ at p and so $X = 0$ (since Ric > 0 at p). Hence X vanishes at p. However, since $\nabla X = 0$, we also get that X vanishes everywhere. □

Corollary 0.57. *Suppose (M, g) is closed and oriented and satisfies* Ric ≥ 0*. Then $b_1(M) \leq n$ with equality if and only if M is the flat torus.*

Proof. We know that $b_1(M) = \dim \mathcal{H}^1(M)$. Also since each element of $\mathcal{H}^1(M)$ is parallel (I.e. its dual vector field has trivial gradient), we get that the map

$$\mathcal{H}^1(M) \longrightarrow T_p^* M, \quad X \longrightarrow X|_p$$

is injective and hence $b_1(M) \leq \dim(T_p^* M) = n$.

Now suppose that $b_1(M) = n$. Since we have n linearly independent parallel fields $X_1, \cdots, X_n$, we get that (M, g) is flat and hence the universal cover of M is $\mathbb{R}^n$ with the standard metric. Hence $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete subgroup of the group of isometries of $\mathbb{R}^n$. The pullbacks $\widetilde{X}_1, \cdots, \widetilde{X}_n$ of the vector fields are constant vector fields on $\mathbb{R}^n$ since they are parallel. Also the action of Γ preserves these constant vector fields. Hence any $\gamma \in \Gamma$ sends any $v \in T_q\mathbb{R}^n$ to the same vector in $T_{\gamma(q)}\mathbb{R}^n$. This implies that Γ acts by translations and so is a discrete subgroup of translations of $\mathbb{R}^n$. Hence $\Gamma \cong \mathbb{Z}^m$ for some m. If the span V of Γ has smaller dimension than $\mathbb{R}^n$ then Γ acts on $V \oplus V^\perp \subset \mathbb{R}^n$ and it acts trivially on

$V^\perp$ which implies that the quotient is of the form $V/\Gamma \times V^\perp$ which is non-compact. Hence the span of Γ is $\mathbb{R}^n$.

Hence a basis of generators $a_1, \cdots, a_m$ for Γ span $\mathbb{R}^n$. They are also linearly independent, since the map $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{R}^n$ must also be injective (why? scaling out $\mathbb{Q}$ linear relations gives us $\mathbb{Z}$ linear relations. Also tensoring a vector space over $\mathbb{Q}$ with $\mathbb{R}$ preserves dimension). Hence the quotient must be a torus. □

1. Bochner Technique In General

Let E be a smooth vector bundle and let $\Gamma(E)$ be the space of smooth sections of E.

Definition 1.1. Recall that a *connection* on E is a morphism of vector spaces (over $\mathbb{R}$):

$$\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{Hom}(TM, E)) = \Gamma(E \otimes T^*M), \quad s \longrightarrow \nabla s$$

satisfying

$$\nabla(fs) = f\nabla s + s \otimes df$$

for each $s \in \Gamma$ and $f \in C^\infty(M)$. We write ∇s as the map

$$X \longrightarrow \nabla_X s.$$

More explicitly ∇ is a map satisfying the following identities:

(1) $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$
(2) $\nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$
(3) $\nabla(fs) = f\nabla s + s \otimes df$ for each smooth function f,
(4) $\nabla_{fX} s = f\nabla_X s$.

Lemma 1.2. *In a local trivialization of E over U, we have that sections correspond to maps $s : U \longrightarrow \mathbb{R}^m$ and $\nabla_X s = D_X s + AX$ where $A : U \longrightarrow \mathrm{Mat}_{m\times m}\mathbb{R}$*

Definition 1.3. A *metric* $\langle , \rangle$ on E is a smooth map

$$E \otimes E \longrightarrow \mathbb{R}, \quad s_1 \otimes s_2 \longrightarrow \langle s_1, s_2\rangle$$

whose restriction to each fiber is an $\mathbb{R}$-linear non-degenerate symmetric bilinear form. A connection ∇ is *compatible* with a metric $\langle , \rangle$ if:

$$D_X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle$$

for each pair of smooth sections s_1, s_2 and each vector X.

We will also define

$$(s_1, s_2) := \int_M \langle s_1, s_2\rangle d\mathrm{vol}$$

for each $s_1, s_2 \in \Gamma(E)$. Also if $S_1, S_2 \in \Gamma(\mathrm{Hom}(TM, E))$, we define

$$\langle S_1, S_2\rangle := \mathrm{tr}(S_1^* S_2) = \sum_{i=1}^n g(S_1^* S_2(e_i), e_i)$$

for any orthonormal basis $e_1, \cdots, e_n$. Here $S_1^* \in \mathrm{Hom}(TM, E)$ is the pointwise adjoint of S_1, defined by:

$$\langle S_1^*(X), s\rangle = g(X, S_1(s)), \quad \forall X \in \mathrm{Vect}(X).$$

We define

$$(S_1, S_2) := \int_M \langle S_1, S_2\rangle d\mathrm{vol}.$$

If M is non-compact, then the above expressions make sense for compactly supported sections.

