1. Multiplicative Sequences and the Hirzebruch Signature Theorem

Definition 1.1. Let $A^* \equiv \oplus_{i\in\mathbb{N}_{\geq 0}} A^i$ be a graded algebra over a commutative ring Λ with unit. We write A^Π to be the ring of formal series $a_0 + a_1 + a_2 + \cdots$ where $a_i \in A^i$ for all $i \in \mathbb{N}_{\geq 0}$. The set of units $(K^\Pi)^\times$ is equal to the set of sequences $1 + a_1 + \cdots$ where $a_i \in A^i$. Let $K_1(x_1), K_2(x_1, x_2), \cdots$ be a sequence of polynomial with coefficients in Λ. so that if the degree of x_i is i for each $i \in \mathbb{N}_{\geq 1}$, then K_i has degree n. For each $a = 1 + a_1 + a_2 + \cdots \in A^\Pi$, define

$$K(a) \equiv 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots .$$

The polynomials $K_1, K_2, \cdots$ is called a **multiplicative sequence of polynomials if** $K(ab) = K(a)K(b)$ for all $a, b \in (A^\Pi)^\times$.

Example 1.2.

$$K_k(x_1, \cdots, x_k) = \lambda^k x_k, \quad \forall\ k \in \mathbb{N}_{\geq 1}$$

is a multiplicative sequence of polynomials for all $\lambda \in \Lambda$.

Example 1.3.

$$K(a) = a^{-1}$$

defines a multiplicative sequence with

$$K_1(x_1) = -x_1$$

$$K_2(x_1, x_2) = x_1^2 - x_2$$

$$K_3(x_1, x_2, x_3) = -x_1^3 - 2x_1x_2 - x_3$$

$$K_4(x_1, x_2, x_3, x_4) = x_1^4 - 3x_1^2x_2 + 2x_1x_3 + x_2^2 - x_4$$

since

$$a^{-1} = 1 - (a_1 + a_2 \cdots) + (a_1 + a_2 \cdots)^2 - \cdots$$
$$= 1 - a_1 + a_1^2 - a_2 - a_1^3 + 2a_1a_2 - a_3 + \cdots$$

In general:

$$K_n = \sum_{i_1 + 2i_2 \cdots + ni_n = n,\ i_j \geq 0} \frac{(i_1 + \cdots + i_n)!}{i_1! i_2! \cdots i_n!} (-x_1)^{i_1} \cdots (-x_n)^{i_n}.$$

These polynomials are use to compare the Chern/Pontryagin/Stiefel Whitney classes of two vector bundles whose Whitney sum is trivial.

Example 1.4. The polynomials

$$K_{2n-1}(x_1, \cdots, x_{2n-1}) = 0$$

$$K_{2n}(x_1, \cdots, x_{2n}) = x_n^2 - 2x_{n-1}x_{n+1} + 2x_{n-2}x_{n+2} \cdots + 2(-1)^{n-1}x_1x_{2n-1} + 2(-1)^n x_{2n}$$

form a multiplicative sequence which compares Pontryagin classes with Chern classes of complex vector bundles.

Suppose that $A^* = \Lambda[t]$ where t has degree 1. Then $A^\Pi = \Lambda[[t]]$ is the ring of formal power series in t.

Lemma 1.5. (Hirzebruch)
Let
$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots \in A^{\Pi} = \Lambda[[t]]$$
be a formal power series in t. Then there is a unique multiplicative sequence $\{K_n\}_{n\in\mathbb{N}}$ satisfying
$$K(1+t) = f(t)$$
(or equivalently, the coefficient of x_1^n in K_n is λ_n).

Definition 1.6. The multiplicative sequence **belonging to** $f(t)$ is the unique multiplicative $\{K_n\}_{n\in\mathbb{N}}$ sequence satisfying $K(1+t) = f(t)$ as in the above lemma.

Example 1.7. The multiplicative sequence belonging to
$$f(t) = 1 + \lambda t + \lambda^2 t^2 + \cdots$$
is the one from Example 1.2.

The multiplicative sequence belonging to
$$f(t) = 1 - t + t^2 - t^3 + \cdots$$
is the one from Example 1.3.

