1. Pontryagin Classes

Before we start defining Pontryagin classes, we need a few more lemmas concerning Chern classes.

**Definition 1.1.** The complexification of a real vector bundle $V$ is the complex vector bundle $V \otimes_{\mathbb{R}} \mathbb{C}$.

This bundle has a complex structure $J$ and it sends the real subbundle $V \subset V \otimes_{\mathbb{R}} \mathbb{C}$ to $JV$. We have that $V \cap JV$ is the zero section and $V + JV = V \otimes_{\mathbb{R}} \mathbb{C}$. Hence we have a canonical isomorphism $V \oplus JV \cong V$. Also the map $J|_V : V \to JV$ is a bundle isomorphism, hence we have a canonical isomorphism

$$\Phi : V \oplus V \to V \otimes_{\mathbb{R}} \mathbb{C}, \quad (x, y) \mapsto x + Jy.$$ 

In particular the complex structure on $V \oplus V$ sends $(x, y)$ to $(-y, x)$.

**Definition 1.2.** If $\pi : E \to B$ is a complex vector bundle with complex structure $J$. The conjugate $\overline{\pi} : \overline{E} \to B$ of $E$ is the complex vector bundle $(E, -J)$. Equivalently if $\Phi_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$ are the transition data, where $GL(n, \mathbb{C})$ acts in the usual way on $\mathbb{C}^n$ then the its conjugate has exactly the same transition data $\Phi^*$ but $GL(n, \mathbb{C})$ acts as follows:

$$GL(n, \mathbb{C}) \times \mathbb{C}^n \to \mathbb{C}^n, \quad (A, z) \mapsto A(\overline{z}).$$

**Lemma 1.3.** If $V$ is a real vector bundle then $V \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic as a complex vector bundle to its conjugate.

**Proof.** Under the canonical isomorphism $V \oplus V \to V \otimes_{\mathbb{R}} \mathbb{C}$, our isomorphism sends $(x, y) \in V \oplus V$ to $(x, -y)$. \qed

**Lemma 1.4.** If $\pi : E \to B$ is a complex vector bundle, we have that $c_k(E) = (-1)^k c_k(\overline{E})$.

**Proof.** We will first prove this for $\gamma^1_\infty$ over $\mathbb{CP}^\infty$. Define

$$\iota : \mathbb{CP}^\infty \to \mathbb{CP}^\infty, \quad [z] \mapsto [\overline{z}].$$

This is a homeomorphism and $\iota^* \gamma^1_\infty$ is isomorphic to $\gamma^1_\infty$ as a complex vector bundle. Hence $c_k(\gamma^1_\infty) = c_k(\gamma^1_\infty)$. Now $\iota|_{\mathbb{CP}^1}$ sends $\mathbb{CP}^1$ to itself. It is the reflection map and hence orientation reversing. Therefore $\iota^* u = -u$. Hence $c_1(\gamma^1_\infty) = -c_1(\gamma^1_\infty) = -u$.

We will now prove our theorem for the canonical bundle $\gamma^1_\infty$ of $Gr_n(\mathbb{C}^\infty)$. Let $h_n : (\mathbb{CP}^\infty)^n \to Gr_n(\mathbb{C}^\infty)$ be the classifying map of $\bigoplus_{i=1}^n p_i^* \gamma^1_\infty$ where $p_i : (\mathbb{CP}^\infty)^n \to \mathbb{CP}^\infty$ is the $i$th projection map. Since $h_n^* : H^*(Gr_n(\mathbb{C}^\infty)) \to H^*((\mathbb{CP}^\infty)^n)$ is injective and since

$$h_n(\gamma^1_\infty) = \bigoplus_{i=1}^n p_i^* \gamma^1_\infty = \mathbb{Z}[u_1, \ldots, u_n],$$

it is sufficient for us to prove our lemma for $\bigoplus_{i=1}^n p_i^* \gamma^1_\infty$. Now

$$c_k(\bigoplus_{i=1}^n p_i^* \gamma^1_\infty) \prod_{I \subset \{1, \ldots, n\}, |I| = k} \left( \cup_{j \in I} p_1^* \gamma^1_\infty \right) = \sum_{I \subset \{1, \ldots, n\}, |I| = k} \prod_{j \in I} u_j$$

by the Whitney product theorem. Therefore

$$c_k(\bigoplus_{i=1}^n p_i^* \gamma^1_\infty) = \sum_{I \subset \{1, \ldots, n\}, |I| = k} \prod_{j \in I} (-u_j) = (-1)^k c_k(\bigoplus_{i=1}^n p_i^* \gamma^1_\infty).$$