Lemma 1.4. $(S_1, S_2) = \sum_{i=1}^n \langle S_1(e_i), S_2(e_i)\rangle$ *for any orthonormal basis* $e_1, \cdots, e_n$.

Definition 1.5. Suppose a connection ∇ is compatible with the metric $\langle , \rangle$ on E, then we define

$$\nabla^* : \Gamma(\mathrm{Hom}(TM, E)) \longrightarrow \Gamma(E)$$

to be the adjoint of ∇. I.e. it is defined by

$$\int_M \langle \nabla^* S, s\rangle d\mathrm{vol} = \int_M \langle S, \nabla s\rangle d\mathrm{vol} \quad \forall\ s \in \Gamma(E), \quad S \in \Gamma(\mathrm{Hom}(TM, E)).$$

The following lemma tells us that ∇^* is basically a connection on $\mathrm{Hom}(TM, E)$,

Lemma 1.6. *(Exercise). Let* $E' := \mathrm{Hom}(TM, E)$ *and define*

$$E'' := \mathrm{Hom}(TM, \mathrm{Hom}(TM, E)).$$

Define

$$\nabla' : \Gamma(E') \longrightarrow \mathrm{Hom}(TM, E'), \quad \nabla'_X s := (Y \longrightarrow \langle X, Y\rangle \nabla^* s).$$

Then ∇' *is a connection on* E'.

Definition 1.7. The *curvature* of ∇

Definition 1.8. If E, E' are vector bundles on M and ∇, ∇' are connections then we define the connection

$$\nabla \otimes \nabla' : \Gamma(E \otimes E') \longrightarrow \mathrm{Hom}(TM, E \otimes E')$$

to be the unique connection satisfying

$$(\nabla \otimes \nabla')_X (s \otimes s') := (\nabla_X s) \otimes s' + s \otimes (\nabla_X s').$$

Definition 1.9. For each $s \in \Gamma(E)$, we define $\nabla^2 s := \nabla(\nabla s)$ where (by abuse of notation) ∇ is the induced connection on $\mathrm{Hom}(TM, E) = E \otimes T^*M$ coming from the Levi-Civita connection on TM and the connection ∇ on E. More explicitly, we define

$$\nabla^2_{X,Y} s := \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s$$

for each pair of vector fields X, Y and each $s \in \Gamma(E)$.

Definition 1.10. The *curvature* of ∇ is the linear map

$$R_\nabla : \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(E) \longrightarrow \Gamma(E)$$

defined by

$$R_\nabla(X, Y)s = \nabla^2_{X,Y} s - \nabla^2_{Y,X} s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Definition 1.11. The *connection Laplacian* of $s \in \Gamma(E)$ is defined to be

$$\nabla^* \nabla s.$$

This does not coincide with our precious notion of Laplacian, say, for vector fields, the only sections satisfying

$$\nabla^* \nabla s = 0$$

are parallel (I.e. $\nabla s = 0$) since:

$$\int_M \langle \nabla^* \nabla s, s\rangle d\mathrm{vol} = \int_M |\nabla s|^2 d\mathrm{vol}.$$

Definition 1.12. We define

$$\mathrm{tr}(\nabla^2 s) := \sum_{i=1}^{n} \nabla^2_{e_i,e_i} s \in \Gamma(E)$$

where $e_1, \cdots, e_n$ is any choice of normal frame (Exercise: this does not depend on the choice of orthonormal frame).

Lemma 1.13. $\nabla^*\nabla s = -\mathrm{tr}\nabla^2 s$.

Proof. Let s_1, s_2 be two compactly supported sections supported in the domain of some orthonormal frame $e_1, \cdots, e_n$. Now let us compute:

$$\langle \mathrm{tr}\nabla^2 s_1, s_2\rangle = \sum_{i=1}^{n} \langle \nabla^2_{e_i,e_i} s_1, s_2\rangle$$

$$= \sum_{i=1}^{n} \langle \nabla_{e_i}\nabla_{e_i} s_1, s_2\rangle - \sum_{i=1}^{n} \langle \nabla_{\nabla_{e_i} e_i} s_1, s_2\rangle$$

$$= -\sum_{i=1}^{n} \langle \nabla_{e_i} s_1, \nabla_{e_i} s_2\rangle + \sum_{i=1}^{n} \nabla_{e_i}\langle \nabla_{e_i} s_1, s_2\rangle - \sum_{i=1}^{n} \langle \nabla_{\nabla_{e_i} e_i} s_1, s_2\rangle$$

