1. Stiefel Whitney Classes

Throughout this section, all cohomology groups have coefficients in $\mathbb{Z}/2\mathbb{Z}$. I will just write $\mathbb{Z}/2 \equiv \mathbb{Z}/2\mathbb{Z}$. The following theorem will be proven later:

Theorem 1.1. To each topological vector bundle $\pi : E \longrightarrow B$ of rank k, there is a sequence of cohomology classes

$$w_i(E) \in H^i(B; \mathbb{Z}/2), \quad i = 0, 1, 2, \cdots$$

where $w_i(E)$ is called the **ith Stiefel-Whitney class** so that:

(Stiefel-1) **rank axiom** $w_0(E) = 1$ and $w_i(E) = 0$ for $i > k$.

(Stiefel-2) **Naturality**: For any continuous map $f : B' \longrightarrow B$, we have that $w_i(f^*E) = f^*(w_i(E))$. Also isomorphic vector bundles have the same Stiefel-Whitney classes.

(Stiefel-3) **The Whitney Product Theorem**: Let $\pi : E \longrightarrow B$, $\pi' : E' \longrightarrow B$ be fiber bundles over the same base B. Then

$$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) \cup w_{k-i}(E').$$

(Stiefel-4) **normalization axiom**: $w_1(\mathcal{O}_{\mathbb{RP}^1}(-1)) \neq 0$ where $\mathcal{O}_{\mathbb{RP}^1}(-1)$ is the natural vector bundle on $\mathbb{RP}^1$ introduced earlier.

For the moment we will assume that such classes w_i exist for each vector bundle. A proof of existence will appear later in the course.

For smooth manifolds this was defined by Stiefel in 1935 and then for general topological spaces by Whitney.

From the naturality axiom we have the following propositions.

Proposition 1.2. If E_1 is isomorphic to E_2 then $w_i(E_1) = w_i(E_2)$.

Proposition 1.3. If $\pi : E \longrightarrow B$ is trivial then $w_i(E) = 0$ for all $i > 0$.

Proof. Here E is isomorphic to $f^*\mathbb{R}^k$ where $\mathbb{R}^k \longrightarrow \text{pt}$ is the trivial vector bundle over a point. Since $H^i(\text{pt}) = 0$ for all $i > 0$ implies that $w_i(\mathbb{R}^k) = 0$ for all $i > 0$ and so $w_i(E) = 0$ for all $i > 0$. $\square$

The following proposition follows from Proposition 1.3 and the Whitney product formula:

Proposition 1.4. If $\pi : E \longrightarrow B$ is any vector bundle and $\pi' : E' \longrightarrow B$ is trivial then $w_i(E) = w_i(E \oplus E')$ for all $i \geq 0$.

Proposition 1.5. Suppose that a rank k vector bundle $\pi : E \longrightarrow B$ admits a continuous section (i.e. a continuous map $s \longrightarrow B \longrightarrow E$ satisfying $\pi \circ s = id_B$). Then $w_k(E) = 0$.

More generally, if we have m sections $s_1, \cdots, s_m$ so that $s_1(x), \cdots, s_m(x)$ are linearly independent for all $x \in B$ then

$$w_k(E) = w_{k-1}(E) = \cdots = w(k-m) = 0.$$

Proof. Choose a metric on E. Let V be the vector subbundle whose fiber at $x \in B$ is $\text{span}(s_1(x), \cdots, s_m(x))$ and let $V^\perp$ be the vector subbundle whose fiber at $x \in E$ consists of the set of vectors in E_x which are orthogonal to V. Then $E = V \oplus V^\perp$. Hence we have an isomorphism:

$$\Phi : V \oplus \mathbb{R}^m \longrightarrow V \oplus V^\perp = E, \quad \Phi(v, (a_1, \cdots, a_m))|_{E_x} = v + \sum_{i=1}^{m} a_i s_i(x) \quad \forall\, x \in B.$$

The result now follows from proposition 1.4 combined with rank axiom. $\square$

If $E \oplus E'$ is trivial then the Whitney product formula combined with Proposition 1.3 gives the following relations:

$$w_1(E) + w_1(E') = 0$$
$$w_2(E) + w_1(E) \cup w_1(E') + w_2(E') = 0$$
$$w_3(E) + w_2(E) \cup w_1(E') + w_1(E) \cup w_2(E') + w_3(E') = 0$$

etc..

