1. Complex Vector bundles and complex manifolds

**Definition 1.1.** A complex vector bundle is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(n, \mathbb{C})$. Equivalently, it is a real vector bundle $E$ together with a bundle automorphism $J : E \rightarrow E$ satisfying $J^2 = -id$. (This is because any vector space with a linear map $J$ satisfying $J^2 = -id$ has a real basis identifying it with $\mathbb{C}^n$ and $J$ with multiplication by $i$). The map $J$ is called a complex structure on $E$.

An almost complex structure on a manifold $M$ is a complex structure on $E$. An almost complex manifold is a manifold together with an almost complex structure.

**Definition 1.2.** A complex manifold is a manifold with charts $\tau_i : U_i \rightarrow \mathbb{C}^n$ which are homeomorphisms onto open subsets of $\mathbb{C}^n$ and chart changing maps $\tau_i \circ \tau_j^{-1} : \tau_j(U_i \cap U_j) \rightarrow \tau_i(U_i \cap U_j)$ equal to biholomorphisms.

Holomorphic maps between complex manifolds are defined so that their restriction to each chart is holomorphic.

Note that a complex manifold is an almost complex manifold. We have a partial converse to this theorem:

**Theorem 1.3.** (Newlander-Nirenberg) (We won’t prove this).

An almost complex manifold $(M, J)$ is a complex manifold if:

$$[J(v), J(w)] = J([v, w]) + J([J(v), w]) + [v, w]$$

for all smooth vector fields $v, w$.

**Definition 1.4.** A holomorphic vector bundle $\pi : E \rightarrow B$ is a complex manifold $E$ together with a complex base $B$ so that the transition data: $\Phi_{ij} : U_i \cap U_j \rightarrow GL(n, \mathbb{C})$ are holomorphic maps (here $GL(n, \mathbb{C})$ is a complex vector space).

**Example 1.5.** We define $\mathbb{C}P^n$ to be the set of complex lines through the origin in $\mathbb{C}^n$. We define the transition maps in the same way as in $\mathbb{R}P^n$.

Coordinates are given equivalence classes of non-trivial vectors $[z_0, \cdots, z_n]$ in $\mathbb{C}^{n+1}$ where two such vectors are equivalent if they are a scalar multiple of each other. We define $U_i = \{z_i \neq 0\}$ and define:

$$\tau_i : U_i \rightarrow \mathbb{C}^n, \quad \tau_i([z_0, \cdots, z_n]) \equiv (\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}).$$

This has a canonical complex line bundle $\mathcal{O}(-1)$ whose fiber over a point $[z_0, \cdots, z_n]$ is the line through this point in $\mathbb{C}^{n+1}$. In other words it is the natural map

$$\pi_{\mathbb{C}P^n} : \mathbb{C}^{n+1} - 0 \rightarrow \mathbb{C}P^n, \quad \pi_{\mathbb{C}P^n}(z_0, \cdots, z_n) = [z_0, \cdots, z_n].$$

These are holomorphic vector bundles with trivializations over $U_i$ given by

$$\tau_i : \pi_{\mathbb{C}P^n}^{-1}(U_i) \rightarrow U_i \times \mathbb{C}, \quad \tau_i(z_0, \cdots, z_n) \equiv ([z_0, \cdots, z_n], z_k/z_i)$$

for some choice of $k \neq i$.

More generally we can define $Gr_k(\mathbb{C})$ to be the set of $k$-dimensional vector spaces in exactly the same way as we did for $Gr_k(\mathbb{R}^n)$. This is a complex manifold with a canonical complex bundle $\gamma_k^n(\mathbb{C})$.

Exercise: Show that the above manifolds and bundles are holomorphic.

**Example 1.6.** If $\pi : E \rightarrow B$ is a real vector bundle then $E \otimes \mathbb{C}$ is a complex vector bundle.

**Lemma 1.7.** If $\pi : E \rightarrow B$ is a complex vector bundle then $E_\mathbb{R}$ (the underlying real vector bundle) is oriented.
Proof. The choice of orientation comes from the fact that $GL(n, \mathbb{C})$ are orientation preserving maps and $\mathbb{C}^n$ has a canonical orientation sending $(x_j + iy_j)_{j \in \{1, \ldots, n\}}$ to $x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$. □

As a result, all complex vector bundles have Euler classes.