$$\overset{\text{see below for justification}}{=} -\langle \nabla s_1, \nabla s_2\rangle + \mathrm{div}(X)$$

where X is the vector field defined by:

$$g(X, v) = \langle \nabla_v s_1, s_2\rangle.$$

Recall that the *divergence* of a vector field X is defined to be $\mathrm{tr}(\nabla X) = \sum_{i=1}^{n} g(\nabla_{e_i} X, e_i)$. Let us now justify the final equality:

$$\sum_i D_{e_i} g(X, e_i) = \sum_i g(\nabla_{e_i} X, e_i) + \sum_i g(X, \nabla_{e_i} e_i) = \mathrm{div}(X) + \sum_i \langle \nabla_{\nabla_{e_i} e_i} s_1, s_2\rangle.$$

Also:

$$\sum_i D_{e_i} g(X, e_i) = \sum_i \nabla_{e_i}\langle \nabla_{e_i} s_1, s_2\rangle$$

and hence

$$\mathrm{div}(X) = -\sum_{i=1}^{n} \nabla_{e_i}\langle \nabla_{e_i} s_1, s_2\rangle - \sum_{i=1}^{n} \langle \nabla_{\nabla_{e_i} e_i} s_1, s_2\rangle.$$

Summarizing, we have:

$$\langle \mathrm{tr}\nabla^2 s_1, s_2\rangle = -\langle \nabla s_1, \nabla s_2\rangle + \mathrm{div}(X).$$

And hence

$$(-\mathrm{tr}\nabla^2 s_1, s_2) = \int_M \langle \nabla s_1, \nabla s_2\rangle + \mathrm{div}(X) d\mathrm{vol} = \int_M \langle \nabla s_1, \nabla s_2\rangle d\mathrm{vol}$$

$$= \langle \nabla^*\nabla s_1, s_2\rangle d\mathrm{vol} = (\nabla^*\nabla s_1, s_2).$$

This proves our lemma. ∎

Now the connection Laplacian and other Laplace operators defined so far are examples of differential operators.

Definition 1.14. Let E_1, E_2 be vector bundles over M. A *differential operator of order $\leq k$* is a map

$$D : \Gamma(E_1) \longrightarrow \Gamma(E_2)$$

so that in local coordinates $x_1, \cdots, x_n$ on M and in a local trivialization of E_1 and E_2 over this coordinate chart, we have

$$D(s)(x) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha s$$

where a_α is a smooth function. Here we are summing over tuples $\alpha = (i_1, \cdots, i_n)$ satisfying $|\alpha| = \sum_j i_j \leq k$ and where $D^\alpha = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} s$. Here s should be viewed as an $m_1 \times m_2$ matrix of functions where m_j is the dimension of the fibers of E_j, $j = 1, 2$. We say that it is of order k if it is not of order $\leq k - 1$.

Lemma 1.15. *Any sort of Laplacian that we have defined so far is a differential operator of order 2.*

Definition 1.16. A differential operator $D^2 : \Gamma(E) \longrightarrow \Gamma(E)$ of order 2 satisfies a *Weitzenböck identity or formula* if it is of the form

$$D^2 s = \nabla^* \nabla s + C(R_\nabla)(s)$$

where C is some trace or contraction of R_∇ (we do not define this precisely - it is usually clear what this looks like).

Proposition 1.17. *Let E be a vector bundle with metric $\langle , \rangle$ and compatible connection ∇. Let D^2 be a differential operator of order 2 as above. Let $s \in \Gamma(E)$. Then*

$$\Delta \left(\frac{1}{2} |s|^2 \right) = \langle \nabla s, \nabla s \rangle - \langle D^2 s, s \rangle + \langle C(R_\nabla)(s), s \rangle.$$

Proof.

$$\Delta \left(\frac{1}{2} |s|^2 \right) = \sum_{i=1}^n \nabla^2_{e_i, e_i} \frac{1}{2} |s|^2$$

$$= \sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \frac{1}{2} |s|^2 - \sum_{i=1}^n \nabla_{\nabla_{e_i} e_i} \frac{1}{2} \langle s, s \rangle$$

$$= \sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \frac{1}{2} |s|^2 - \sum_{i=1}^n \langle \nabla_{\nabla_{e_i} e_i} s, s \rangle$$

$$= \sum_{i=1}^n \Big(\langle \nabla_{e_i} \nabla_{e_i} s, s \rangle + \langle \nabla_{e_i} \nabla_{e_i} s, s \rangle - \langle \nabla_{\nabla_{e_i} e_i} s, s \rangle \Big)$$

$$= \langle \nabla s, \nabla s \rangle + \langle \sum_{i=1}^n \nabla^2_{e_i, e_i} s, s \rangle$$

$$\overset{Lemma 1.13}{=} \langle \nabla s, \nabla s \rangle - \langle \nabla^* \nabla s, s \rangle = \langle \nabla s, \nabla s \rangle - \langle D^2 s, s \rangle + \langle C(R_\nabla)(s), s \rangle.$$