As a result $w_i(E')$ is a polynomial expression in the Stiefel-Whitney classes of E.

Define $H^{\Pi}(B; \mathbb{Z}/2)$ to be the ring of formal infinite series:

$$a_0 + a_1 + a_2 + a_3 + \cdots, \quad a_i \in H^i(B; \mathbb{Z}/2) \quad \forall i \in \mathbb{Z}_{>0}$$

with product

$$(a_0 + a_1 + \cdots) \cdot (b_0 + b_1 + \cdots) = a_0 \cup b_0 + (a_0 \cup b_1 + a_1 \cup b_0) + (a_0 \cup b_2 + a_1 \cup b_1 + a_2 \cup b_0) + \cdots.$$

The product is commutative since we are working mod 2 and is also associative. Note $H^{\Pi}(V; \mathbb{Z}/2) = \prod_{i \in \mathbb{N}} H^i(V; \mathbb{Z}/2)$. If V is a finite dimensional CW complex such a smooth manifold then $H^{\Pi}(V; \mathbb{Z}/2) = H^*(V; \mathbb{Z}/2)$.

Definition 1.6. The **total Stiefel Whitney class** of a vector bundle of $\pi : V \to B$ is given by

$$w(V) \equiv 1 + w_1(V) + w_2(V) + \cdots.$$

Note that the Whitney product theorem now says:

$$w(V \oplus W) = w(V) w(W).$$

Lemma 1.7. The subset

$$R \equiv \{1 + a_1 + \cdots \in H^{\Pi}(V; \mathbb{Z}/2)\} \subset H^{\Pi}(V; \mathbb{Z}/2)$$

is a commutative subgroup under multiplication. In fact it is the group of units of $H^{\Pi}(V; \mathbb{Z}/2)$.

Proof. This follows from the (Taylor) formula:

$$(1 + (a_1 + a_2 + \cdots))^{-1} = 1 - (a_1 + a_2 + \cdots) + (a_1 + a_2 + \cdots)^2 - \cdots$$
$$= 1 - a_1 + (a_1^2 - a_2) + (-a_1^3 + a_1 a_2 + a_2 a_1 - a_3) \cdots.$$

Exercise: prove this properly. $\square$

This means that we can solve the equation $w(V \oplus W) = w(V) w(W)$ giving us $w(V) = w(V)^{-1} w(W)$.

Theorem 1.8. (Whitney Duality Theorem) Let $B \subset M$ be a submanifold of a manifold M. Then $w(\mathcal{N}_M(B)) = w(TB)^{-1} w(TM|_B)$.

Proof. This follows from the above discussion combined with the fact that $\mathcal{N}_M(B) \oplus TB = TM|_B$. $\square$

Example 1.9. Since $\mathcal{N}_{\mathbb{R}^n} S^{n-1}$ is trivial (as it has a section given by radial the vector field pointing outwards), we have that $w(TS^{n-1}) = w(\mathbb{R}^n|_{S^{n-1}}) = 0$.

Lemma 1.10. (Homework)

$$H^i(\mathbb{RP}^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$

Also if $a \in H^i(\mathbb{RP}^n;\mathbb{Z}/2) - \{0\}$ and $b \in H^j(\mathbb{RP}^n;\mathbb{Z}/2) - \{0\}$ and $i + j \le n$ then $a \cup b \in H^{i+j}(\mathbb{RP}^n;\mathbb{Z}/2) - \{0\}$.

In other words: $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$ is isomorphic to the algebra $\mathbb{Z}/2[a]/a^{n+1}$ where a has degree 1.

Combining the above lemma with (Stiefel-4) and (Stiefel-1):

Corollary 1.11. $w(\mathcal{O}_{\mathbb{RP}^1}(-1)) = 1 + a$ where $a \in H^1(\mathbb{RP}^n) - \{0\}$ is the unique non-zero element.

Definition 1.12. To make things easier we will define $\gamma_n^1 \equiv \gamma_n^1$.

Example 1.13. Since γ_n^1 is a subbundle of the trivial bundle $\nu \equiv \mathbb{RP}^n \times \mathbb{R}^{n+1}$, we have

$$\begin{aligned} w(\nu/\gamma_n^1) &= (1+a)^{-1} \\ &= 1 - a + a^2 - a^3 + \cdots + (-1)^n a^n n = 1 + a + \cdots + a^n. \end{aligned}$$

This is an example of a rank n vector bundle with the property that all that Stiefel-Whitney classes which can be non-zero are in fact non-zero.