Finally we prove our lemma in general. Let $f : B \to Gr_k(\mathbb{C}^\infty)$ be the classifying map for $E$. Then $E \cong f^*(\gamma^1_\infty)$

$$c_k(f) = f^*(c_k(\gamma^1_\infty)) = f^*(-1)^k c_k(\gamma^1_\infty) = (-1)^k c_k(E).$$
Corollary 1.5. \(2c_k(V \otimes_R C) = 0\) for all odd \(k\).

Proof. Since \(V \otimes_R C\) is isomorphic as a complex vector bundle to \(V \otimes_R C\), we get that
\[c_k(V \otimes_R C) = c_k(V \otimes_R C) = -c_k(V \otimes_R C).\]
Therefore ignoring these odd Chern classes, we have the following definition:

Definition 1.6. Let \(V \longrightarrow B\) be real vector bundle then the \(i\)th Pontryagin class is
\[p_i(V) \equiv c_{2i}(V \otimes_R C) \in H^{4i}(B).\]

The total Pontryagin class of \(V\) is the class \(p(V) \equiv p_0(V) + p_1(V) + \cdots\).

If \(X\) is a complex manifold then we define \(p_i(X) \equiv p_i(TX)\).

Lemma 1.7. If \(f : B' \longrightarrow B\) is a continuous map and \(\pi : V \longrightarrow B\) is a real vector bundle then \(f^*(p_i(V)) = p_i(f^*(V))\).

Proof. This follows immediately from the naturality property of Chern classes. \(\square\)

Theorem 1.8. Let \(V, V' \longrightarrow B\) be two vector bundles over the same base then \(p(V \oplus V')\) is equal to \(p(V)p(V') \mod 2\). In other words, \(2(p(V \oplus V') - p(V)p(V')) = 0\).

Proof. Since \(2c_k(V \otimes_R C) = 2c_k(V' \otimes_R C) = 0\) for all odd \(k\),
\[2(p(V \oplus V') - p(V)p(V')) = 2(c(V \otimes_R C \oplus V' \otimes_R C) - c(V \otimes_R C)c(V' \otimes_R C)) = 0\]
by the Whitney sum formula. \(\square\)

Lemma 1.9. (Exercise:) Let \(\pi : E \longrightarrow B\) be a complex vector bundle and let \(E(\mathbb{R})\) be its underlying real vector bundle. Then \(E(\mathbb{R}) \otimes_R C\) is isomorphic as a complex vector bundle to \(E \oplus \overline{E}\).

Proposition 1.10. For any complex \(n\) bundle \(\pi : E \longrightarrow B\), the Chern classes of \(E\) determine the Pontryagin classes of \(E\) by the following formula:
\[1 - p_1(E) + p_2(E) - \cdots + (-1)^n p_n(E) = (1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E))(1 + c_1(E) + \cdots + c_n(E)).\]
Hence \(p_k(E)\) is equal to:
\[c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \cdots + 2(-1)^{j-k} c_{k-j}(E)c_{k+j}(E) + \cdots + 2(-1)^k c_{2k}(E)c_1(E) + 2(-1)^{k+1} c_{2k}(E).\]

Proof. Since \(E(\mathbb{R}) \otimes_R C\) is isomorphic as a complex vector bundle to \(E \oplus \overline{E}\), our theorem follow from the Whitney sum formula and the fact that \(c_k(\overline{E}) = (-1)^k c_k(E)\). \(\square\)

Let us compute the Pontryagin classes of \(\mathbb{C}P^n\). We will use the above proposition to do this. Before we do this we need to compute the Chern classes of \(\mathbb{C}P^n\) first using the following Lemma.

Lemma 1.11. We have that \(c(T\mathbb{C}P^n) = (1 - u)^{n+1}\) where \(u \in H^2(\mathbb{C}P^n)\) is Poincaré-dual to \([\mathbb{C}P^1] \in H_2(\mathbb{C}P^n)\).