□

Corollary 1.18. *If $C(R_\nabla) \geq 0$ (I.e $\langle C(R_\Delta)(s'), s' \rangle \geq 0$ for each s') and $D^2 s = 0$ then $\nabla s = 0$ which implies that s is parallel, and hence the dimension of the space of sections satisfying $D^2 s = 0$ is $\leq$ the dimension of the fiber of E. If, in addition, $C(R_\nabla) > 0$ at some point of M then the only sections s satisfying $D^2 s = 0$ are 0.*

Proof.

$$0 = \int \Delta\left(\frac{1}{2}|s|^2\right) d\mathrm{vol} = \int_M |\nabla s|^2 d\mathrm{vol} + \int \langle C(R_\delta)s, s\rangle$$

which implies that $\nabla s = 0$. Since s is parallel, it is determined by its value at a single point giving us our dimension restriction. Now suppose $C(R_\nabla) > 0$ at some point p. Then the above inequality forces our section s to be zero at p. Since s is parallel, this implies $s = 0$ everywhere. □

Now let us apply this to p-forms. In this case, our differential operator of order 2 is the Hodge Laplacian. Recall that the *curvature operator* $\mathfrak{R}$ is defined to be the map

$$\mathfrak{R} : \wedge^2 TM \longrightarrow \wedge^2 TM, \quad \mathfrak{R}(X \wedge Y, Z \wedge W) := R(X, Y, Z, W).$$

Fact: This is symmetric. I.e. swapping (X, Y) and (Z, W) leaves $\mathfrak{R}$ unchanged.

Theorem 1.19. *(D. Meyer 1971) Let $\Delta : \Omega^p(M) \longrightarrow \Omega^p(M)$ be the Hodge Laplacian for p-forms. Then $\Delta = \nabla^*\nabla + C(R_\nabla)$ for some $C(R_\nabla)$. If $\mathfrak{R} \geq 0$ then $C(R_\nabla) \geq 0$.*

Corollary 1.20. *If $\mathfrak{R} \geq 0$ and M is closed and orientable then*

$$b_k(M) \leq \binom{n}{k} = b_k(\mathbb{T}^m).$$

If $\mathfrak{R} > 0$ at some point of M then $b_k(M) = 0$ for each $0 < k < n$.

Proof. Let $p \in M$ be a point. The dimension of $\mathcal{H}^k(M)$ is at most the dimension of $\wedge^k T_p^* M$ by the previous theorem and corollary. This gives us our bound on b_k. If, in addition, $\mathfrak{R} > 0$ at p, then $\mathcal{H}^k(M) = 0$ by the previous theorem and corollary. □

We will now focus on the proof of the theorem by Meyer above. To do this, we need to talk about Clifford multiplication on forms. First of all, let us just talk about the abstract Clifford algebra and then later on see how this is realized on differential forms.

Definition 1.21. Let V be a vector space over a field K (in our case this will always be $\mathbb{R}$). The *free algebra* generated by V is the algebra

$$TV := \oplus_{i=0}^{\infty} V^{\otimes i} = K \oplus V \oplus V^{\otimes 2} \oplus \cdots$$

with multiplication given by $(a, b) \longrightarrow a \otimes b$ (the 0th tensor power is defined to be K itself).

Let $Q : V \longrightarrow \mathbb{R}$ be a quadratic form on V. Then the *Clifford algebra* $\mathrm{Cl}(V, Q)$ is defined to be the quotient TV/I where I is the ideal generated by $\{v \in V \ : \ v \otimes v - Q(v)1 = 0\}$. If h is the bilinear form associated to Q then we sometimes write $\mathrm{Cl}(V, h)$.

Example 1.22. If $Q = 0$ then the Clifford algebra is just the exterior algebra.

Example 1.23. Suppose $V = \mathbb{R}$ and $Q(x) = -x^2$. Then I is generated by $x \otimes x + x^2 1$. In other words x is a square root of -1. Hence $\mathrm{Cl}(Q, V) \cong \mathbb{C}$.

Example 1.24. $\mathrm{Cl}(\mathbb{R}^2, Q)$ where $Q(x, y) = -x^2 - y^2$ gives us the quaternions. Here i, j, k corresponds to e_1, e_2 and $e_1 \otimes e_2$ where e_1, e_2 are the basis vectors for $\mathbb{R}^2$.

