
We will first compute the cohomology ring in the case when \( n = 1 \) (this is in the homework)

**Lemma 1.1.** We have the following graded algebra isomorphism

\[
H^*(Gr_1(\mathbb{R}^\infty), \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[a]
\]

where \( a \in H^2(\mathbb{RP}^\infty, \mathbb{Z}/2\mathbb{Z}) \) has degree 2.

**Proof.** \( \mathbb{RP}^k \) is constructed as a CW complex by attaching a \( k - 1 \) to \( \mathbb{RP}^{k-1} \) via the double covering map \( S^{k-1} \to \mathbb{RP}^{k-1} \). The cellular cohomology with \( \mathbb{Z}/2\mathbb{Z} \) coefficients is then the direct sum decomposition and the ordering of the line bundles preserves the direct sum decomposition and the ordering of the line bundles. Hence we just need to compute the ring structure. This follows from the following commutative diagram where \( i + j = n \):

\[
\begin{array}{cccc}
H^i(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \times H^j(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^n(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \\
\uparrow & \cong & \uparrow & \cong \\
H^i(\mathbb{RP}^n; \mathbb{RP}^n - \mathbb{RP}^j; \mathbb{Z}/2\mathbb{Z}) \times H^j(\mathbb{RP}^n; \mathbb{RP}^n - \mathbb{RP}^j; \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^n(\mathbb{RP}^n; \mathbb{RP}^n - \mathbb{RP}^0; \mathbb{Z}/2\mathbb{Z}) \\
\uparrow & \cong & \uparrow & \cong \\
H^i(\mathbb{R}^n; \mathbb{R}^n - \mathbb{R}^j; \mathbb{Z}/2\mathbb{Z}) \times H^j(\mathbb{R}^n; \mathbb{R}^n - \mathbb{R}^j; \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^n(\mathbb{R}^n; \mathbb{R}^n - 0; \mathbb{Z}/2\mathbb{Z}) \\
\uparrow & \cong & \uparrow & \cong \\
H^i(\mathbb{R}^i; \mathbb{R}^i - 0; \mathbb{Z}/2\mathbb{Z}) \times H^j(\mathbb{R}^j; \mathbb{R}^j - 0; \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^n(\mathbb{R}^n; \mathbb{R}^n - 0; \mathbb{Z}/2\mathbb{Z})
\end{array}
\]

\( \square \)

**Corollary 1.2.** \( H^*((\mathbb{RP}^\infty)^n) \cong (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n] \).

Note that \( (\mathbb{RP}^\infty)^n \) classifies vector bundles of the form \( \oplus_{i=1}^n \gamma_i \) where \( \gamma_i \) is a line bundle up to isomorphism which preserve the direct sum decomposition and the ordering of the line bundles \( \gamma_1, \cdots, \gamma_n \).

**Theorem 1.3 (Leray Hirsch Theorem).** Let \( \pi : E \to B \) be a fiber bundle (all our spaces are CW complexes). Let \( \iota : F \to E \) be the natural inclusion map of the fiber and suppose that there is a linear map

\[
s : H^*(F; \Lambda) \to H^*(E; \Lambda)
\]

satisfying \( \iota^* \circ s = id_{H^*(F)} \). Then the natural linear map

\[
H^*(F; \Lambda) \otimes H^*(B; \Lambda) \to H^*(E; \Lambda), \quad \alpha \otimes \beta \to s(\alpha) \cup \pi^* \beta
\]

is an isomorphism.

In particular the natural map

\[
\pi^* : H^*(B; \Lambda) \to H^*(E; \Lambda)
\]

is injective.

Later on we will also need a proof of a relative version of the Leray-Hirsch theorem.

**Theorem 1.4 (Relative Leray Hirsch Theorem).** Let \( \pi : E \to B \) be a fiber bundle (all our spaces are CW complexes) and let \( E_0 \subseteq E \) be a subbundle. Let \( \iota : F \to E \) be the natural inclusion map of the fiber and let \( F_0 \subseteq F \) be the fiber of \( E_0 \). Suppose that there is a linear map

\[
s : H^*(F; F_0; \Lambda) \to H^*(E, E_0; \Lambda)
\]
satisfying $\iota^* \circ s = \text{id}_{H^*(F,F_0)}$. Then the natural linear map
$$H^*(F,F_0; \Lambda) \otimes H^*(B; \Lambda) \longrightarrow H^*(E,E_0; \Lambda), \quad \alpha \otimes \beta \longrightarrow s(\alpha) \cup \pi^* \beta$$
is an isomorphism.