The multiplicative sequence belonging to
$$f(t) = 1 + t^2$$
is the one from Example 1.4.

Proof of Lemma 1.5. Uniqueness:

Let $\Lambda[t_1, \cdots, t_n]$ be the polynomial ring where t_i has degree 1 for all i. Let $\sigma = \prod_{i=1}^n (1+t_i)$. Then the ith elementary symmetric polynomial σ_i is the homogeneous part of σ of degree i. Hence
$$\sigma = 1 + \sigma_1 + \sigma_2 + \cdots .$$

Therefore
$$\begin{aligned} K(1 + \sigma_1 + \sigma_2 + \cdots) &= 1 + K_1(\sigma_1) + K_2(\sigma_2) + \cdots \\ &= K(\prod_{i=1}^n (1+t_i)) = \prod_{i=1}^n K(1+t_i) = \prod_{i=1}^n f(t_i). \end{aligned}$$
Therefore $K_n(\sigma_1, \cdots, \sigma_n)$ is the homogeneous part of $\prod_{i=1}^n f(t_i)$ of degree n. Since $\sigma_1, \cdots, \sigma_n$ are algebraically independent, this uniquely determines K_n.

Existence:

For any partition $i_1, \cdots, i_r$ of n, we define $\lambda_I \equiv \lambda_1 \lambda_2 \cdots \lambda_r$. Define
$$K_n(\sigma_1, \cdots, \sigma_n) \equiv \sum_I \lambda_I s_I(\sigma_1, \cdots, \sigma_n)$$
where we sum over all partitions I of n. Here $s_I(\sigma_1, \cdots, \sigma_n)$ is the unique polynomial in the elementary symmetric polynomials equal to $\sum_p t_{\sigma(1)}^{t_1} \cdots t_{\sigma(r)}^{t_r}$ where we sum over all permutations p of $\{1, \cdots, r\}$.

Define
$$s_I(1 + l_1 t + l_2 t^2 + \cdots) \equiv s_I(l_1 t, l_2 t^2, \cdots, l_n t^n).$$
Then
$$s_I(ab) = \sum_{HJ=I} s_J(a) s_H(b)$$

for all $a, b \in (\Lambda[[t]])^\times$. Therefore

$$K(ab) = \sum_I \lambda_I s_I(ab) = \sum_I \lambda_I \sum_{HJ=I} s_H(a) s_J(b) = \sum_I \sum_{HJ=I} \Lambda_H s_H(a) \lambda_J s_J(b) = K(a)K(b)$$

for all $a, b \in (\Lambda[[t]])^\times$. Hence K is multiplicative.

The coefficient of σ_1^n of $K_n(\sigma_1, \cdots, \sigma_n)$ is λ_n. $\square$

Definition 1.8. Let $\{K_n\}_{n\in\mathbb{N}}$ be a multiplicative sequence of polynomials. Let M^m be an oriented m-manifold. The **K-genus** $K[M^m]$ is 0 of m is not divisible by 4. If $m = 4k$ then $K[M^m] \equiv p_1(TM^m) \cup \cdots p_k(TM^m)[\mu_M]$.

Lemma 1.9. The map $M \longrightarrow K[M]$ descends to a ring homomorphism

$$\Omega_* \longrightarrow \mathbb{Q}.$$

Hence we get an induced map

$$\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Proof. Since Pontryagin numbers are cobordism invariants, this descends to a map $\Omega_* \longrightarrow \mathbb{Q}$. This map is additive since addition is given by disjoint union. If p (resp. p') is the total Pontryagin class of M (resp. M') then the total Pontryagin class of $M \times M'$ is $p \times p'$ modulo 2. Also $K(p \times p') = K(p) \times K(p')$ since $(K_n)_{n\in\mathbb{N}}$ is a multiplicative sequence modulo 2. Hence $K(p \times p')[M \times M'] = (-1)^{mm'} K(p)[M]K(p')[M']$ where $m = \dim(M)$ and $m' = \dim(M')$. Since these numbers are non-zero only when m, m' are divisible by 4, we get $K(p \times p')[M \times M'] = K(p)[M]K(p')[M']$ and hence we get a ring homomorphism. $\square$