Lemma 1.25. *The dimension of $\mathrm{Cl}(V, Q)$ is 2^m where $m = \dim(V)$.*

Proof. Let $P : TV \twoheadrightarrow \mathrm{Cl}(V,Q)$ be the natural surjection. Let $F^k := F^k\mathrm{Cl}(V,Q) \subset \mathrm{Cl}(V,Q)$ be the image of $\oplus_{i=0}^{k} TV^{\otimes k}$. Then if $e \in F^k$ and $f \in F^l$ then $ef \in F^{kl}$. This is an example of a *filtration* on the algebra $\mathrm{Cl}(V,Q)$. Define $F^{-1} = 0$. Consider $\mathrm{Gr} := \oplus_{k=0}^{\infty} F^k/F_{k-1}$ Then the product on $\mathrm{Cl}(V,Q)$ gives us an induced product on Gr. Also $\mathrm{Gr} \cong \wedge^* E$ with the wedge product (Exercise). Since the dimension of Gr is 2^m and equal to the dimension of $\mathrm{Cl}(V,Q)$ we are done. ◻

Definition 1.26. The *transpose* x^t of an element $x \in \mathrm{Cl}(V,Q)$ is defined as follows: Consider the vector space isomorphism $\Phi : TV \longrightarrow TV$ sending $v_1 \otimes \cdots v_k$ to $v_k \otimes v_{k-1} \cdots \otimes v_1$. Since Φ preserves the kernel of the natural map $TV \longrightarrow \mathrm{Cl}(V,Q)$, we get an automorphism of the vector space $\mathrm{Cl}(V,Q)$ sending x to x^t (this automorphism does not preserve the product, but it is an anti-homomorphism).

For each $x \in \mathrm{Cl}(V,Q)$ let $(x)_0$ be the degree 0 part of x. We can extend Q to a quadratic form $Q : \mathrm{Cl}(V,Q) \longrightarrow K$ as follows:

$$Q(x) := Q((x^t x)_0).$$

Lemma 1.27. *(Exercise). Q is a positive definite quadratic form on $\mathrm{Cl}(V,Q)$ if the characteristic of K is not equal to 2. The associated symmetric bilinear form is $\langle x,y\rangle = Q((x^t y)_0)$. Also, for each $v_1,\cdots,v_k \in V$, we have $Q(v_1\cdots v_k) = Q(v_1)\cdots Q(v_k)$.*

Lemma 1.28. *(Exercise)*

$$\langle ax,y\rangle = \langle x, a^t y\rangle$$
$$\langle xa,y\rangle = \langle x, ya^t\rangle$$

for each $x,a,y \in \mathrm{Cl}(V,Q)$.

Definition 1.29. If $\theta \in V^*$, we define *interior product* $i_\theta : \wedge^* V \longrightarrow \wedge^* V$ to be the unique linear map sending $v_1 \wedge \cdot \wedge v_k$ to $\sum_{i=1}^{k}(-1^k)\theta(v_i)v_1 \cdots \wedge v_{i-1} \wedge v_{i+1} \cdots \wedge v_k$.

If $\langle,\rangle$ is a metric on V, then for each $v \in V$, we define $v^\# \in V^*$ to be the unique element satisfying $\langle v,f\rangle = v^\# f$ for each $f \in V$.

Lemma 1.30. *Let V be a vector space and Q a positive definite quadratic form. Consider the unique bilinear and associative product $\cdot$ on $\wedge^* V$ satisfying:*

$$\theta \cdot \omega = \theta \wedge \omega - i_{\theta^\#}\omega, \quad \forall\ \theta \in \wedge^1 V, \quad \omega \in \wedge^p(V)$$
$$\omega \cdot \theta = (-1)^p(\theta \wedge \omega + i_{\theta^\#}\omega), \quad \forall\ \theta \in \wedge^1 V, \quad \omega \in \wedge^p(V)/$$

Then $\mathrm{Cl}(V,-\langle,\rangle)$ is naturally isomorphic to this product $\cdot$ on $\wedge^* V$.

The word natural means that this isomorphism is in fact a natural transformation between functors. The first functor sends a bundle with metric to its associated Clifford bundle. The second functor is $\wedge^*$ together with the product described above.

Proof. We have $\mathrm{Cl}(V,\langle,\rangle)|_x$ is generated as an algebra by elements of $V = \wedge^1 V$. Similarly, $(\wedge^* V, \dot{)}$ is generated by $\wedge^1 V$. Let $T(V) = \oplus_{i=0}^{\infty} V^{\otimes i}$ and let

$$\Phi : (TV,\otimes) \longrightarrow (\wedge^* V,\cdot)$$

be the unique algebra map whose restriction to V is the identity map. Since

$$\theta \cdot \theta = \theta \wedge \theta - i_{\theta^\# \theta} = -g(\theta,\theta) = |\theta|^2,$$

and since $\mathrm{Cl}(V, -\langle,\rangle) = TV/I$ where I is the ideal in TV generated by elements $e \in V$ satisfying $e^2 = -\langle e, e\rangle$, we have that Φ descends to an injective bundle map

$$\Psi : (\mathrm{Cl}(V, \langle,\rangle)) \longrightarrow (\wedge^* V, \cdot).$$

Since the dimensions of both vector spaces are the same, this is an isomorphism. $\square$

Lemma 1.31. *(Exercise). Let $e_1, \cdots, e_n$ be an orthonormal basis of V. Then*

$$e_{i_1} \cdot \cdots \cdot e_{i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

$i_1, \cdots, i_k \in \mathbb{N}$ is an orthonormal basis for $\mathrm{Cl}(V, Q)$ under the identification above.