We will only prove the Leray-Hirsch theorem as the relative version of this theorem has exactly the same proof.

**Proof.** This argument would be straightforward if we knew about spectral sequences, but we don’t. As a result we will do this a different (but directly related) way. Now $B$ is a direct limit of compact sets $K_0 \subset K_1 \subset \cdots$. Therefore is sufficient for us to show that
$$H^*(F; \Lambda) \otimes H^*(K_i; \Lambda) \longrightarrow H^*(E|K_i; \Lambda), \quad \alpha \otimes \beta \longrightarrow s(\alpha) \cup \pi^* \beta$$
is an isomorphism for all $i$.

So from now on we will assume that $B$ is compact. Let $U_1, \ldots, U_m$ be open subsets of $B$ so that $E|U_i$ is trivial. Define $U_{<i} \equiv \bigcup_{j<i} U_i$. Suppose (by induction) we have shown that the map
$$F_{<i} : H^*(F; \Lambda) \otimes H^*(U_{<i}; \Lambda) \longrightarrow H^*(E|U_{<i}; \Lambda), \quad F_{<i}(\alpha \otimes \beta) \equiv s(\alpha) \cup \pi^* \beta$$
is an isomorphism for some $i$. We now wish to show that the corresponding map $F_{<i+1}$ is an isomorphism. Consider the following commutative diagram:

$$
\begin{array}{cccccc}
\alpha \otimes \beta & \longrightarrow & (s(\alpha)|U_{<i} \cap U_{i+1}) \cup \beta \\
H^*(F; \Lambda) \otimes H^*(U_{<i} \cap U_{i+1}; \Lambda) & \longrightarrow & H^*(E|U_{<i} \cap U_{i+1}; \Lambda) \\
\alpha \otimes \beta & \longrightarrow & (s(\alpha)|U_{<i}) \cup \beta \\
H^*(F; \Lambda) \otimes H^*(U_{<i}; \Lambda) & \longrightarrow & H^*(E|U_{<i}; \Lambda) \\
\alpha \otimes \beta & \longrightarrow & (s(\alpha)|U_{<i+1}) \cup \beta \\
H^*(F; \Lambda) \otimes H^*(U_{<i+1}; \Lambda) & \longrightarrow & H^*(E|U_{<i+1}; \Lambda)
\end{array}
$$

The vertical arrows form a Mayor-Vietoris long exact sequence. Also the horizontal arrows are isomorphisms at the top and the bottom for all $i$. Hence by the five lemma we get our isomorphism.

We have the following corollary of the Leray-Hirsch theorem:

**Theorem 1.5. Thom Isomorphism Theorem over $\mathbb{Z}/2$** Let $\pi : E \longrightarrow B$ be a rank $n$ vector bundle and define $E_0 \equiv E - B$ where $B \subset E$ is the zero section. Then there is a class $\alpha \in H^n(E,E_0; \mathbb{Z}/2\mathbb{Z})$ so that the map
$$H^*(B; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{n+n}(E,E_0; \mathbb{Z}/2\mathbb{Z}), \quad \beta \longrightarrow \beta \cup \alpha$$
is an isomorphism.

This theorem is true over any coefficient field if we assumed that $E$ is an oriented vector bundle.

**Definition 1.6.** The **unoriented Euler class** of a vector bundle $\pi : E \longrightarrow B$ as above is a class $e(E; \mathbb{Z}/2\mathbb{Z}) \in H^n(E; \mathbb{Z}/2\mathbb{Z})$ given by the image of the class $\alpha$ under the composition $H^n(E,E_0; \Lambda) \longrightarrow H^n(E; \Lambda) \longrightarrow H^n(B; \Lambda)$. 

Note that if $E$ is an oriented vector bundle then we can define the Euler class $e(E; \Lambda)$ over any coefficient ring $\Lambda$. Usually when people talk about the Euler class, they are talking about $e(E; \mathbb{Z})$ (we will call this the Euler class) and we will write $e(E)$.