We define $Gr_k(\mathbb{C}^\infty)$ as the direct limit of $Gr_k(\mathbb{C}^n)$ as $n$ goes to infinity. This is a complex vector bundle (it is no longer holomorphic). The following theorem has exactly the same proof as the corresponding theorem over $\mathbb{R}$:

**Theorem 1.8.** $Gr_k(\mathbb{C}^n)$ is the classifying space for complex vector bundles.

More precisely: Let $[B, Gr_n(\mathbb{C}^\infty)]$ be the set of continuous maps $B \to K$ up to homotopy for some CW complex $B$. Let $Vect_n^C(B)$ be the set of isomorphism classes of complex vector bundles over $B$ of rank $k$. In other words the map:

$$i : [B, Gr_n(\mathbb{C}^\infty)] \to Vect_n^C(B), \quad i(f) \equiv f^*\gamma_k(\mathbb{C}), \quad \forall f : B \to Gr_n(\mathbb{C}^\infty)$$

is a bijection.

**Theorem 1.9.** Let $h_n : (\mathbb{C}P^\infty)^n \to Gr_n(\mathbb{C}^\infty)$ be the classifying map for the bundle $\bigoplus_{i=1}^n p_i^*\gamma_k(\mathbb{C})$ where $p_i : (\mathbb{C}P^\infty)^n \to \mathbb{C}P^\infty$ is the projection map to the $i$th factor.

Then $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[u]$ as a ring where $u$ has degree 2 and hence $H^*((\mathbb{C}P^\infty)^n; \mathbb{Z}) = \mathbb{Z}[u_1, \ldots, u_n]$ as a ring where $u_1, \ldots, u_n$ has degree 2.

Also the natural map $h_n^* : H^*(Gr_n(\mathbb{C}^\infty); \mathbb{Z}) \to H^*((\mathbb{C}P^\infty)^n; \mathbb{Z}) = \mathbb{Z}[u_1, \ldots, u_n]$ is injective with image equal to $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ where $\sigma_j$ is the $j$th symmetric polynomial in $u_1, \ldots, u_k$.

**Definition 1.10.** The $k$-th Chern class $c_k(E)$ of a complex vector bundle $\pi : E \to B$ is defined to be $f^*\sigma_k \in H^k(B; \mathbb{Z})$ where $f : B \to Gr_k(\mathbb{C}^\infty)$ is the classifying map for $E$.

We define $c(E) \equiv c_1(E) + c_2(E) + \cdots \in H^*(B; \mathbb{Z})$ to be the total Chern class of $E$.

**Proposition 1.11.** The Chern classes $c_k(E) \in H^{2k}(B)$ satisfy the following axioms and are uniquely characterized by them:

- **Dimension:** $c_0(E) = 1$ and $c_k(E) = 0$ for all $k > 2n$ where $n$ is the rank of our bundle.
- **Naturality:** Any two isomorphic complex bundles have the same chern classes. Also if $f : B' \to B$ is continuous then $c_k(f^*(E)) = f^*(c_k(E))$.
- **Whitney Sum:** For two complex vector bundles $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ we have that

$$c_k(E_1 \oplus E_2) = \sum_{j=0}^k c_j(E_1) \cup c_{k-j}(E_2).$$

- **Normalization:** $c_1(\mathbb{C}P^1(-1)) = -u$ where $H^*(\mathbb{C}P^1; \mathbb{Z}) = \mathbb{Z}[u]/u^2$, where $u$ has degree 2.

The proof is very similar to the analogous proof for Stiefel Whitney classes. (Exercise).

We will now classify all complex vector bundles over $\mathbb{C}P^1$. We need some preliminary lemmas.

**Lemma 1.12.** Let $G$ be a lie group and let $H$ be a closed lie subgroup. Then the coset space $G/H$ is a manifold and the quotient map $G \to G/H$ is a fiber bundle with fiber diffeomorphic to $H$.

We won’t prove this, we will just use it in the next lemma.

**Lemma 1.13.** The determinant map $det : GL(k, \mathbb{C}) \to \mathbb{C}^*$ is an isomorphism on $\pi_1$ and hence $\pi_1(GL(k, \mathbb{C})) = \mathbb{Z}$. 
Proof. By induction it is sufficient for us to show that $Gl(k - 1, \mathbb{C}) \rightarrow Gl(k, \mathbb{C})$ is an isomorphism on $\pi^1$ for $k > 1$. Now $Gl(k, \mathbb{C})$ acts transitively on $\mathbb{C}^k - 0$ and has stabilizer subgroup is isomorphic to the subgroup $G \subset GL(k, \mathbb{C})$ consisting of invertible matrices of the form:

$$
\begin{pmatrix}
1 & * & \cdots & * \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}.
$$

This means that the quotient $Gl(k, \mathbb{C})/G$ is diffeomorphic to $\mathbb{C}^k - 0$ and hence $Gl(k, \mathbb{C})$ is a fiber bundle over $\mathbb{C}^k - 0$ with fiber homotopic to $G$.