Lemma 1.32. *(Exercise). Let $x \in \wedge^k V$. Then $x^t = (-1)^{f(k)} x$ where*

$$f(k) = \left(\begin{array}{ll} 0 & \text{if } k = 0 \bmod 4 \\ 0 & \text{if } k = 1 \bmod 4 \\ 1 & \text{if } k = 2 \bmod 4 \\ 1 & \text{if } k = 3 \bmod 4. \end{array} \right.$$

Lemma 1.33. *Let V be a vector space over K and $\langle,\rangle$ a positive definite bilinear form on V and Q its associated quadratic form. We also write $\langle,\rangle$ to be the induced metric on $\mathrm{Cl}(V, Q)$ as described above. Then for each $\theta \in V$, $\psi \in \wedge^2 V$ and $\omega_1, \omega_2 \in \mathrm{Cl}(V, Q) = \wedge^* V$, we have*

$$\langle \theta \cdot \omega, \omega_2 \rangle = -\langle \omega_1, \theta \cdot \omega_2 \rangle$$

$$\langle [\psi, \omega_1], \omega_2 \rangle = -\langle (\omega_1, [\psi, \omega_2])$$

where $[-,-]$ is the commutator.

This lemma follows immediately from Lemma 1.28 and the previous lemma.

Now let us assume that $K = \mathbb{R}$.

Definition 1.34. Let $\pi : E \longrightarrow B$ be a smooth vector bundle with a metric $\langle,\rangle$. Define

$$Q : E \longrightarrow \mathbb{R}, \quad Q(v) := \langle v, \rangle$$

be the associated quadratic form. The *Clifford bundle* associated to (E, Q), denoted by $\mathrm{Cl}(E, Q)$ or $\mathrm{Cl}(E, \langle,\rangle)$, is the bundle whose fiber over $x \in M$ is $\mathrm{Cl}(E|_x, Q|_{E|_x})$. More precisely, if $\rho_{ij} : U_{ij} \longrightarrow O(\mathbb{R}^n)$ are transition functions defining Q then the induced maps $U_{ij} \longrightarrow \mathrm{Cl}(\mathbb{R}^n, |\cdot|^2)$ are the transition maps defining this Clifford bundle.

The following lemma gives a more explicit description of $\mathrm{Cl}(E, Q)$.

We will be interested in the Clifford bundle $\mathrm{Cl}(T^*M, -g)$. The lemmas above tells us that this is isomorphic to $\wedge^* T^* M$ with the unique bilinear and associative product $\cdot$ satisfying

$$\theta \cdot \omega = \theta \wedge \omega - i_{\theta\#}\omega, \quad \forall\ \theta \in \wedge^1 T^* M, \quad \omega \in \wedge^p(T^* M)$$

$$\omega \cdot \theta = (-1)^p(\theta \wedge \omega + i_{\theta\#}\omega), \quad \forall\ \theta \in \wedge^1 T^* M, \quad \omega \in \wedge^p(T^* M).$$

The metric on $\mathrm{Cl}(T^*M, -g)$ is identical to the usual metric on $\wedge^* TM$ by one of the lemmas above.

Definition 1.35. For an $(\cdot, r)$ tensor S and vector fields X, Y, we define

$$R(X, Y)(S) := \nabla^2_{X,Y} S - \nabla^2_{Y,X} S = \nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{[X,Y]} S.$$

Lemma 1.36. *For $\omega_1, \omega_2 \in \Omega^*(M)$ and vector fields X, Y, we have:*

$$\nabla_X(\omega_1 \cdot \omega_2) = (\nabla_X \omega_1) \cdot \omega_2 + \omega_1 \cdot (\nabla_X \omega_2)$$
$$R(X,Y)(\omega_1 \cdot \omega_2) = (R(X,Y))(\omega_1) \cdot \omega_2 + \omega_1 \cdot (R(X,Y)(\omega_2).$$

Proof. Let us prove the first formula. We only need to prove this when $\omega_1 = \theta$ is a 1-form by induction. We have:

$$\nabla_X(\theta \wedge \omega) = (\nabla_X \theta) \wedge \omega + \theta \wedge (\nabla_X \omega)$$
$$\nabla_X(i_{\theta^\#}\omega) = i_{\theta^\#}\omega + i_{\nabla_X \theta^\#}(\nabla_X \omega).$$

