1. **The Oriented Cobordism Ring**

**Definition 1.1.** Let $M$ be an oriented manifold with boundary. Then the boundary $\partial M$ also has a natural orientation as follows: If we have any local oriented chart

$$\tau : U \rightarrow \mathbb{H}^n \equiv \{(x_1, \cdots, x_n : x_1 \geq 0)\}$$

then $x_2, \cdots, x_n$ is an oriented chart for $\partial M$.

Another way of describing this for smooth manifolds is as follows: Let $V$ be a vector field defined near $\partial M$ which points outwards. In other words, in any chart $\tau$ as above, $V$ is equal to $f(x_1, \cdots, x_n) \frac{\partial}{\partial x_1} + V_2$ in this chart where $f(0, x_2, \cdots, x_n) < 0$ and $V_2$ is tangent to $\partial M$. Let $E \subset TM|_{\partial M}$ be the one dimensional sub-bundle spanned by $V$. Then

$$TM|_{\partial M}/E \cong T\partial M$$

and hence $TM|_{\partial M} \cong E \oplus T\partial M$. Since we have a natural trivialization $T : E \rightarrow \partial M \times \mathbb{R}$ sending $V$ to 1, and since $TM|_{\partial M}$ is oriented, we get that $T\partial M$ has a natural orientation and hence $\partial M$ is oriented.

Here is a third way of describing this. An orientation on a smooth $n$-manifold $M$ corresponds a choice of $n$-form $\Omega$ which does not vanish anywhere. Let $V$ be the vector field as above. Then $i_V(\Omega)|_{\partial M}$ is a nowhere vanishing $n-1$ form on $\partial M$ and hence gives us a natural orientation on $\partial M$.

(Exercise: show that these three definitions are equivalent).

**Theorem 1.2.** (Collar Neighborhood Theorem) Let $M$ be a smooth paracompact manifold with boundary. Then there is a neighborhood of $\partial M$ diffeomorphic to $(0, 1] \times \partial M$.

**Oriented Cobordism**

**Definition 1.3.** If $M$ is an oriented manifold then we write $-M$ for the same manifold but with opposite orientation.

Two smooth manifold $M, M'$ are said to be **oriented cobordant** or **belong to the same cobordism class** if if there is an oriented compact manifold with boundary $X$ and an orientation preserving diffeomorphism

$$\Phi : M \sqcup (-M') \rightarrow \partial X.$$  

**Example 1.4.** Suppose that there is an orientation preserving diffeomorphism $\Psi : M \rightarrow M'$ then $M$ and $M'$ are oriented cobordant by the cobordism $X = [0, 1] \times M$ and the diffeomorphism

$$\Phi : M \sqcup (-M') \rightarrow X, \quad \left\{ \begin{array}{ll} \Phi(x) = (0, x) & \text{if } x \in M \\ \Phi(x) = (1, \Psi(x)) & \text{if } x \in M' \end{array} \right.$$  

**Definition 1.5.** We define $\Omega_n$ to be the set of all oriented cobordism classes of $n$ manifolds. If $M$ is an oriented manifold, then we write $[M]$ for the corresponding element in $\Omega_n$.

Note, one may wonder if $\Omega_n$ is actually a set at all. Since every $n$-manifold can be embedded in to $\mathbb{R}^{2n}$ by Whitehead’s theorem, one sees that every $n$-manifold is diffeomorphic submanifold of $\mathbb{R}^{2n}$. This implies that each manifold is oriented cobordant to a manifold diffeomorphic to a submanifold of $\mathbb{R}^{2n}$. Therefore the size of $\Omega_n$ is at most the power set of $\mathbb{R}^{2n}$ and hence must be a set.
Lemma 1.6. (Exercise). Being oriented cobordant is a reflexive, symmetric and transitive relation. Also \( \Omega_n \) becomes an abelian group where the group operation is disjoint union.

