1. The Oriented Cobordism Ring

Definition 1.1. Let M be an oriented manifold with boundary. Then the boundary ∂M also has a natural orientation as follows: If we have any local oriented chart

$$\tau : U \longrightarrow \mathbb{H}^n \equiv \{(x_1, \cdots, x_n \ : \ x_1 \geq 0)\}$$

then $x_2, \cdots, x_n$ is an oriented chart for ∂M.

Another way of describing this for smooth manifolds is as follows: Let V be a vector field defined near ∂M which points outwards. In other words, in any chart τ as above, V is equal to $f(x_1, \cdots, x_n)\frac{\partial}{\partial x_1} + V_2$ in this chart where $f(0, x_2, \cdots, x_n) < 0$ and V_2 is tangent to ∂M. Let $E \subset TM|_{\partial M}$ be the one dimensional sub-bundle spanned bay V. Then

$$TM|_{\partial M}/E \cong T\partial M$$

and hence $TM|_{\partial M} \cong E \oplus T\partial M$. Since we have a natural trivialization $T : E \longrightarrow \partial M \times \mathbb{R}$ sending V to 1, and since $TM|_{\partial M}$ is oriented, we get that $T\partial M$ has a natural orientation and hence ∂M is oriented.

Here is a third way of describing this. An orientation on a smooth n-manifold M corresponds a choice of n-form Ω which does not vanish anywhere. Let V be the vector field as above. Then $i_V(\Omega)|_{\partial M}$ is a nowhere vanishing $n-1$ form on ∂M and hence gives us a natural orientation on ∂M.

(Exercise: show that these three definitions are equivalent).

Theorem 1.2. (Collar Neighborhood Theorem) Let M be a smooth paracompact manifold with boundary. Then there is a neighborhood of ∂M diffeomorphic to $(0, 1] \times \partial M$.

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Definition 1.3. If M is an oriented manifold then we write $-M$ for the same manifold but with opposite orientation.

Two smooth manifold M, M' are said to be **oriented cobordant** or **belong to the same cobordism class** if if there is an oriented compact manifold with boundary X and an orientation preserving diffeomorphism

$$\Phi : M \sqcup (-M') \longrightarrow \partial X.$$

Example 1.4. Suppose that there is an orientation preserving diffeomorphism $\Psi : M \longrightarrow M'$ then M and M' are oriented cobordant by the cobordism $X = [0, 1] \times M$ and the diffeomorphism

$$\Phi : M \sqcup (-M') \longrightarrow X, \quad \begin{cases} \Phi(x) = (0, x) & \text{if } x \in M \\ \Phi(x) = (1, \Psi(x)) & \text{if } x \in M' \end{cases}$$

Definition 1.5. We define Ω_n to be the set of all oriented cobordism classes of n manifolds. If M is an oriented manifold, then we write $[M]$ for the corresponding element in Ω_n.

Note, one may wonder if Ω_n is actually a set at all. Since every n-manifold can be embedded in to $\mathbb{R}^{2n}$ by Whitehead's theorem, one sees that every n-manifold is diffeomorphic submanifold of $\mathbb{R}^{2n}$. This implies that each manifold is oriented cobordant to a manifold diffeomorphic to a submanifold of $\mathbb{R}^{2n}$. Therefore the size of Ω_n is at most the power set of $\mathbb{R}^{2n}$ and hence must be a set.

Lemma 1.6. (Exercise). Being oriented cobordant is a reflexive, symmetric and transitive relation. Also Ω_n becomes an abelian group where the group operation is disjoint union.

Also $\Omega_* \equiv \sqcup_{n\geq 0}\Omega_n$ is a ring with addition equal to disjoint union and multiplication corresponds to the cross product. The identity element is the positively oriented point $\{\star\}$ in Ω_0. Also $[M_1^n] \times [M_2^m] = (-1)^{mn}[M_2^m] \times [M_1^n]$ which means that Ω_* is a **graded commutative ring**.

Definition 1.7. Ω_* is called the **oriented cobordism ring**.

Lemma 1.8. (Pontryagin) If M and M' are oriented cobordant $4k$ manifolds then they have the same Pontryagin numbers.

Proof. Since $M \sqcup -M'$ is the oriented boundary of a $4k+1$ manifold, we get that all the Pontryagin numbers of $M \sqcup -M'$ are trivial. Let $p_I(M), p_I(M')$ be two Pontryagin numbers where I is a partition of k. Then

$$0 = p_I(M \sqcup -M') = p_I(M) + p_I(-M') = p_I(M) - p_I(M')$$

and hence they have the same Pontryagin numbers. □

Corollary 1.9. For any partition I of k, we get a group homomorphism

$$\Omega_{4k} \longrightarrow \mathbb{Z}, \quad [M] \longrightarrow p_I(M).$$

Corollary 1.10. The products

$$\mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_r}$$

as $i_1, \cdots, i_r$ range over all partitions of k are linearly independent inside the group Ω_{4k}. Hence Ω_{4k} has rank greater than or equal to $p(k)$ which is the number of partitions of k.

Proof. This follows from the fact (from the previous section) that the $p(k) \times p(k)$-matrix

$$\left[p_{i_1} \cdots p_{i_r}[\mathbb{CP}^{2j_1} \times \cdots \times \mathbb{CP}^{2j_s}]\right]$$

where $i_1, \cdots, i_r$ and $j_1, \cdots, j_s$ run over all partitions of k.

Hence we get a surjective group homomorphism

$$\Omega_{4k} \longrightarrow \mathbb{Z}^{P_k}, \quad M \longrightarrow (p_{i_1} \cdots p_{i_r}[M])_{i_1,\cdots,i_r \in P_k}$$

where P_k is the set of partitions of k. □

Here is Ω_k for some small k:

- $\Omega_0 = \mathbb{Z}$ since every 0 manifold is a set of signed points.
- $\Omega_1 = 0$ since every compact oriented 1-manifold is the boundary of a disjoint union of disks.
- $\Omega_2 = 0$ since every compact oriented 2-manifold is a genus g surface and hence is the boundary of a 3 manifold with g handles.
- $\Omega_3 = 0$. (Rohlin).
- $\Omega_4 = \mathbb{Z}$ and is generated by $\mathbb{CP}^2$.
- $\Omega_5 = \mathbb{Z}/2$ generated by Y^5, a non-singular hypersurface of degree $(1,1)$ inside $\mathbb{RP}^2 \times \mathbb{RP}^4$.
- $\Omega_6 = 0$
- $\Omega_7 = 0$
- $\Omega_8 = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\mathbb{CP}^4$ and $\mathbb{CP}^2 \times \mathbb{CP}^2$
- $\Omega_9 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $Y^5 \times \mathbb{CP}^2$ and Y^9, a non-singular hypersurface of degree $(1,1)$ inside $\mathbb{RP}^2 \times \mathbb{RP}^8$.
- $\Omega_{10} = \mathbb{Z}/2$ generated by $Y^5 \times Y^5$

- $\Omega_{11} = \mathbb{Z}/2$ generate by Y^{11}, a non-singular hypersurface of degree $(1, 1)$ inside $\mathbb{RP}^4 \times \mathbb{RP}^8$.