**Proof.** We will only prove our theorem when the coefficient field is $\mathbb{Z}/2\mathbb{Z}$. The proof is exactly the same if we have oriented vector bundles and another coefficient ring.

Our fiber is $F = \mathbb{R}^n$ and the fiber of $\pi|_{p_0}$ is $F_0 = \mathbb{R}^n - 0$. By the relative Leray-Hirsch theorem it is sufficient to show that there is a class $\alpha \in H^n(E, E_0; \Lambda)$ whose restriction to $H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is the unit $1 \in \mathbb{Z}/2\mathbb{Z}$. We assume that $B$ is connected.

Let $(U_i)_{i \in \mathbb{N}}$ be an open cover by relatively compact sets where $E|_{U_i}$ is trivial. Define $U_{<i} = \bigcup_{j<i} U_j$. We'll suppose that $U_{<i}$ and $U_i$ is connected for all $i \in \mathbb{N}$ and that $F = \mathbb{R}^n \subset E|_{U_0}$.

Suppose (by induction) there is a class $\alpha_i \in H^n(E|_{U_{<i}}; E_0|_{U_{<i}}; \mathbb{Z}/2\mathbb{Z})$ whose restriction to $H^n(\mathbb{R}^n; \mathbb{Z}/2\mathbb{Z})$ is 1. Consider the Mayer-Vietoris sequence:

$$H^n(E|_{U_{<i+1}}; E_0|_{U_{<i+1}}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{a}$$

$$H^n(E|_{U_{<i}}; E_0|_{U_{<i}}; \mathbb{Z}/2\mathbb{Z}) \oplus H^n(E|_{U_{i+1}}; E_0|_{U_{i+1}}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{b} H^n(E|_{U_{<i}\cap U_{i+1}}; E_0|_{U_{<i}\cap U_{i+1}}; \mathbb{Z}/2\mathbb{Z}).$$

Since

$$H^*(E|_{U_{i+1}}; E_0|_{U_{i+1}}; \mathbb{Z}/2\mathbb{Z}) = H^*(U_{i+1}; \mathbb{Z}/2\mathbb{Z}) \oplus H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z}) \cong H^{*-n}(U_{i+1}; \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}$$

we get a class $\alpha' \in H^*(E|_{U_{i+1}}; E_0|_{U_{i+1}}; \mathbb{Z}/2\mathbb{Z})$ mapping to 1 under the above isomorphism and hence whose restriction to $H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z})$ is 1. Also since $E|_{U_{<i}\cap U_{i+1}}$ is trivial, we get using similar reasoning that the images of $\alpha_i$ and $\alpha'$ in $H^*(E|_{U_{<i}\cap U_{i+1}})$ are equal. Hence $b(\alpha_i + \alpha') = 0$. Hence there is a class $\alpha_{i+1} \in H^n(E|_{U_{<i+1}}; E_0|_{U_{<i+1}}; \mathbb{Z}/2\mathbb{Z})$ so that $a(\alpha_i + \alpha')$. This class maps to 1 in $H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z})$.

The Euler class satisfies the following properties:

1. **(Functoriality)** If $\pi : E \rightarrow B$ is isomorphic to $f^*E'$ for some other bundle $\pi' : E' \rightarrow B'$ and function $f : B \rightarrow B'$ then $e(E; \Lambda) = f^*(e(E'; \Lambda))$.
2. **(Whitney Sum Formula)** If $\pi : E \rightarrow B$ and $\pi' : E' \rightarrow B$ are two vector bundles over the same base then $e(E \oplus E'; \Lambda) = e(E; \Lambda) \cup e(E'; \Lambda)$.
3. **(Normalization)** If $E$ admits a nowhere zero section then $e(E; \Lambda) = 0$.
4. **(Orientation)** If $E$ is an oriented vector bundle and $\overline{E}$ is the same bundle with the opposite orientation then $e(E) = -e(\overline{E})$.

The following is a geometric interpretation of the Euler class when the base is a compact manifold. We need a definition first.

