HW 5

2.6 ② (0 pts) From fig. 7, and since the initial condition is $f(x)=0$, the maximum will be the maximum between $u(0,t)=T_0$ and $u(a,t)=$

$$= -\frac{K}{h}\frac{\partial u}{\partial x}(a,t)+T_1 \approx -\frac{K}{h}\cdot\frac{h(T_1-T_0)}{K+ha}+T_1 =$$

$$=(T_0-T_1)\cdot\frac{z}{z+1}+T_1 \quad \text{for } t>0 \quad (\text{and } 0 \text{ for } t=0).$$

(5 pts)
⑧ By Ex. 7, $b_m = \dfrac{2(1-\cos(\lambda_m a))}{\lambda_m\left(a+\frac{K}{h}\cos^2(\lambda_m a)\right)}$.

So in Ex. 8, the new coefficients are $B_m = T\cdot b_m$.

(10 pts)
⑩ This is about proving orthogonality for the eigenvalues λ_n (not their approximation).

$$\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx =$$

$$=\frac{1}{2}\int_0^a\left(\cos((\lambda_n-\lambda_m)x)-\cos((\lambda_n+\lambda_m)x)\right)dx =$$

$$=\frac{1}{2(\lambda_n-\lambda_m)}\left(\sin(\lambda_n a)\cos(\lambda_m a)-\cos(\lambda_n a)\sin(\lambda_m a)\right)+$$

$$-\frac{1}{2(\lambda_n+\lambda_m)}\cdot\left(-\frac{K\lambda_n}{h}\cos(\lambda_n a)\cos(\lambda_m a)-\frac{K\lambda_m}{h}\cos(\lambda_n a)\cos(\lambda_m a)\right)$$

and apply eq. (11).

(10 pts)
2.10 ④ $u(x,t)=\phi(x)T(t)$

$$\frac{\partial u}{\partial x}(0,t)=\phi'(0)\cdot T(t)=0 \implies \phi'(0)=0$$

$$\phi'(x)=-c_1\lambda\sin(\lambda x)+c_2\lambda\cos(\lambda x).$$

Thus $\phi'(0)=0 \Rightarrow C_2=0$. So $\phi(x;\lambda)=\cos(\lambda x)$.

As in book, we get $u(x,t)=\int_0^\infty A(\lambda)\cos(\lambda x)e^{-\lambda^2 kt}\,d\lambda$,

$A(\lambda)=\frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx$.

(6) (10 pts) (a) $\frac{\partial^2 u}{\partial x^2}=p^2e^{-px}\left(2\cos(\omega t-px)\right)=$

$=\frac{1}{k}\cdot\left(\omega e^{-px}\cos(\omega t-px)\right)=\frac{1}{k}\frac{\partial u}{\partial t}$

(c) (10 pts) Half a year corresponds to $t=\frac{\pi}{\omega}$.

Need x such that $u(0,0)=u(x,\frac{\pi}{\omega})\Rightarrow$

$\Rightarrow e^{-px}\sin(\pi-px)=0 \Rightarrow x=\frac{\pi}{p}=\pi\cdot\sqrt{\frac{2k}{\omega}}$.

(8) (10 pts) $\phi(x)=c_1e^{px}+c_2e^{-px}$

$\phi(0)=0 \Rightarrow c_2=-c_1 \Rightarrow \phi(x)=c_1(e^{px}-e^{-px})$ only bounded if $c_1=0$ (for $p>0$, $\lim(e^{px}-e^{-px})=\infty-0=\infty$, similarly for $p<0$).

For $p>0$, $\phi(x)$ bounded $\Rightarrow c_1=0 \Rightarrow \phi(x)=c_2e^{-px}$ so $\phi(0)=0\Rightarrow\phi\equiv0$ (similarly for $p<0$).

$\boxed{2.11}$ (4) (10 pts) The problem with condition $u(x,0)=f_0(x)$ for $x\in\mathbb{R}$ has solution given by eq. 7., which can be written:

$$u(x,t) = \frac{1}{\sqrt{4\kappa\pi t}} \cdot \left(\int_0^\infty f_0(x') \exp\left[\frac{-(x'-x)^2}{4\kappa t} \right] dx' + \right.$$

$$\left. + \underbrace{\int_{-\infty}^0 f_0(x') \exp\left[\frac{-(x'-x)}{4\kappa t} \right] dx'}_{} \right)$$

now just make a change of variable $y = -x'$.

(6) (10 pts).

• odd: $u(-x,t) = \frac{1}{\sqrt{4\kappa\pi t}} \int_{-\infty}^{\infty} f(x') \exp\left[\frac{-(x'+x)^2}{4\kappa t} \right] dx' =$

$$= \frac{1}{\sqrt{4\kappa\pi t}} \cdot \int_{\infty}^{-\infty} (-1) f(-y) \exp\left[\frac{-(x-y)^2}{4\kappa t} \right] dy \qquad y = -x'$$

$$= \left(\quad \right) \int_{-\infty}^{\infty} - f(y) \exp\left[\frac{-(y-x)^2}{4\kappa t} \right] dy = -u(x,t).$$

• periodic: $u(x+2a,t) = \frac{1}{\sqrt{4\kappa\pi t}} \int_{-\infty}^{\infty} f(x') \exp\left[\frac{-(x'-x-2a)^2}{4\kappa t} \right] dx'$

$= u(x,t)$ by change of variable $y = x' - 2a$.

(7) (5 pts) By eq. 7, and change of var. $y = -x'$,

$$u(x,t) = \frac{1}{\sqrt{4\kappa\pi t}} \left(\int_0^\infty \exp\left[\frac{-(x'-x)^2}{4\kappa t} \right] dx' + \right.$$

$$\left. + \int_0^\infty (-1) \exp\left[\frac{-(y+x)^2}{4\kappa t} \right] dy \right).$$