1. Cell decomposition of Grassmannian.

We will first describe the cell structure. We have natural inclusions:

$$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^{m-1} \subset \mathbb{R}^m.$$

An n-plane $X \subset \mathbb{R}^m$ gives us a sequence of integers:

$$\dim(X \cap \mathbb{R}^0) = 0 \leq \dim(X \cap \mathbb{R}^1) \leq \cdots \dim(X \cap \mathbb{R}^{m-1}) \leq \dim(X \cap \mathbb{R}^m) = n.$$

Two consecutive integers in this sequence differ by at most one due to the fact that $\dim(\mathbb{R}^i) - \dim(\mathbb{R}^{i-1}) = 1$. Hence the above sequence contains n-jumps of size 1.

Definition 1.1. A **Schubert symbol** is a sequence of n integers $0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq m$. We define $e(\sigma) \subset Gr_n(\mathbb{R}^m)$ to be the set of $X \subset Gr_n(\mathbb{R}^m)$ so that $\dim(X \cap \mathbb{R}^{\sigma_i}) = i$ and $\dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i - 1$. In other words, σ_i is the point where the dimension 'jumps'. The closure $\overline{e(\sigma))}$ is called a **Schubert variety**.

We will show later that this is an open cell of dimension $d(\sigma) = \sum_{i=1}^n (\sigma_i - i)$. Define

$$H_k \equiv \left\{(x_1, \cdots, x_k) \in \mathbb{R}^k \ : \ x_k > 0\right\}.$$

This is the **upper half plane**. We have that $X \in e(\sigma)$ if and only if it has a basis $v_1, \cdots, v_n \in \mathbb{R}^m$ so that $v_i \in H^{\sigma_i}$ for all $i \in \{1, \cdots, n\}$.

We can rescale the basis so that the last non-zero coordinate in v_i is 1. This means that $X \in e(\sigma)$ if and only if the basis $v_1, \cdots, v_n$ for X can be described as the row space of the $n \times m$ matrix:

$$\begin{bmatrix} * & \cdots & *10 & \cdots & 000 & \cdots & 000 & \cdots & 0 \\ * & \cdots & *** & \cdots & *10 & \cdots & 000 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & *** & \cdots & *** & \cdots & *10 & \cdots & 0 \end{bmatrix}$$

Here the i-th row has $\sigma_i - 1$ entries with anything in them followed by a 1 and then followed by $m - (\sigma_i - 1)$ zeros.

Lemma 1.2. For each $X \in e(\sigma)$, there is a unique orthonormal basis

$$(v_1, \cdots, v_n) \in \prod_{i=1}^{n} H^{\sigma_i}$$

of X.

Proof. Since $\dim(X \cap \mathbb{R}^{\sigma_1}) = 1$, there are only two unit vectors inside $X \cap \mathbb{R}^{\sigma_1}$ and only one inside $X \cap H^{\sigma_1}$. Hence v_1 is unique.

Now $\dim(X \cap \mathbb{R}^{\sigma_2}) = 2$. There are at most two unit vectors inside $X \cap \mathbb{R}^{\sigma_2}$ which are orthogonal to v_1. Hence there is only one such vector inside $X \cap H^{\sigma_2}$. Hence v_2 is unique.

We now continue by induction giving us our result. □

Definition 1.3. Let $e'(\sigma)$ be the set of orthonormal n-frames $(v_1, \cdots, v_n)$ so that $v_i \in H^{\sigma_i}$. $\overline{e}'(\sigma)$ be the set of orthonormal n-frames $(v_1, \cdots, v_n)$ so that v_i is in the closure of H^{σ_i}.

Note that $\overline{e'(\sigma)} = \overline{e}'(\sigma)$. The discussion above tells us that $e'(\sigma)$ is homeomorphic to $e(\sigma)$.

We have the following lemma:

Lemma 1.4. The set $\overline{e}'(\sigma)$ is a topologically closed cell of dimension $d(\sigma) = \sum_{i=1}^n (\sigma_i - i)$ whose interior maps homeomorphically to $e(\sigma)$. and

As a result, $e(\sigma)$ is an open cell of dimension $d(\sigma)$ and the map $\partial(\overline{e}'(\sigma)) \longrightarrow Gr_n(\mathbb{R}^m)$ is the gluing map for the boundary of the corresponding n-cell.

Proof. We proceed by induction on n. The set $\overline{e}'(\sigma_1)$ is the set of vectors $(x_1, \cdots, x_{\sigma_1}, 0, \cdots, 0)$ so that $\sum_{i=1}^{\sigma_1} x_i^2 = 11$ and $x_{\sigma_1} \geq 0$. This is a closed hemisphere of dimension $\sigma_1 - 1$ and hence is homeomorphic to the disk of dimension $\sigma_1 - 1$.

Now suppose $\overline{e}'(\sigma_1, \cdots, \sigma_n)$ is homeomorphic to a disk of dimension $\sum_{i=1}^{n}(\sigma_i - i)$ and consider $\overline{e}'(\sigma_1, \cdots, \sigma_{n+1})$. The key idea here is to construct a homeomorphism

$$\beta : \overline{e}'(\sigma_1, \cdots, \sigma_n) \times D \longrightarrow \overline{e}'(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})$$

where D is a dimension $\sigma_{n+1} - (n+1)$ ball.