**Definition 1.7.** Let $M_1, M_2$ be submanifolds of a manifold $X$. Then $M_1$ is transverse to $M_2$ if for every point $x \in M_1 \cap M_2$, we have that

$$\text{codim}(T_xM_1 \cap T_xM_2 \subset T_xX) = \text{codim}(T_xM_1 \subset T_xX) + \text{codim}(T_xM_2 \subset T_xX).$$

Let $\pi : E \rightarrow B$ be a smooth vector bundle over a smooth compact manifold $B$. A smooth section $s : B \rightarrow E$ is transverse to 0 if the submanifold $s(B) \subset E$ is transverse to the zero section $B \subset E$.

Note that if $M_1$ intersects $M_2$ transversely then $M_1 \cap M_2$ is a manifold. Also if $X$, $M_1$ and $M_2$ are oriented (in other words $TX, TM_1$ and $TM_2$ are oriented) then $M_1 \cap M_2$ has an
orientation defined as follows: Let \( N(M_1 \cap M_2) \), \( NM_1 \) and \( NM_2 \) be the normal bundles of \( M_1 \cap M_2 \), \( M_1 \) and \( M_2 \) inside \( X \). Then we have isomorphisms
\[
TM_1 \oplus NM_1 \cong TX|_{M_1}, \quad TM_2 \oplus NM_2 \cong TX|_{M_2},
\]
\[
T(M_1 \cap M_2) \oplus N(M_1 \cap M_2) \cong TX|_{M_1 \cap M_2},
\]
\[
N(M_1) \oplus N(M_2) \cong NM_1|_{M_1 \cap M_2} \oplus NM_2|_{M_1 \cap M_2}.
\]
The first two isomorphisms give us an orientation on \( NM_1 \) and \( NM_2 \) and the last one gives us an orientation on \( N(M_1 \cap M_2) \). The third isomorphism then gives us an orientation on \( T(M_1 \cap M_2) \) called the intersection orientation.

Also note that any compact manifold \( M \) (whether oriented or not) has a fundamental class \([M] \in H^n(M; \mathbb{Z}/2\mathbb{Z})\) over \( \mathbb{Z}/2\mathbb{Z} \).

It turns out that a ‘generic’ section is transverse (‘generic’ will be defined precisely later in the course). I won’t prove this for the moment (maybe later).

**Lemma 1.8.** Let \( \pi : E \rightarrow B \) be a smooth vector bundle over a smooth compact manifold \( B \) with a smooth section \( s \) transverse to 0. Then \( e(E; \mathbb{Z}/2\mathbb{Z}) \) is Poincaré dual to \([s^{-1}(0)] \in H^*(B; \mathbb{Z}/2\mathbb{Z})\).

If \( E \) and \( B \) are oriented then \( s^{-1}(0) \) has the intersection orientation and the above lemma makes sense in this case over any coefficient field \( \Lambda \).

Our goal is to compute the cohomology of \( Gr_n(\mathbb{R}^\infty) \) and so we must continue....

**Definition 1.9.** Let \( \pi : E \rightarrow B \) be a rank \( n \) vector bundle. The **projective bundle** \( \mathbb{P}(E) \rightarrow B \) is the fiber bundle whose fiber at a point \( b \in B \) is \( \mathbb{P}(\pi^{-1}(b)) \) (i.e. the set of lines inside \( \pi^{-1}(b) \)).

**Lemma 1.10.** Let \( \pi : E \rightarrow B \) be a rank \( n \) vector bundle. The natural map \( H^*(B) \rightarrow H^*(\mathbb{P}(E)) \) is injective. In fact \( H^*(\mathbb{P}(E)) \cong H^*(\mathbb{RP}^{n-1}) \otimes H^*(B) \) and the natural map \( H^*(B) \rightarrow H^*(\mathbb{P}(E)) \) is the inclusion map into the first factor.

**Proof.** Now \( \mathbb{P}(E) \) as a canonical line bundle \( \gamma_E \) whose fiber at a point \( x \in E \) is the line \( l \) passing through \( x \) inside \( \pi^{-1}(\pi(x)) \). Let \( f : \mathbb{P}(E) \rightarrow \mathbb{RP}^\infty \) be the classifying map for this line bundle. Recall that \( H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[a] \) where \( a \in H^2(\mathbb{RP}^\infty) = 0 \). We will also write \( H^*(\mathbb{RP}^{n-1}) = (\mathbb{Z}/2\mathbb{Z})[b]/(b^n) \). Recall that our fiber \( F \) is equal to \( \mathbb{RP}^{n-1} \).