Also \( \Omega_n \equiv \sqcup_{n \geq 0} \Omega_n \) is a ring with addition equal to disjoint union and multiplication corresponds to the cross product. The identity element is the positively oriented point \( \{ \star \} \) in \( \Omega_0 \). Also \( [M_1^n] \times [M_2^m] = (-1)^{mn}[M_2^m] \times [M_1^n] \) which means that \( \Omega_* \) is a graded commutative ring.

Definition 1.7. \( \Omega_* \) is called the oriented cobordism ring.

Lemma 1.8. (Pontryagin) If \( M \) and \( M' \) are oriented cobordant \( 4k \) manifolds then they have the same Pontryagin numbers.

Proof. Since \( M \sqcup -M' \) is the oriented boundary of a \( 4k + 1 \) manifold, we get that all the Pontryagin numbers of \( M \sqcup -M' \) are trivial. Let \( p_I(M), p_I(M') \) be two Pontryagin numbers where \( I \) is a partition of \( k \). Then

\[
0 = p_I(M \sqcup -M') = p_I(M) + p_I(-M') = p_I(M) - p_I(M')
\]

and hence they have the same Pontryagin numbers. \( \square \)

Corollary 1.9. For any partition \( I \) of \( k \), we get a group homomorphism

\[
\Omega_{4k} \rightarrow \mathbb{Z}, \quad [M] \rightarrow p_I(M).
\]

Corollary 1.10. The products

\[
\mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_r}
\]

as \( i_1, \cdots, i_r \) range over all partitions of \( k \) are linearly independent inside the group \( \Omega_{4k} \). Hence \( \Omega_{4k} \) has rank greater than or equal to \( p(k) \) which is the number of partitions of \( k \).

Proof. This follows from the fact (from the previous section) that the \( p(k) \times p(k) \)-matrix

\[
[p_{i_1} \cdots p_{i_r} [\mathbb{CP}^{2j_1} \times \cdots \times \mathbb{CP}^{2j_s}]]
\]

where \( i_1, \cdots, i_r \) and \( j_1, \cdots, j_s \) run over all partitions of \( k \).

Hence we get a surjective group homomorphism

\[
\Omega_{4k} \rightarrow \mathbb{Z}^{P_k}, \quad M \rightarrow (p_{i_1} \cdots p_{i_r} [M])_{i_1, \cdots, i_r \in P_k}
\]

where \( P_k \) is the set of partitions of \( k \). \( \square \)

Here is \( \Omega_k \) for some small \( k \):

- \( \Omega_0 = \mathbb{Z} \) since every 0 manifold is a set of signed points.
- \( \Omega_1 = 0 \) since every compact oriented 1-manifold is the boundary of a disjoint union of disks.
- \( \Omega_2 = 0 \) since every compact oriented 2-manifold is a genus \( g \) surface and hence is the boundary of a 3 manifold with \( g \) handles.
- \( \Omega_3 = 0 \) (Rohlin).
- \( \Omega_4 = \mathbb{Z} \) and is generated by \( \mathbb{CP}^2 \).
- \( \Omega_5 = \mathbb{Z}/2 \) generated by \( Y^5 \), a non-singular hypersurface of degree \( (1, 1) \) inside \( \mathbb{RP}^2 \times \mathbb{RP}^4 \).
- \( \Omega_6 = 0 \)
- \( \Omega_7 = 0 \)
- \( \Omega_8 = \mathbb{Z} \oplus \mathbb{Z} \) generated by \( \mathbb{CP}^4 \) and \( \mathbb{CP}^2 \times \mathbb{CP}^2 \)
- \( \Omega_9 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) generated by \( Y^9 \) and \( Y^9 \), a non-singular hypersurface of degree \( (1, 1) \) inside \( \mathbb{RP}^2 \times \mathbb{RP}^8 \).
- \( \Omega_{10} = \mathbb{Z} \) generated by \( Y^5 \times Y^5 \)
\[ \Omega_{11} = \mathbb{Z}/2 \] generate by \( Y^{11} \), a non-singular hypersurface of degree \((1, 1)\) inside \( \mathbb{R}P^4 \times \mathbb{R}P^8 \).