Let

$$T_{u,v} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

be the unique rotation sending u to v and fixing all vectors orthogonal to u and v. This has the following properties:

(1) $T_{u,v}x$ is continuous in u, v and x and
(2) if $u, v \in \mathbb{R}^k$ then $T_{u,v}(x) = x + a$ where $a \in \mathbb{R}^k$. In other words, it fixes x mod $\mathbb{R}^k$.

Define $b_i \equiv e_{\sigma_i}$ for all $i = 1, \cdots, n$ (in other words, the σ_ith coordinate is 1 and all the other coordinates are 0). So $(b_1, \cdots, b_n) \in \overline{e}'(\sigma_1, \cdots, \sigma_n)$. For each $x = (x_1, \cdots, x_n) \in \overline{e}'(\sigma_1, \cdots, \sigma_n)$, define

$$T_x : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad T_x \equiv T_{b_n, x_n} \circ \cdots \circ T_{b_1, x_1}.$$

Let

$$D \equiv \{u \in H^{\sigma_{n+1}} \ : \ u \cdot u = 1, \quad u \cdot b_i = 0 \quad \forall\ i = 1, \cdots, n\}.$$

Here D is homeomorphic to a closed hemisphere inside $H^{\sigma_{n+1}}$ and hence is homeomorphic to a ball of dimension $\sigma_{n+1} - (n+1)$. Define:

$$\beta : \overline{e}'(\sigma_1, \cdots, \sigma_n) \times D \longrightarrow \overline{e}'(\sigma_1, \cdots, \sigma_n, \sigma_{n+1}), \quad \beta(x, u) = (x, T_x u).$$

This map is well defined since:

- T fixes $H^{\sigma_{n+1}}$ and since
- $((x_1, \cdots, x_n), T_{x_1, \cdots, x_n} u)$ is an orthonormal basis as:

$$T^{-1}_{x_1, \cdots, x_n}((x_1, \cdots, x_n), T_{x_1, \cdots, x_n} u) = (b_1, \cdots, b_n, u)$$

  which is orthonormal and T is an isometry.

Also β is an invertible continuous map and so is a homeomorphism. Hence we are done by induction.

A similar induction process tells us that the interior of $\overline{e}'(\sigma)$ is the interior of $\overline{e}'(\sigma)$ for all σ.

We now need to show that the interior of $\overline{e}'(\sigma)$ maps homeomorphically onto $e(\sigma)$ for all σ. The interior of $e(\sigma_1, \cdots, \sigma_n)$ corresponds to orthonormal vectors $v_1, \cdots, v_n$ so that $v_i \in H^{\sigma_i}$ for all $i = 1, \cdots, n$. These are precisely the elements in the interior of $\overline{e}'(\sigma)$.

□

Theorem 1.5. The $\binom{m}{n}$ sets $e(\sigma_1, \cdots, \sigma_n)$ for all n form a cell complex for $Gr_n(\mathbb{R}^m)$. Also taking the limit as $m \to \infty$, one gets an infinite cell decomposition of $Gr_n(\mathbb{R}^\infty)$

Proof. Basically we need to show that the boundary of $\overline{e}'(\sigma_1, \cdots, \sigma_n)$ gets mapped to images of cells of lower dimension.

Let $(v_1, \cdots, v_n) \in \overline{e}'(\sigma_1, \cdots, \sigma_n) - e'(\sigma_1, \cdots, \sigma_n)$. Now $v_i \in H^{\sigma_i}$ for all $i \in 1, \cdots, n$. Also since $(v_1, \cdots, v_n) \notin e'(\sigma_1, \cdots, \sigma_n)$, there is some $j \in \{1, \cdots, n\}$ so that $v_j \in H^{\sigma_j - 1}$. Define $\sigma'_i \equiv \sigma_i$ for all $i \neq j$ and $\sigma'_j \equiv \sigma_j - 1$. Then $v_i \in H^{\sigma'_i}$ for all $i = 1, \cdots, n$ and hence $(v_1, \cdots, v_n)$ maps to the image of $\overline{e}'(\sigma'_i)$ which is the image of a lower dimensional cell.

We have that $Gr_n(\mathbb{R}^\infty)$ has the corresponding direct limit cell complex with the direct limit topology. $\square$

Definition 1.6. A **partition** of an integer $r \geq 0$ is an unordered sequence of positive integers $i_1, \cdots, i_s$ which sum to r. The number of partitions of r is denoted by $p(r)$.

e.g. The partitions of 4 are

$$1,1,1,1, \quad 1,1,2, \quad 1,3, \quad ,2,2, \quad 4$$

and so $p(4) = 5$. Zero has 1 partition which is the vacuous partition.

Corollary 1.7. The number of r cells in $Gr_n(\mathbb{R}^m)$ is the number of partitions of r which each number in the partition is $\leq m - n$.

In particular, the number of r cells in $Gr_n(\mathbb{R}^\infty)$ is $p(r)$.

Proof. The r cells correspond to sequences

$$1 \leq \sigma_1 < \cdots < \sigma_n \leq m$$

so that $\sum_{i=1}^n (\sigma_i - i) = r$. Let l be the number of terms where $\sigma_i - i = 0$. Hence $\sigma_l - l, \cdots, \sigma_n - n$ is our partition of r. Also since $\sigma_i \leq m - (n - i)$ for all i, we get that $\sigma_i - i \leq m - n$ for all i.

Conversely if $0 \leq j_1 \leq j_s \leq m - n$ is a partition of r so that $j_i \leq m - n$ for all i then we define $\sigma_i \equiv i$ for all $i \leq n - s$ and $\sigma_i \equiv j_{i+s-n}$ for all $i > n - s$.

$\square$