Proof. Let \(\omega \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}\) be the orthogonal complement of \(\gamma_{1,n}^1\). Claim: (Exercise) \(T\mathbb{C}P^n \cong Hom_{\mathbb{C}}(\gamma_{1,n}^1, \omega)\) (using the same reasoning as with \(\mathbb{R}P^n\)).

Let \(\underline{C}\) be the trivial \(\mathbb{C}\) bundle \(\mathbb{C}P^n \times \mathbb{C}\). Since \(\underline{C} \cong Hom_{\mathbb{C}}(\gamma_{1,n}^1, \gamma_{1,n}^1)\). Then
\[T\mathbb{C}P^n \oplus \underline{C} \cong Hom_{\mathbb{C}}(\gamma_{1,n}^1, \omega \oplus \gamma_{1,n}^1) = Hom_{\mathbb{C}}(\gamma_{1,n}^1, \oplus_{j=1}^n \underline{C}) \cong ((\gamma_{1,n}^1)^*)^n.\]
Since $(\gamma_1^*)^n|_{\mathbb{C}P^1} = 0|_{\mathbb{C}P^1}(-1)$, we get that $c_1((\gamma_1^*)^n) = -c_1(\gamma_1)$ and so $c_1((\gamma_1^*)^n) = -c_1(\gamma_1)$. Hence $c(T\mathbb{C}P^n) = (1-c_1(\gamma_1^*))^{n+1} = (1-u)^{n+1}$. □

Hence by the above lemma and the above proposition, we have:

$$1 - p_1(\mathbb{C}P^n) + p_2(\mathbb{C}P^n) - \cdots = c(T\mathbb{C}P^n) \oplus c(T\mathbb{C}P^n) = (1+u)^{n+1}(1-u)^{n+1} = (1-u^2)^{n+1}.$$ Therefore

$$p_k(\mathbb{C}P^n) = \binom{n+1}{k}u^k.$$ E.g.

$$p(\mathbb{C}P^5) = 1 + 6u^2 + 15u^4.$$ 

**Lemma 1.12.** Let $\pi: V \to B$ be an oriented rank $n$ vector bundle. Then the real $2n$-plane bundle $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ (i.e. the real structure underlying $V \otimes \mathbb{C}$) is isomorphic to $V \oplus V$ and the natural orientation on $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ coming from the complex structure gets sent to natural sum orientation on $V \oplus V$ if and only if $n(n-1)/2$ is even.

**Proof.** Let $J$ be the natural complex structure on $V \oplus \mathbb{C}$. If $v_1, \cdots, v_n$ is an oriented basis for a fiber of $V$ then $v_1, Jv_1, v_2, Jv_2, \cdots, v_n, Jv_n$ is an oriented real basis for the corresponding fiber of $(E \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ and

$$v_1 \oplus 0, v_2 \oplus 0, \cdots, v_n \oplus 0, 0 \oplus v_1, \cdots, 0 \oplus v_n$$

is an oriented basis for the corresponding fiber of $V \oplus V$. The orientations of these bases agree if and only if $n(n-1)/2$ is even. □

We have the following immediate corollary:

**Corollary 1.13.** If $V$ is an oriented rank $2n$ vector bundle. Then $p_n(V)$ is equal to the square of the Euler class $e(V)$.

Let $\tilde{G}_n(\mathbb{R}^\infty)$ be the oriented Grassmannian. i.e. the space parameterizing oriented $n$-planes inside $\mathbb{R}^\infty$. Let $\tilde{\gamma}_n^\infty$ be the corresponding canonical oriented bundle over this Grassmannian. We have the following theorem which we won’t prove (see Theorem 15.9 in Milnor and Stasheff’s Characteristic classes book.)

**Theorem 1.14.** If $\Lambda$ is an integral domain containing $\frac{1}{2}$, then $H^*(\tilde{G}_{2k+1}(\mathbb{R}^\infty))$ over $\Lambda$ is generated by the Pontryagin classes

$$p_1(\tilde{\gamma}_\infty^{2k+1}), \cdots, p_k(\tilde{\gamma}_\infty^{2k+1})$$

and $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))$ is generated by

$$p_1(\tilde{\gamma}_\infty^{2k}), \cdots, p_{k-1}(\tilde{\gamma}_\infty^{2k}), e(\tilde{\gamma}_\infty^{2m}).$$