Since \( \gamma \) restricted to each fiber is 0(−1), we get that \( f^*a \) restricted to the fiber \( F = \mathbb{RP}^{n-1} \) is \( b \). Hence \( f^*(a^n)|_F = b^n \) which implies that that map \( H^*(E) \rightarrow H^*(F) \) is surjective. Hence by the Leray-Hirsch theorem, \( H^*(\mathbb{P}(E)) \cong H^*(\mathbb{RP}^{n-1}) \otimes H^*(B) \) and the natural map \( H^*(B) \rightarrow H^*(\mathbb{P}(E)) \) is the inclusion map into the first factor. \( \square \)

**Definition 1.11.** Let \( \pi : E \rightarrow B \) be a real vector bundle. A **splitting map** for \( E \) is a map \( f : B' \rightarrow B \) so that \( f^*E \cong \bigoplus_{i=1}^n \gamma_i \) where \( \gamma_i \) are line bundles over \( B' \) and where \( f^* : H^*(B) \rightarrow H^*(B') \) is injective.

**Lemma 1.12.** Let \( \pi : E \rightarrow B \) be a vector bundle. Let \( P : \mathbb{P}(E) \rightarrow B \) be the associated projective bundle. Then there is a line subbundle \( \gamma \subset P^*E \).

**Proof.** Here \( \gamma \) is defined to be the line in \( P^*E \) which sends a point \( x \in \mathbb{P}(E) \) to the corresponding line in \( E \). \( \square \)

**Lemma 1.13.** Every real vector bundle \( \pi : E \rightarrow B \) of rank \( n \) has a splitting map.
Proof. Suppose (inductively) we have constructed a map \( P_k : B_k \rightarrow B \) for some \( 0 \leq k < n \) so that \( P_k^* : V \oplus \bigoplus_{i=1}^{k} \gamma_i \rightarrow B_k \) where \( V \) is a vector bundle and \( \gamma_i \) are line bundles and so that \( P_k^* : H^*(B) \rightarrow H^*(B_k) \) is injective. Define \( B_{k+1} \equiv \mathbb{P}(V) \) and let \( p : \mathbb{P}(V) \rightarrow B' \) be the natural map. Then by Lemma 1.12 we have that \( p^*(V) \cong V' + \gamma_{k+1} \) where \( \gamma_{k+1} \) is a line subbundle of \( V \). Define

\[
\begin{align*}
P_{k+1} : B_{k+1} & \rightarrow B, \quad P_{k+1} \equiv P_k \circ p.
\end{align*}
\]

Then \( P_{k+1}^* E = V' \oplus \bigoplus_{i=1}^{k+1} \gamma_i \). Also \( p^* : H^*(B_k) \rightarrow H^*(B_{k+1}) \) is injective by Lemma 1.10. Hence \( P_{k+1}^* : H^*(B) \rightarrow H^*(B_{k+1}) \) is injective. Therefore we are done by induction. \( \square \)

**Definition 1.14.** A polynomial \( p(a_1, \cdots, a_n) \in (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n] \) is **symmetric** if \( p(a_1, \cdots, a_n) = p(a_{\sigma(1)}, \cdots, a_{\sigma(n)}) \) for any permutation \( \sigma \) of \( \{1, \cdots, n\} \).

The **\( n \)th symmetric function** \( \sigma_i \in (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n] \) is the polynomial

\[
\sum_{0 \leq j_1 < j_2 < \cdots < j_i \leq n} \prod_{k=1}^{i} a_{j_k}.
\]

We have the following lemma (which we won’t prove):

**Lemma 1.15.** The subring \( R^\sigma \subset R \equiv (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n] \) of symmetric polynomials is freely generated by elementary symmetric functions \( \sigma_1, \cdots, \sigma_n \). Hence

\[
R \cong (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n] \subset (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n].
\]

**Theorem 1.16.** Let

\[
h_n : (\mathbb{RP}^\infty)^n \rightarrow Gr_n(\mathbb{R}^\infty)
\]

be the classifying map for the rank \( n \) bundle \( \bigoplus_{i=1}^{n} \gamma_i \). Then

\[
h_n^* : H^*(Gr_n(\mathbb{R}^\infty)) \rightarrow H^*((\mathbb{RP}^\infty)^n) \cong (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]
\]

is injective and its image is the free algebra generated by the elementary symmetric functions \( \sigma_1, \cdots, \sigma_n \).