We have that $G$ deformation retracts on to $Gl(k - 1, \mathbb{C})$ and this deformation retraction $h_t : G \rightarrow G, t \in [0, 1]$ is given by

$$
\begin{pmatrix}
1 & x_1 & \cdots & x_k \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix} = \begin{pmatrix}
1 & tx_1 & \cdots & tx_k \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}.
$$

Here $GL(n, k)$ is identified with invertible matrices of the form:

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}.
$$

Since $Gl(k, \mathbb{C})$ is a fiber bundle over $\mathbb{C}^k - 0$ with fiber homotopic to $GL(k - 1, \mathbb{C})$ we get a long exact sequence:

$$
\pi_2(\mathbb{C}^k - 0) \rightarrow \pi_1(Gl(k - 1, \mathbb{C})) \rightarrow \pi_1(Gl(k, \mathbb{C})) \rightarrow \pi_1(\mathbb{C}^k - 0).
$$

Since $\mathbb{C}^k - 0$ is homotopic to a sphere of dimension $2k - 1$, we get that $\pi_j(\mathbb{C}^k - 0) = 0$ for $j = 1, 2$ as $k > 1$. Therefore the map

$$
\pi_1(Gl(k - 1, \mathbb{C})) \rightarrow \pi_1(Gl(k, \mathbb{C}))
$$

is an isomorphism and we are done by induction. \qed

Lemma 1.14. Complex vector bundles of rank $n$ over $\mathbb{CP}^1$ are classified by their first Chern class. There is exactly one such bundle with Chern class $mu$ for each $m \in \mathbb{Z}$ and this is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(m) \oplus \mathbb{C}^{n-1}$ where $\mathcal{O}_{\mathbb{CP}^1}(m) \equiv \mathcal{O}_{\mathbb{CP}^1}(-1)^{\oplus -m}$.

Proof. All such bundles are classified by homotopy classes of maps from $\mathbb{CP}^1 = S^2$ to $Gr_n(\mathbb{C}^\infty)$ and hence by $\pi_2(Gr_n(\mathbb{C}^\infty))$. Let $V_n \rightarrow Gr_n(\mathbb{C}^\infty)$ be the frame bundle of $\gamma^\infty_n(\mathbb{C})$ (i.e. the bundle whose fiber at a point is the set of bases of that fiber). This is a principal $GL(n, \mathbb{C})$ bundle and since $Gr_n(\mathbb{C}^\infty)$ is a classifying space, we have that $V_n$ is contractible. Hence we have a homotopy long exact sequence:

$$
\pi_2(V_n) \rightarrow \pi_2(Gr_n(\mathbb{C}^\infty)) \rightarrow \pi_1(GL(n, \mathbb{C})) \rightarrow \pi_1(V_n).
$$

Since $\pi_i(V_n) = 0$ for $i = 1, 2$, we get that $\pi_2(Gr_n(\mathbb{C}^\infty)) = \pi_1(GL(n, \mathbb{C})) = \mathbb{Z}$ by the previous lemma.

Since $\pi_2(Gr_n(\mathbb{C}^\infty)) = \mathbb{Z}$ we have that complex vector bundles of rank $n$ over $\mathbb{CP}^1$ are classified by $\mathbb{Z}$. A bundle representing $m \in \mathbb{Z}$ is built using the cluching construction:
Let \([z, w]\) be homogeneous coordinates for \(\mathbb{CP}^1\) and let \(U_1 = \{z \neq 0\}\) and \(U_2 = \{w \neq 0\}\). Then \(U_1 \cap U_2\) is homotopic to the equator \(S^1 \subset \mathbb{CP}^1 = S^2\). Therefore the transition maps

\[
\Phi_{12} : U_1 \to U_2 \to GL(n, \mathbb{C})
\]

are classified by elements of \(\pi_1(GL(n, \mathbb{C}))\). A bundle representing \(m \in \mathbb{Z}\) is therefore given by a map \(\Phi_{12}\) as above so that \(\det \circ \Phi_{12} : U_1 \cap U_2 \to \mathbb{C}^*\) represents \(m \in \pi_1(\mathbb{C}^*)\).

The bundle \(O_{\mathbb{CP}^1}(-1)\) has transition map

\[
\Phi_{12} : U_1 \cap U_2 \to \mathbb{C}^*, \quad \Phi_{12}([z, w] = z/w).
\]

Therefore the bundle \(O_{\mathbb{CP}^1}(m)\) has transition map

\[
\Phi_{12} : U_1 \cap U_2 \to \mathbb{C}^*, \quad \Phi_{12}([z, w] = (z/w)^{-m}).
\]

These bundles represent \(-m \in \mathbb{Z}\).

Therefore the bundles \(O_{\mathbb{CP}^1}(m) \oplus \mathbb{C}^{n-1}\) represent \(-m \in \mathbb{Z}\) as well and they represent all complex bundles of rank \(n\) up to isomorphism since \(\pi_2(GL(n, \mathbb{C})) = \mathbb{Z}\).