Lemma 1.14. There is a canonical isomorphism:

$$T\mathbb{RP}^n \cong Hom\left(\gamma_n^1,\ \nu/\gamma_n^1\right).$$

Proof. Since

$$TS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \ : \ x \cdot v = 0\}$$

and since the map $x \to \pm x$ gives us our double cover $S^n \to \mathbb{RP}^n$, we have that

$$T\mathbb{RP}^n = S^n \times \mathbb{R}^{n+1}/\pm 1.$$

In other words, it is the set of pairs $\{(x,v),(-x,-v)\}$ where $x \in S^n$ and $v \in \mathbb{R}^{n+1}$. Such a pair gives us a unique linear map

$$L_x : \gamma_n^1|_x \to (\gamma_n^1)^\perp|_x$$

sending x to v. Conversely any such linear map determines a pair $\{(x,v),(-x,-v)\}$ as above. $\square$

Theorem 1.15.

$$T\mathbb{RP}^n \oplus (\mathbb{RP}^n \times \mathbb{R}) \cong \oplus_{i=1}^{n+1} \gamma_n^1.$$

Hence:

$$w(T\mathbb{RP}^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \cdots$$

Proof. We have that $Hom(\gamma_n^1, \gamma_n^1)$ is trivial as it has a nonwhere zero section given by the identity map. Hence:

$$\begin{aligned} T\mathbb{RP}^n \oplus (\mathbb{RP}^n \times \mathbb{R}) &\cong T\mathbb{RP}^n \oplus Hom(\gamma_n^1, \gamma_n^1) \\ \cong Hom(\gamma_n^1, (\nu/\gamma_n^1) \oplus \gamma_n^1) &\cong Hom(\gamma_n^1, \mathbb{RP}^n \times \mathbb{R}^n) \\ &\cong \oplus_{i=1}^{n+1} \gamma_n^1. \end{aligned}$$

$\square$

Corollary 1.16. (Stiefel) The class $w(\mathbb{RP}^n)$ equals 1 if and only if n is a power of 2.

Proof. **if**: This is easy by inductively using the relation $(a+b) = a^2 + 2ab + b^2$.
only if: Suppose $n+1 = 2^r m$ where $m > 1$ is odd. Then

$$w(\mathbb{RP}^n) = (1 + a^{2^r})^m) = 1 + ma^{2^r} + m(m-1)2a^{2.2^r} + \cdots \neq 1.$$

□

Division Algebras.

Definition 1.17. A manifold M is **parallelizable** if its tangent space is a trivial vector bundle.

Example 1.18. S^1 is parallelizable since $\frac{\partial}{\partial \theta}$ is a non-trivial section.

Theorem 1.19. Suppose that there is a bilnear product operation

$$p : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n$$

without zero divisors (I.e. $p(x \otimes y) \neq 0$ for all $x, y \in \mathbb{R}^n - 0$). Then $w(\mathbb{RP}^{n-1}) = 1$. Hence n cannot be a power of 2.

This product does not need to be associative or have an identity element. In fact such division algebras only exist when $n = 1, 2, 4, 8$ (Bott Milnor Kevaire 1958).

Proof. Let $e_1, \cdots, e_n$ be basis vectors for $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$ define the right multiplication map

$$R_x : \mathbb{R}^n \to \mathbb{R}^n, \quad R_x(y) \equiv p(y, x).$$

This map is invertible if $x \neq 0$.
For each $x \in \mathbb{R}^n - \{0\}$,

$$x = R_1(R_1^1(x)), R_2(R_1^1(x)), \cdots, R_k(R_1^1(x))$$

form a basis for $\mathbb{R}^n$ with the first vector equal to x (this is because multiplication on the left is also an isomorphism). Hence we have $n-1$ linearly indpendent sections

$$\mathbb{RP}^{n-1} \to Hom(\gamma^1_{n-1}, (\mathbb{RP}^n \times \mathbb{R}^n)/\gamma^1_{n-1}), \quad [\pm x] \to (y \to [R_i(R_1^{-1}(y))]), \forall\ y \in \mathbb{R}x$$

$$\forall\ [\pm x] \in \mathbb{RP}^n,\ i \in \{1, \cdots, n-1\}.$$

□

Note that if a manifold M of dimension n can be immersed in to $\mathbb{R}^{n+k}$ then its dual class $\overline{w}(TM) := w(TM)^{-1}$ satisfies

$$\overline{w}_i(TM) = 0, \quad \forall i > k$$

since

(1) $dim(\mathcal{N}_{\mathbb{R}^{n+k}} M) = k$,
(2) $\overline{w}(TM) = w(\mathcal{N}_{\mathbb{R}^{n+k}}(M))$
(3) and $w(T\mathbb{R}^{n+k}) = 0$.