Definition 1.10. The **signature** $\sigma(M)$ of a compact oriented manifold M^m is defined to be 0 if m is not divisible by 4. If $m = 4k$ then it is defined as follows: Choose a basis $a_1, \cdots, a_r$ of $H^{2k}(M;\mathbb{Q})$ so that the symmetric matrix

$$(a_i \cup a_j)[M]$$

is diagonal. Then $\sigma(M)$ is defined to be the number of positive entries minus the number of negative entries in this diagonal matrix (in other words, it is the signature of the quadratic form

$$Q_M : H^{2k}(M;\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad Q_M(a) \equiv (a \cup a)[M].$$

.)

Lemma 1.11. (Thom) The signature $\sigma(M)$ satisfies:

(1) $\sigma(M \sqcup M') = \sigma(M) + \sigma(M')$,
(2) $\sigma(M \times M') = \sigma(M)\sigma(M')$ and
(3) if M is the oriented boundary of a manifold then $\sigma(M) = 0$.

Part (1) and (2) from this lemma are left as an exercise. We will focus on proving part (3). We need some preliminary lemmas and definitions.

Definition 1.12. Let $B : V \otimes V \longrightarrow \mathbb{Q}$ be a non-degenerate bilinear form. For any subspace $W \subset V$, we define $W^\perp \equiv \{v \in V \ : \ B(v, w) = 0 \ \forall \ w \in W\}$.

A subspace $L \subset V$ is **isotropic** if $B|_{L\otimes L} = 0$. It is **Lagrangian** if $\dim(L) = \frac{1}{2}\dim(V)$. Equivalently L is Lagrangian if and only if L and $L^\perp$ are isotropic. (exercise).

We leave the proof of this lemma as a linear algebra exercise.

Lemma 1.13. Suppose that $B : V \otimes V \longrightarrow \mathbb{Q}$ is a non-degenerate bilinear form and suppose that V admits a Lagrangian subspace. Then the signature of the associated quadratic form $B(v, v)$ is zero.

Lemma 1.14. Let M^{4k} be a $4k$-manifold which is the boundary of an oriented $4k+1$-manifold W. Let $\iota : M \longrightarrow W$ be the natural inclusion map. Then the image of

$$\iota^* : H^{2k}(W; \mathbb{Q}) \longrightarrow H^{2k}(M; \mathbb{Q})$$

is isotropic with respect to the quadratic form Q_M.

Proof. Let $c, c' \in H^{2k}(W; \mathbb{Q})$. Then

$$\iota^* c \cup \iota^* c'([M]) = \iota^*(c \cup c')(\partial[W]) = \delta \circ \iota^*(c \cup c')([W]) = 0.$$

□

Proof of Lemma 1.11. Suppose M is the oriented boundary of an oriented $4k+1$-manifold W and let $\iota : M \longrightarrow W$ be the inclusion map. We write $PD(a)$ for the Poincaré-dual of a class $a \in H_*(M; \mathbb{Q})$ or $a \in H^*(M; \mathbb{Q})$. Also we have that the map

$$D_W : H^{2k}(W; \mathbb{Q}) \longrightarrow H_{2k+1}(W, M; \mathbb{Q}), \quad \alpha \longrightarrow \alpha \cap [W].$$

is an isomorphism (Lefschetz duality). Again we write $LD(a)$ for the Lefschetz dual of $a \in H^{2k+1}(W, \partial W; \mathbb{Q})$.

Consider the commutative diagram:

$$\begin{array}{ccccc}
H^{2k}(W;\mathbb{Q}) & \xrightarrow{\iota^*} & H^{2k}(M;\mathbb{Q}) & \longrightarrow & H^{2k+1}(W,M;\mathbb{Q} \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H_{2k+1}(W,M;\mathbb{Q}) & \rightarrow & H_{2k}(M;\mathbb{Q}) & \xrightarrow{\iota_*} & H_{2k}(N;\mathbb{Q})
\end{array}$$

Here the vertical arrows are Poincaré or Lefschetz duality maps and the horizontal arrows form a long exact sequence. This means that the Poincaré dual of $\ker(\iota_*)$ is equal to the image of ι^*. Hence $\dim \ker(\iota_*) = \dim \mathrm{Im}(\iota^*)$.