We now need to compute the first Chern class of these bundles. This is done as follows: It is sufficient for us to computing \(c_1(O_{\mathbb{CP}^1}(m))\). Since \(O_{\mathbb{CP}^1}(m) \oplus O_{\mathbb{CP}^1}(-m)\) is trivial, we get that \(c_1(O_{\mathbb{CP}^1}(m)) = -c_1(O_{\mathbb{CP}^1}(-m))\). Therefore we can assume that \(m < 0\).

Now \(O_{\mathbb{CP}^1}(m) = f_m^* \Omega_{\mathbb{CP}^1}(-1)\) where \(f_m\) is the map

\[
f_m : \mathbb{CP}^1 \to \mathbb{CP}^1, \quad f_m([z, w]) = [z^{-m}, w^{-m}].
\]

(this is well defined since \(-m > 0\)). Since \(f_m^*(u) = mu\), we get that \(c_1(O_{\mathbb{CP}^1}(m)) = m\). Hence \(O_{\mathbb{CP}^1}(m) = m\) for all \(m \in \mathbb{Z}\). \(\square

**Lemma 1.15.** The bundle \(O_{\mathbb{CP}^n}(-1)\) has no holomorphic sections other than the zero section.

**Proof.** If the bundle did have such a section then by restricting to \(\mathbb{CP}^1\) we would see that \(O_{\mathbb{CP}^1}(-1)\) has a holomorphic section. Therefore we can assume that \(n = 1\).

Define \(O_{\mathbb{CP}^1}(n) \equiv O_{\mathbb{CP}^1}(-1)^{\otimes n}\). Let \(U_1 = \{z \neq 0\}\) and \(U_2 = \{w \neq 0\}\). We have two trivializations \(\tau_j : O_{\mathbb{CP}^1}(n)|_{U_j} \to U_j \times \mathbb{C}, j = 1, 2\). The bundle \(O_{\mathbb{CP}^1}(n)\) is characterized by the transition data

\[
\Phi_{12} : U_1 \cap U_2 \to GL(1, \mathbb{C}) \equiv \mathbb{C}^*, \quad \Phi_{12}([z, w]) \equiv (z/w)^{-n}.
\]

This means that if \(n = 0\) then \(O_{\mathbb{CP}^1}(0)\) is isomorphic as a holomorphic bundle to the trivial bundle \(\mathbb{CP}^1 \times \mathbb{C}\).

If \(n = 1\) then we have a section \(s\) satisfying

\[
\tau_1 \circ (s|_{U_1})([z, w]) = ([z, w], z/w) \quad \text{and} \quad \tau_2 \circ (s|_{U_2})([z, w]) = ([z, w], 1).
\]

Now suppose that \(O_{\mathbb{CP}^1}(-1)\) has a section \(\sigma\). Since

\[
\iota : O_{\mathbb{CP}^1}(-1) \to O_{\mathbb{CP}^1}(1) = O_{\mathbb{CP}^1}(0) = \mathbb{CP}^1 \times \mathbb{C}
\]

is an isomorphism we get a section \(\iota(\sigma \otimes s)\) of \(\mathbb{CP}^1 \times \mathbb{C}\). Since all holomorphic functions on \(\mathbb{CP}^1\) are constant this implies that \(pr(\sigma \otimes s)\) is constant where \(pr : \mathbb{CP}^1 \times \mathbb{C} \to \mathbb{C}\) is the projection map.

Since \(s([0, 1]) = 0\) we then get that \(\sigma \otimes s([0, 1]) = 0\) which implies that \(\sigma \otimes s\) is the zero section. Since \(s\) is nonzero along \(U_2\) this implies that \(\sigma\) must be zero along \(U_2\). Since \(U_2\) is dense in \(\mathbb{CP}^1\), we then get that \(\sigma\) must be zero. Hence \(O_{\mathbb{CP}^n}(-1)\) only has one section given by the zero section. \(\square

**Lemma 1.16.** There exists a non-trivial holomorphic vector bundle which is trivial as a complex vector bundle.
Proof. We have that $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$ is trivial as a complex vector bundle.

Suppose that it was trivial as a holomorphic vector bundle. Then it would admit two holomorphic sections $s, s'$ which form a basis at each fiber. Since such sections are of the form $s = s_1 \oplus s_2$ and $s' = s'_1 \oplus s'_2$ where $s_1, s'_1$ are sections of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ and $s_2, s'_2$ are sections of $\mathcal{O}_{\mathbb{CP}^1}(1)$, we get that either $s_1$ or $s_2$ is a non-trivial section of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ which is impossible. □