Example 1.20. We have $w(\mathbb{RP}^9) = (1+a)^{10} = 1 + a^2 + a^8$ (since $10 = 5 \times 2$). Hence $\overline{w}(\mathbb{RP}^9) = 1 + a^2 + a^4 + a^6$. This means

Theorem 1.21. $\mathbb{RP}^{2^r}$ cannot be immersed in to $\mathbb{R}^{n+k}$ for $k < 2^r - 1$.

Proof.
$$w(T\mathbb{RP}^{2^r}) = (1+a)^{2^r+1} = (1+a)(1+2^r) = 1 + a + a^{2^r}.$$
Therefore:
$$\overline{w}(T\mathbb{RP}^{2^r}) = 1 + a + a^2 + \cdots + a^{2^r-1}.$$
□

Whitney proved that every smooth compact manifold of dimension n can be immersed in $\mathbb{R}^{2n-1}$ and hence the above theorem gives us the best possible estimate. Note that the above theorem tells us that $\mathbb{RP}^8$ cannot be immersed in $\mathbb{R}^{14}$ and hence $\mathbb{RP}^9$ cannot be immersed in $\mathbb{R}^{14}$ (as proven above) since $\mathbb{RP}^8$ is a submanifold of $\mathbb{RP}^9$.

Stiefel-Whitney Numbers.

Recall, every manifold has a fundamental class $[M] \in H^n(M;\mathbb{Z}/2)$ with $\mathbb{Z}/2$ coefficients. Also recall for any $\nu \in H^*(M;\mathbb{Z}/2)$ we have the evaluation $\nu([M]) \in \mathbb{Z}/2$ called the **Kronecker index** (this is defined to be zero if $* \neq n$).

Definition 1.22. For any tuple $(r_1, \cdots, r_n) \in \mathbb{N}^n_{\geq 0}$, the number
$$(\cup_{i=1}^n w_i^{r_i})[M] \in \Lambda$$
is called a **Stiefel-Whitney number**. We will write
$$w_1^{r_1} \cdots w_n^{r_n}[M]$$
for such a number.

Two manifolds M, M' have the **same Stiefel-Whitney numbers** if
$$w_1^{r_1} \cdots w_n^{r_n}[M] = w_1^{r_1} \cdots w_n^{r_n}[M']$$
for all tuples $(r_1, \cdots, r_n) \in \mathbb{N}^n_{\geq 0}$.

Note that for such a number to be non-zero, we need $\sum_i ir_i = n$.

Let us compute some Stiefel-Whitney numbers of $\mathbb{RP}^n$. We have two cases: (1) n is even, (2) n is odd.

(1) If n is even then $w_n(T\mathbb{RP}^n) = (n+1)a^n = a^n$ and hence $w_n[\mathbb{RP}^n] = 1$. Also $w_1(T\mathbb{RP}^n) = a$ and so $w_1^n[\mathbb{RP}^n] = 1$. If n is a power of 2 then $w(T\mathbb{RP}^n) = 1 + a + a^n$ which means that all other Stiefel-Whitney numbers are 0.
(2) If $n = 2k-1$ is odd then $w(T\mathbb{RP}^{2k-1}) = (1+a)^{2k} = (1+a^2)^k$. Hence all odd Stiefel-Whitney classes vanish. This means that we can only consider cup products of even Stiefel-Whitney classes. But their degrees will never add up to n (which is odd). Hence all Stiefel-Whitney classes vanish.

Theorem 1.23. If B is the boundary of a smooth compact manfiold then all the Stiefel-Whitney numbers of B vanish.

Proof. Let $B = \partial W$ for some compact W. Then the fundamental class $[W] \in H_{n+1}(W;B)$ gets sent to $[B] \in H_n(B)$ under the map
$$\partial : H_{n+1}(W;B) \to H_n(B).$$
Also
$$\nu[B] = \nu[\partial B] = (\delta\nu)[B] \tag{1}$$
for all $\nu \in H^n(B)$ where $\delta : H^n(B) \to H^{n+1}(W;B)$ is the natural map coming from the long exact sequence of the pair (W, B).