Hence

$$\nabla_X(\theta\dot{\omega}) = \nabla_X(\theta \wedge \omega - i_{\theta^\#}\omega)$$
$$= (\nabla_X \theta) \wedge \omega + \theta \wedge (\nabla_X \omega) - i_{\theta^\#}\omega - i_{\nabla_X \theta^\#}(\nabla_X \omega)$$
$$= (\nabla_X \theta) \wedge \omega - i_{\theta^\#}\omega + \theta \wedge (\nabla_X \omega) - i_{\nabla_X \theta^\#}(\nabla_X \omega)$$
$$\nabla_X \theta) \cdot \omega + \theta \cdot (\nabla_X \omega).$$

The second formula follows from the first formula (we won't do the calculation here). □

Definition 1.37. Define the *Dirac operator* on forms to be the map

$$D : \Omega^* M \longrightarrow \Omega^* M$$
$$D(\omega) := \sum_{i=1}^{n} \theta^i \nabla_{e_i} \omega$$

where $e_1, \cdots, e_n$ is any frame and $\theta^1, \cdots, \theta^n \in T^*M$ is the dual frame.

More generally, if you have a bundle with a 'Clifford action' (I.e. a morphism of bundles $\mathrm{Cl}(T^*M, -g) \longrightarrow \mathrm{End}(E)$ which is a fiberwise algebra morphism), then you can similarly define a Dirac operator.

Proposition 1.38. *(Exercise).*

$$d\omega = \theta^i \wedge \nabla_{e_i} \omega$$
$$\delta\omega = -i_{(\theta^i)^\#} \nabla_{e_i} \omega$$
$$D = d + \delta.$$

To prove the proposition above, you show that the first two formulas do not depend on the choice of frame and then prove it for an orthonormal frame.

Corollary 1.39. $D^2 = \Delta$ *where* Δ *is the Hodge Laplacian.*

Definition 1.40. If Y is a vector field, then define $Y^\flat$ to be the corresponding dual 1-form. Recall that:

$$\mathrm{Ric}(v) := \sum_{i=1}^{n} R(v, e_i) e_i.$$

Theorem 1.41. *(Weitzenbök formula for* 1*-forms only).*
Let X be a vector field and θ the dual 1*-form. Then*

$$\Delta\theta = \nabla^* \nabla \theta + \mathrm{Ric}(X)^\flat.$$

Proof. Let $p \in M$. We consider an orthonormal frame e_i which satisfies $\nabla e_i|_p = 0$. We will use the following formula for the exterior derivative:

$$d\omega(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \cdots, \widehat{X_i}, \cdots, X_k).$$

Also

$$\delta\omega = -\sum_{i=1}^{n} i_{e_i} \nabla_{e_i} \omega.$$

(We leave these formulas as exercises - or refer to the appendix of Peterson). Let $Z = \sum_{i=1}^{n} a_i e_i$ be a constant linear combination of e_i's. Then

$$(\Delta\theta)(Z) = (d\delta\theta)(Z) + (\delta d\theta)(Z) = \nabla_Z \delta\theta - \sum_{i=1}^{n} (\nabla_{e_i} d\theta)(e_i, Z) =$$

$$= -\sum_{i=1}^{n} \nabla_Z ((\nabla_{e_i}\theta)(e_i)) - \sum_{i=1}^{n} (\nabla_{e_i} d\theta)(e_i, Z)$$

$$-\sum_{i=1}^{n} (\nabla^2_{Z,e_i}\theta)(e_i)) - \nabla_{e_i} \sum_{i=1}^{n} (d\theta)(e_i, Z)$$

(by the flatness conditions $\nabla e_i|_p = 0$)

$$-\sum_{i=1}^{n} (\nabla^2_{Z,e_i}\theta)(e_i)) - \nabla_{e_i} \sum_{i=1}^{n} (\nabla_{e_i}\theta)(Z) - (\nabla_Z\theta)(e_i)$$