Also: $x \in \mathrm{Im}(\iota^*)^\perp$ iff $x \cup \iota^*(c)([M]) = 0, \ \forall\ c \in H^{2k}(W; \mathbb{Q})$ iff $\iota^*(c)(PD(x)) = 0\ \forall\ c \in H^{2k}(W; \mathbb{Q})$ iff $c(\iota_*(PD(x))) = 0\ \forall\ c \in H^{2k}(W; \mathbb{Q})$ iff $\iota_*(PD(x)) = 0$. Which implies that $\mathrm{Im}(\iota^*)^\perp = PD(\ker(\iota_*))$. Hence $\dim \ker(\iota_*) = \dim(H^{2k}(M; \mathbb{Q})) - \dim(\mathrm{Im}(\iota^*))$. Therefore $\dim(\mathrm{Im}(\iota^*)) = \dim(H^{2k}(M; \mathbb{Q}))/2$. Also be the previous lemma, $\mathrm{Im}(\iota^*)$ is isotropic and hence it is Lagrangian. Hence the signature is 0. □

Theorem 1.15. (Hirzebruch Signature Theorem)

Let $(L_k(x_1, \cdots, x_k))_{k \in \mathbb{N}}$ be the multiplicative sequence of polynomials belonging to the power series

$$\sqrt{t}/\tanh(\sqrt{t}) = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1} 2^{2k} B_k t^k/(2k)! \cdots.$$

Then the signature $\sigma(M^{4k})$ of any smooth compact oriented $4k$-manifold M is equal to the L-genus of $[M]$.

Here B_k is the k**th Bernoulli number**. They are defined using the series: $\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$.

The first three L-polynomials are

$$L_1(p_1) = \frac{1}{3} p_1$$

$$L_2(p_1,p_2) = \frac{1}{45}(7p_2 - p_1^2)$$
$$L_3(p_1,p_2,p_3) = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3).$$

Proof of the Hirzebruch signature theorem. Since the correspondences $M \longrightarrow \sigma(M)$ and $M \longrightarrow L(M)$ induce algebra homomorphisms

$$\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$$

it is sufficient for us to check the theorem for the generators $[\mathbb{CP}^{2k}]_{k\in\mathbb{N}}$ of this algebra.

The signature of $\mathbb{CP}^{2k}$ is 1 (Exercise).

We now need to compute $L_k[\mathbb{CP}^{2k}]$. The Pontryagin class of $\mathbb{CP}^{2k}$ is $p = (1+u^2)^{2k+1}$. Also the multiplicative sequence $(L_k)_{k\in\mathbb{N}}$ by definition satisfies

$$L(1+u^2) = \sqrt{u^2}/\tanh(\sqrt{u^2}) = u/\tanh(u).$$

Therefore

$$L(p)[M] = L((1+u^2)^{2k+1})[M] = (L(1+u^2))^{2k+1}[M].$$

This is the coefficient of u^{2k} in the power series for $(u/\tanh(u))^{2k+1}$. By Cauchy's integral formula this coefficient is equal to:

$$\frac{1}{2\pi_i}\oint \frac{1}{z^{2k+1}}\frac{z^{2k+1}dz}{\tanh(z)^{2k+1}} = \frac{1}{2\pi_i}\oint \frac{dz}{\tanh(z)^{2k+1}} \overset{v=\tanh(z)}{=} \frac{1}{2\pi_i}\oint \frac{dv}{(1-v^2)v^{2k+1}} =$$
$$\frac{1}{2\pi i}\oint \frac{(\sum_{j=1}^{\infty} v^{2i})}{v^{2k+1}}dv = 1.$$

Hence $L[\mathbb{CP}^{2k}] = \sigma(\mathbb{CP}^{2k}) = 1$ which implies that $L[M] = \sigma(M)$ for all oriented manifolds M. $\square$

Corollary 1.16. The L-genus is always an integer.

This is because the signature is always an integer.

Corollary 1.17. The L-genus is a homotopy invariant of M.

Again this is true since the signature is a homotopy invariant.