Let $i : B \hookrightarrow W$ be the inclusion map. Since $B = \partial W$, we have that $TW|_B = TB \oplus (B \times \mathbb{R})$ and so $i^* w_i(TW) = w_i(TB)$ for all i. The long exact sequence:

$$H^n(W) \xrightarrow{i} H^n(B) \xrightarrow{\delta} H^{n+1}(W, B)$$

tells us that

$$\delta w_i(TB) = \delta(i^* w_i(TW)) = 0. \tag{2}$$

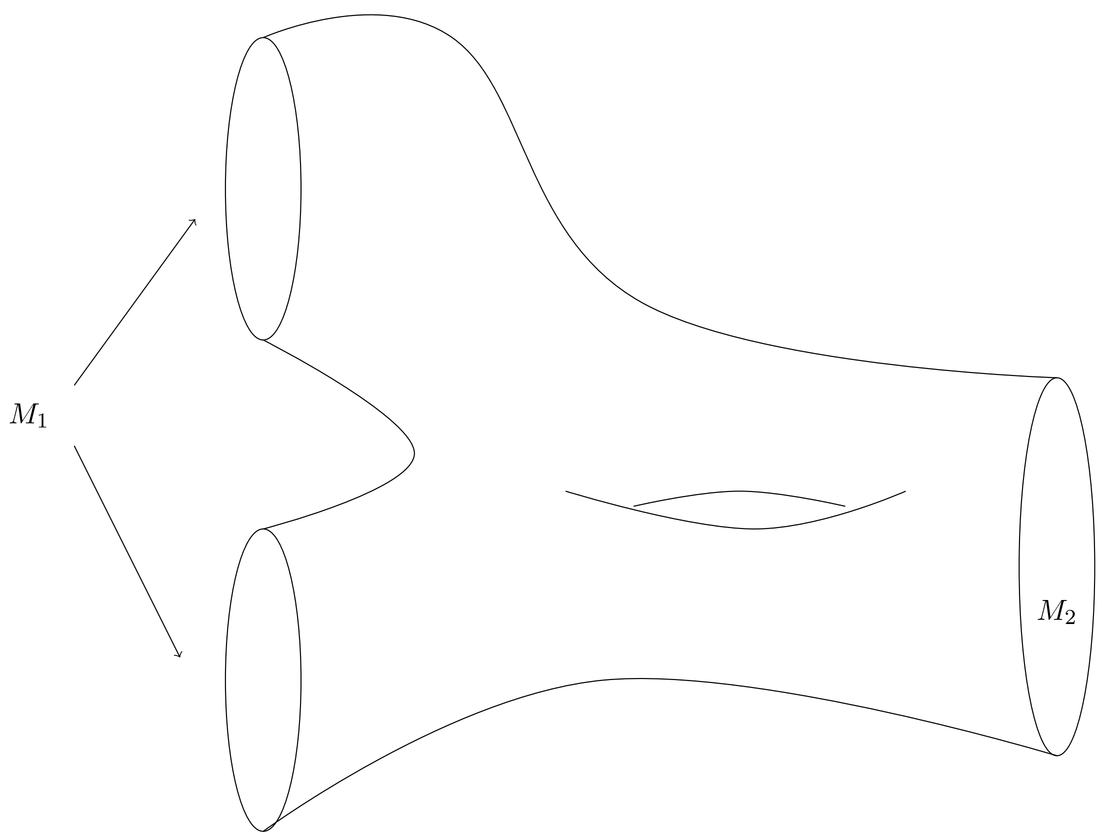
Hence

$$w_1^{r_1} \cdots w_n^{r_m}([B]) \stackrel{(1)}{=} \delta\left(w_1^{r_1} \cup \cdots \cup w_n^{r_m}\right)([W]) \stackrel{(2)}{=} 0.$$

□

Theorem 1.24. (Thom) If all the Stiefel-Whitney numbers vanish for a compact manifold B then B is the boundary of a manifold W.

Definition 1.25. Two manifolds M_1, M_2 are **cobordant** if there is a manifold W so that $\partial W = M_1 \sqcup M_2$.



Corollary 1.26. Cobordant manifolds have the same Stiefel-Whitney numbers.