(by our formula for d)

$$= \sum_{i=1}^{n} (\nabla^2_{e_i,Z}\theta - \nabla_{Z,e_i}\theta)(e_i) - \sum_{i=1}^{n} (\nabla^2_{e_i,e_i}\theta)(Z)$$

$$= \sum_{i=1}^{n} (R(e_i, Z)\theta)(e_i) + (\nabla^* \nabla\theta)(Z).$$

We now need to compute the curvature term. We will use the following identity:

$$(R(X, Y)\theta)(W) = R(X, Y)(\theta(W)) - \theta(R(X, Y)W) = -\theta(R(X, Y)W)$$

for all $\theta \in \Omega^1(M)$, $X, Y, W \in \mathrm{Vect}(M)$. This is because $R(X, Y)f = \nabla_X\nabla_Y f - \nabla_Y\nabla_X f - \nabla_{[X,Y]} f = d(df(X))(Y) - d(df(Y))(X) - df([X, Y]) = d^2 f(X, Y) = 0$ for any smooth function f. Hence

$$\sum_{i=1}^{n} (R(e_i, Z)\theta)(e_i) = -\sum_{i=1}^{n} \theta(R(e_i, Z)e_i) = \sum_{i=1}^{n} \theta(R(Z, e_i)e_i)$$

$$= \sum_{i=1}^{n} g(R(Z, e_i)e_i, X) = \sum_{i=1}^{n} R(Z, e_i, e_i, X) \overset{\text{since } \mathfrak{R} \text{ is symmetric}}{=} \sum_{i=1}^{n} R(e_i, X, Z, e_i) = \sum_{i=1}^{n} R(X, e_i, e_i, Z)$$

$$g(\mathrm{Ric}(X), Z) = \mathrm{Ric}(X)^\flat(Z).$$

Combining everything gives us our lemma. □

Now we wish to consider all forms. We need the following lemma.

Lemma 1.42. *Let e_i be a frame with dual coframe θ^i*
$$D^2\omega = \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega = \sum_{i,j=1}^{n} \left(\nabla^2_{e_i,e_j}\omega\right) \cdot \theta^j \cdot \theta^i.$$

Proof. Let $p \in M$. Recall
$$\nabla^2_{e_i,e_j} = \nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i}e_j}$$
is tensorial in i and j and so these expressions are invariantly defined (tensorial means linear over $C^\infty(M)$). Hence we can choose our frame so that $\nabla e_i|_p = 0$. We will use Einstein summation conventions. Then
$$\begin{aligned} D^2\omega = \theta^i \cdot (\nabla_{e_i}(\theta^j \cdot \nabla_{e_j}\omega)) &= \theta^i \cdot (\nabla_{e_i}\theta^j) \cdot \nabla_{e_j}\omega + \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \\ &= \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \end{aligned}$$
by flatness
$$= \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega - \theta^i \cdot \theta^j \cdot \nabla_{\nabla_{e_i}e_j}\omega = \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega.$$
Now define
$$\widehat{D}\omega := (\nabla_{e_i}\omega) \cdot \theta^i.$$
One can show (in a similar way that we did for D), that
$$\widehat{D}\omega = (-1)^p(d - \delta)\omega$$
if ω is a p-form. Hence
$$\widehat{D}^2 = \Delta = D^2.$$
Hence, an identical proof as above, but with D replaced by $\widehat{D}$ gives us the second identity. □

Theorem 1.43. *(Weitzenbök identities for p-forms). Let e_i be a frame and θ^i its dual coframe. Then for any form ω,*
$$D^2\omega = \nabla^*\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot R(e_i,e_j)\omega = \nabla^*\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} R(e_i,e_j)\omega \cdot \theta^j \cdot \theta^i$$

Proof. Using the identities above, it is sufficient to check:
$$\nabla^*\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot R(e_i,e_j)\omega = \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega$$
$$\nabla^*\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} R(e_i,e_j)\omega \cdot \theta^j \cdot \theta^i = \sum_{i,j=1}^{n} \left(\nabla^2_{e_i,e_j}\omega\right) \cdot \theta^j \cdot \theta^i.$$
Since these formulas are established in the same way, we will only focus on the first formula. Let $p \in M$. We can assume e_i is orthonormal at p and $\nabla e_i|_p = 0$. We have identities:
$$\theta^i \cdot \theta^i = -1, \quad \theta^i \cdot \theta^j = -\theta^j \cdot \theta^i.$$
Hence
$$\begin{aligned} \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega &= -\sum_{i=1}^{n} \nabla^2_{e_i,e_i}\omega + \sum_{i\neq j} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega \\ &= -\nabla^*\nabla\omega + \sum_{i\neq j} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega \end{aligned}$$

$$= -\nabla^*\nabla\omega + \sum_{i<j} \theta^i \cdot \theta^j \cdot (\nabla^2_{e_i,e_j}\omega - \nabla^2_{e_j,e_i}\omega)$$

$$= -\nabla^*\nabla\omega + \sum_{i<j} \theta^i \cdot \theta^j \cdot R(e_i,e_j)\omega = -\nabla^*\nabla\omega + \frac{1}{2}\sum_{i\neq j} \theta^i \cdot \theta^j \cdot R(e_i,e_j)\omega$$

$$= -\nabla^*\nabla\omega + \frac{1}{2}\sum_{i\neq j} \theta^i \cdot \theta^j \cdot R(e_i,e_j)\omega.$$

□