Hence

\[
H^*(Gr_n(\mathbb{R}^\infty)) \cong (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n]
\]

for natural classes

\[
\sigma_1 \in H^1(Gr_n(\mathbb{R}^\infty)), \cdots, \sigma_n \in H^n(Gr_n(\mathbb{R}^\infty)).
\]

**Proof.** First of all the natural map \( h_n^* : H^*(Gr_n(\mathbb{R}^\infty)) \rightarrow H^*((\mathbb{RP}^\infty)^n) \) is injective for the following reason:

Let \( f : B \rightarrow Gr_n(\mathbb{R}^\infty) \) be the splitting map. Let \( g : B \rightarrow (\mathbb{RP}^\infty)^n \) be the corresponding classifying map for \( f^* \gamma_i \). Then since \( (g \circ h_n)^* \gamma_i \cong f^* \gamma_i \) and since \( Gr_n(\mathbb{R}^\infty) \) is a classifying space, we can homotope \( f \) so that \( f = g \circ h_n \). Since \( f^* : H^*(Gr_n(\mathbb{R}^\infty)) \rightarrow H^*(B) \) is injective, we get that \( h_n^* : H^*(Gr_n(\mathbb{R}^\infty)) \rightarrow H^*((\mathbb{RP}^\infty)^n) \) is injective.

The image of the map must be contained inside \( (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n] \) since permuting linear bundles does not change the isomorphism type of their direct sum decomposition. This means that if we compose \( h_n \) with a map permuting the factors inside \( (\mathbb{RP})^n \), we get a map which is homotopic to \( h_n \).

Hence it is sufficient for us to show that \( \sigma_i \) is in the image of \( h_n^* \) for all \( i \). This is done in the following way: We have that \( h_n^*(e(\gamma_i)) = e(\bigoplus_{i=1}^{n} \gamma_1) = \prod_{i=1}^{n} a_i = \sigma_i \). Hence \( \sigma_i \in \text{Im}(h_n^*) \).
We have: \( H^*\left((\mathbb{RP}^{n-1})^{n-1}\right) = (\mathbb{Z}/2\mathbb{Z})[a'_1, \cdots, a'_{n-1}] \). Let \( \sigma'_k \in H^*\left((\mathbb{RP}^{n-1})^{n-1}\right) \) be the \( k \)th symmetric function in \( a'_1, \cdots, a'_{n-1} \).

Now suppose (by induction) that the image of

\[
h_{n-1} : H^*(\text{Gr}_{n-1}(\mathbb{R}^\infty)) \rightarrow H^*(\mathbb{RP}^\infty)^{n-1}
\]

contains \( \sigma'_k \) for all \( k \).

Consider the commutative diagram:

\[
\begin{array}{ccc}
H^*\left((\mathbb{RP}^\infty)^{n-1}\right) & \xrightarrow{h^*_{n-1}} & H^*(\text{Gr}_{n-1}(\mathbb{R}^\infty)) \\
\downarrow{\iota^*_{n-1}} & & \downarrow{h^*_n} \\
H^*\left((\mathbb{RP}^\infty)^n\right) & \xrightarrow{h^*_n} & H^*(\text{Gr}_{n}(\mathbb{R}^\infty))
\end{array}
\]

Consider the restricted map:

\[
A' \equiv \iota^*_{n-1}|_{(\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_{n-1}]} : (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_{n-1}] \rightarrow (\mathbb{Z}/2\mathbb{Z})[\sigma'_1, \cdots, \sigma'_{n-1}].
\]

This is an isomorphism since \( \iota^*_{n-1}(a_n) = 0 \). Since \( \sigma'_k \in \text{Im}(h^*_{n-1}) \) we then get that \( \sigma_k \in \text{Im}(h^*_n) \) by looking at the above commutative diagram and the fact that \( A' \) is an isomorphism. Hence by induction we have that \( \text{Im}(h^*_n) = (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_{n-1}, \sigma_n] \).

\( \square \)